

On double gaussian sums and theta
transformation formula

C. Y. LIN

Introduction

It is known that in the Theta Transformation Formula

$$\theta_{\sigma \cdot m}(\sigma, \tau) = k(\sigma) e(\phi_m(\sigma) \det(\gamma\tau + \delta)^{\frac{1}{2}}) \theta_m(\tau) \cdots \cdots \cdots (1)$$

$k^8(\sigma) = 1$ [I, P.176, P.182]. A formula for $k(\sigma)$ would be of values in the theory of modular forms. For the genus one theta group $\Gamma_1(1,2)$ we have the following formula

$$k^2(\sigma) = \begin{cases} i^{d-1} = i^{a-1} = (-1)^{\frac{1}{2}(d-1)} = (-1)^{\frac{1}{2}(a-1)} & \text{if } a, d \text{ are odd} \\ i^c = -i^b & \text{if } b, c \text{ are odd} \end{cases}$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(1,2)$. [II]

In this paper, a formula of $k(\sigma)$ for a special kind of element in genus two theta group $\Gamma_2(1,2)$ is obtained.

§1. Theta group of genus two $\Gamma_2(1,2)$.

$$\text{Let } \Gamma_2(1) = \left\{ \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(4, \mathbb{Z}) \mid \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t_\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

where $\alpha, \beta, \gamma, \delta, 0, 1$ are all two by two integral matrices (last two being zero and identity). $\Gamma_2(1)$ is the Siegel modular group of genus two. A matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $M(4, \mathbb{Z})$ belongs to $\Gamma_2(1)$ if and only if $\alpha^t \beta, \gamma^t \delta$ are symmetric and $\alpha^t \delta - \beta^t \gamma = 1$.

Let

$$\Gamma_2(1,2) = \left\{ \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_2(1) \mid (\alpha^t \beta)_0 \equiv (\gamma^t \delta)_0 \equiv 0 \pmod{2} \right\}$$

where $\begin{pmatrix} x & y \\ z & u \end{pmatrix}_0 = \begin{pmatrix} x \\ u \end{pmatrix}$. $\Gamma_2(1,2)$ is a subgroup of $\Gamma_2(1)$, usually called Theta group of genus 2. In this paper we shall consider only those σ in $\Gamma_2(1,2)$ whose γ is a scalar matrix. In below we shall state without proof some properties of $\Gamma_2(1,2)$ which we need in the sequel.

Lemma 1. Let $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_2(1,2)$, $\gamma = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$, c is a positive integer. Then

$$(i) \quad \delta \text{ is symmetric, hence } \delta = \begin{pmatrix} d_1 & d \\ d & d_2 \end{pmatrix}. \text{ Similarly,}$$

$$\alpha \text{ is symmetric, hence } \alpha = \begin{pmatrix} a_1 & a \\ a & a_2 \end{pmatrix}.$$

- (ii) cd_1 and cd_2 are both even
- (iii) $(c, d, d_1) = (c, d, d_2) = 1$.
- (iv) $(c, \det \delta) = (c, \det \alpha) = 1$.

Lemma 2. (classification)

In the notation of Proposition 1, σ can be classified into the following four cases:

- (I) c is odd; then d_1, d_2 are both even.
- (II) c is even, d_1 and d_2 are both even; hence d is odd.
- (III) c is even, d_1 and d_2 are of opposite parity; hence

d is odd. Without loss of generality, we may assume

d_1 is odd and d_2 is even.

(IV) c is even, d_1 and d_2 are both odd; hence d even.

§2. Theta transformation formula

In [III, P.181] Krazer gave another form of theta transformation formula, for $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_2(1)$, $\det \gamma \neq 0$,

$$\theta_m(\tau) = (\det \gamma)^{-1/2} \det(\gamma\tau + \delta)^{-1/2} (\pi i)^{-1} |\det \gamma|^{-1} G \underline{e}(-\frac{\phi}{2}) \underline{e}(\frac{\psi(m)}{2}) \theta_{\sigma \cdot m}(\sigma \cdot \tau) \dots \dots \dots (2)$$

Equating formula (1) and (2), then put $m = 0$, we get

$$k(\sigma) = -i |\det \gamma|^{\frac{3}{2}} G^{-1} \underline{e}(-\frac{\phi}{2}) \dots \dots \dots (3)$$

where

$$\left\{ \begin{array}{l} \phi = -\frac{1}{4} {}^t(\gamma^t \delta)_0 \alpha \gamma^{-1} (\gamma^t \delta)_0 - \frac{1}{2} {}^t(\gamma^t \delta)_0 (\alpha^t \beta)_0 \dots \dots \dots (4) \end{array} \right.$$

$$\left\{ \begin{array}{l} G = \sum_{\xi_1, \xi_2=0}^{|\det \gamma|-1} \underline{e}(-\frac{1}{2} \xi (\gamma^{-1} \delta)^t \xi + \frac{1}{2} \xi \gamma^{-1} (\gamma^t \delta)_0) \dots \dots \dots (5) \end{array} \right.$$

Lemma 3. If $\sigma \in \Gamma_2(1,2)$, $\gamma = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$, c is a positive integer, then

$$\underline{e}(-\frac{\phi}{2}) = \begin{cases} 1 & \text{for case (I) and (II)} \\ (-1)^{(c/2)(a_1/2)} & \text{for case (III)} \dots \dots \dots (6) \\ (-1)^{(c/2)(a_1+a_2)/2} & \text{for case (IV)} \end{cases}$$

§3. Degree two gaussian density G .

Lemma 4. With notations as in Lemma 3, G defined by (5), we have

$$G = c^2 \sum_{\text{mod } c} \underline{e}(-\frac{1}{2c} \xi \delta^t \xi + \frac{1}{2} \xi (\delta)_0)$$

Let $H = H(c, \delta) = \sum_{\xi \bmod c} e(-\frac{1}{2c} \xi \delta^t \xi + \frac{1}{2} \xi(\delta)_0)$ and let $\psi(n, T) = \sum_{\xi \bmod n} e(-\frac{1}{n} \xi T^t \xi)$ where n is a positive integer and T is a two by two integer symmetric matrix.

Lemma 5.

$$H = \begin{cases} \frac{1}{4} \psi(2c, -\delta) & \text{for case (I), } d \text{ is odd} \\ \psi(c, -\frac{\delta}{2}) & \text{for case (I), } d \text{ is even} \\ \frac{1}{4} \psi(2c, -\delta) & \text{for case (II)} \\ \psi(c, -\delta') - \frac{1}{4} \psi(2c, -\delta) & \text{for case (III)} \\ H(c, \delta'') & \text{for case (IV)} \end{cases}$$

where

$$\delta' = \begin{pmatrix} 2d_1 & d \\ d & \frac{d_2}{2} \end{pmatrix} \quad \delta'' = \begin{pmatrix} d_1 & d_1+d \\ d_1+d & d_1+d_2+2d \end{pmatrix}.$$

§4. Separation and reduction of double gaussian sums $\psi(n, T)$.

Double gaussian sums $\psi(n, T)$ was considered by Weber in [IV], the following three propositions are keys for the computation of $\psi(n, T)$:

Lemma 6. [IV, P.36] If $n > 0$ is an odd integer, and if

$$(\det T, n) = 1, \text{ then } \psi(n, T) = \left(\frac{\det T}{n} \right) (-1)^{\frac{n-1}{2}} n$$

where $(-)$ is the Legendre symbol.

Lemma 7. (Separation) [IV, P.17] If $(n_1, n_2) = 1$ then

$$\psi(n_1 n_2, T) = \psi(n_1, n_2 T) \psi(n_2, n_1 T).$$

Lemma 8. (Reduction) [IV, P.44]

If $\det T$ is odd, then $\psi(2^p, T) = 4 \psi(2^{p-2}, T)$ if $p \geq 4$ or if $p \geq 3$ and $T_0 \equiv 0 \pmod{2}$.

Lemma 9. Let T be an integral symmetric matrix,

then $\psi(1, T) = 1$

$$\psi(2, T) = [1 + (-1)^{t_1}] \times [1 + (-1)^{t_2}]$$

$$\psi(2^2, T) = 2^2 (1 + i^{t_1} + i^{t_2} + i^{t_1+t_2+2t})$$

$$\psi(2^3, T) = 2^4 \cdot i^{t_2/2} \quad \text{if } t_1, t_2 \text{ odd, } t_2 \text{ even.}$$

It follows from Lemma 9 that

$$\psi(2, T) = \begin{cases} 2^2 & \text{if } (T)_0 \equiv 0 \pmod{2}. \\ 0 & \text{otherwise} \end{cases}$$

$$\psi(2^2, T) = \begin{cases} 2^3 (-1)^{(t_1/2) \cdot (t_2/2)} & \text{if } t_1 \equiv t_2 \equiv 0, t \not\equiv 0 \pmod{2} \\ 2^3 & \text{if } t_1 \equiv t \equiv 1 \pmod{2}, t_2 \equiv 0 \pmod{4}. \\ 2^3 \cdot i^{-t_1} & \text{if } t_1 \equiv t \equiv 0 \pmod{2}, t_2 \equiv 2 \pmod{4}. \end{cases}$$

§5. Calculations and final results

(Case I) If d is even then $H = \psi(c, -\frac{\delta}{2}) = (\frac{\det(-\frac{\delta}{2})}{c})$.
 $(-1)^{\frac{c-1}{2}} c = (\frac{\det \delta}{c}) (-1)^{\frac{c-1}{2}} c$, by Lemmas 5 and 6 and the fact
 $(\frac{\det \delta}{c}) = (\frac{2^2 \det(\delta/2)}{c}) = (\frac{\det(\delta/2)}{c}) = (\frac{\det(-\delta/2)}{c})$.

If d is odd, $H = \frac{1}{4} \psi(2c, -\delta) = \frac{1}{4} \psi(c, -2\delta) \psi(2, -c\delta)$
 $= \frac{1}{4} (\frac{\det(-2\delta)}{c}) (-1)^{(c-1)/2} c \times 4 = (\frac{\det \delta}{c}) (-1)^{(c-1)/2} c$, by
 Lemma 5 and 6 and the fact that

$$\psi(2, -c\delta) = \sum_{\xi \pmod{2}} e(-\frac{c}{2} \xi \delta^t \xi) = 4.$$

Therefore, in case (I), $H = \left(\frac{\det \delta}{c}\right) (-1)^{(c-1)/2} c$.

(Case II)

Let $c = 2^{p-1} \cdot q$

$$\begin{aligned}
 H &= \frac{1}{4} \psi(2c, -\delta) \\
 &= \frac{1}{4} \psi(2^p q, -\delta) \\
 &= \frac{1}{4} \psi(2^p, -q\delta) \psi(q, -2^p \delta) \\
 &= \frac{1}{4} \left(\frac{\det(-2^p \delta)}{q}\right) (-1)^{(q-1)/2} q \times \psi(2^p, -q\delta) \\
 &= \frac{1}{4} \left(\frac{\det \delta}{q}\right) (-1)^{(q-1)/2} q \times \psi(2^p, -q\delta) \dots \dots \dots (6)
 \end{aligned}$$

in which $p \geq 2$ for c is even.

Suppose p is odd, $p = 2k+1$, then by reduction k times and Lemma 9,

$$\begin{aligned}
 \psi(2^p, -q\delta) &= \psi(2^{2k+1}, -q\delta) \\
 &= 4^k \psi(2, -q\delta) \\
 &= 4^k \times 4 \\
 &= 4^{k+1} \\
 &= 2^{p+1} \dots \dots \dots (7)
 \end{aligned}$$

Suppose p is even, $p = 2k$, then by reduction $k-1$ times and Lemma 9,

$$\begin{aligned}
 \psi(2^p, -q\delta) &= \psi(2^{2k}, -q\delta) \\
 &= 4^{k-1} \psi(2^2, -q\delta) \\
 &= 4^{k-1} \times 2^3 \times (-1)^{(d_1/2) \cdot (d_2/2)} \\
 &= 2^{2k+1} \cdot (-1)^{(d_1/2) \cdot (d_2/2)}
 \end{aligned}$$

$$= 2^{p-1} \cdot (-1)^{(d_1/2) \cdot (d_2/2)} \dots \dots \dots (8) \quad 7$$

(7) and (8) can be combined in one formula:

$$\psi(2^p, -q\delta) = 2^{p+1} \cdot (-1)^{(p-1) \cdot (d_1/2) \cdot (d_2/2)} \dots \dots \dots (9)$$

Put (9) into (6), we get

$$\begin{aligned} H &= \left(\frac{\det \delta}{q}\right) (-1)^{(q-1)/2} \cdot q \cdot 2^{p-1} \cdot (-1)^{(p-1) \cdot (d_1/2) \cdot (d_2/2)} \\ &= \left(\frac{\det \delta}{q}\right) (-1)^{(q-1)/2 + (p-1) \cdot (d_1/2) \cdot (d_2/2)} \cdot c. \end{aligned}$$

(Case III)

$$H = \psi(c, -\delta') - \frac{1}{4} \psi(2c, -\delta) \dots \dots \dots (10)$$

$$\begin{aligned} \psi(2c, -\delta) &= \psi(2^p q, -\delta) \\ &= \psi(q, -2^p \delta) (2^p, -q\delta) \dots \dots \dots (11) \end{aligned}$$

where $\psi(q, -2^p \delta) = \left(\frac{\det \delta}{q}\right) (-1)^{(q-1)/2} q$ by proposition 6

Suppose p is odd, $p = 2k+1$, then by reduction $k-1$ times and Lemma 9

$$\begin{aligned} \psi(2^p, -q\delta) &= \psi(2^{2k+1}, -q\delta) \\ &= 4^{k-1} \psi(2^3, -q\delta) \\ &= 4^{k+1} i^{-q(d_2/2)} \\ &= 2^{p+1} i^{-q(d_2/2)} \dots \dots \dots (12) \end{aligned}$$

Suppose p is even, $p = 2k$, then by reduction $k-1$ times and Lemma 9,

$$\begin{aligned} \psi(2^p, -q\delta) &= \psi(2^{2k}, -q\delta) \\ &= 4^{k-1} \psi(2^2, -q\delta) \\ &= 2^{p+1} \times \begin{cases} 1 & \text{if } d_2/2 \text{ even} \\ i^{-2d_1} & \text{if } d_2/2 \text{ odd,} \end{cases} \dots \dots \dots (13) \end{aligned}$$

Put (12), (13) into (11), we get

$$\begin{aligned}
 \psi(2c, -\delta) &= \left(\frac{\det \delta}{q}\right) (-1)^{(q-1)/2} q \times 2^{p+1} \\
 &\times \begin{cases} i^{-q(d_2/2)} & \text{if } p \text{ is odd} \\ 1 & \text{if } p, d_2/2 \text{ are even} \\ i^{-2d_1} & \text{if } p \text{ is even, } d_2/2 \text{ is odd.} \end{cases} \\
 &= 4 \left(\frac{\det \delta}{q}\right) (-1)^{(q-1)/2} c \\
 &\times \begin{cases} i^{-q(d_2/2)} & p \text{ odd} \\ 1 & p, d_2/2 \text{ even} \\ i^{-qd_1} & p \text{ even, } d_2/2 \text{ odd.} \dots\dots (14) \end{cases}
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \psi(c, -\delta') &= \psi(2^{p-1} \cdot q, -\delta') \\
 &= \psi(q, -2^{p-1} \delta') \psi(2^{p-1}, -q\delta') \dots\dots\dots (15)
 \end{aligned}$$

where $\psi(q, -2^{p-1} \delta') = \left(\frac{\det \delta}{q}\right) (-1)^{(q-1)/2} q$ by Lemma 6 and the fact $\det \delta' = \det \delta$.

Suppose p is odd, $p = 2k+1$, then by reduction $k-1$ time and Lemma 9, we get

$$\begin{aligned}
 \psi(2^{p-1}, -q\delta') &= 4^{k-1} \psi(2^2, -q\delta') \\
 &= 2^p i^{-(d_2/2) \cdot q} \dots\dots\dots (16)
 \end{aligned}$$

Suppose p is even, and $p \geq 4$, $p = 2k$, then by reduction $k-2$ times and Lemma 9, we get

$$\begin{aligned}
 \psi(2^{p-1}, -q\delta') &= 4^{k-2} \psi(2^3, -q\delta') \\
 &= 2^p \times \begin{cases} 1 & \text{if } d_2/2 \text{ even} \\ i^{-qd_1} & \text{if } d_2/2 \text{ odd.} \dots\dots\dots (17) \end{cases}
 \end{aligned}$$

Put (16), (17) into (15), we get

$$\begin{aligned}
 \psi(c, -\delta') &= \left(\frac{\det \delta}{q}\right) (-1)^{(q-1)/2} \cdot q \times 2^p \\
 &\quad \times \begin{cases} i^{-(d_2/2)q} & p \text{ odd} \\ 1 & p, d_2/2 \text{ even} \\ i^{-qd_1} & p \text{ even, } d_2/2 \text{ odd.} \end{cases} \\
 &= 2 \left(\frac{\det \delta}{q}\right) (-1)^{(q-1)/2} \cdot c \\
 &\quad \times \begin{cases} i^{-(d_2/2)q} & p \text{ odd} \\ 1 & p, d_2/2 \text{ even} \\ i^{-qd_1} & p \text{ even, } d_2/2 \text{ odd.} \end{cases} \dots\dots\dots (18)
 \end{aligned}$$

Put (18) and (14) into (10), we get

$$H = \left(\frac{\det \delta}{q}\right) (-1)^{(q-1)/2} \cdot c \times \begin{cases} i^{(d_2/2) \cdot q} & p \text{ odd} \\ 1 & p, d_2/2 \text{ even} \\ i^{-qd_1} & p \text{ even, } d_2/2 \text{ odd} \end{cases}$$

In case $p = 2$, i.e $c = 2q$, we get from Lemma 9 that

$$\psi(2, -q\delta) = \begin{cases} 4 & \text{if } d_2/2 \text{ is even} \\ 0 & \text{otherwise} \end{cases} \dots\dots\dots (19)$$

Put (19) into (15), we get

$$\psi(c, -\delta') = 2 \left(\frac{\det \delta}{q}\right) (-1)^{(q-1)/2} \cdot c \times \begin{cases} 1 & d_2/2 \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Put (20) and (14) into (10), we get

$$H = \left(\frac{\det \delta}{q} \right) (-1)^{(q-1)/2} c \times \begin{cases} 2 - 1 = 1 & \text{if } p=2, d_2/2 \text{ even} \\ 0 - i^{-qd_1} = -i^{-qd_1} & \text{if } p=2, d_2/2 \text{ odd} \end{cases}$$

(Case IV):

By Lemma 5, $G = G(c, \delta) = G(c, \delta'')$ where $\delta'' = \begin{pmatrix} d_1 & d + d_1 \\ d + d_1 & d_1 + d_2 + d \end{pmatrix}$ belongs to case III, hence Case IV follows from case III.

In summary, we have

Proposition 1. If $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_2(1, 2)$, $\gamma = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$,

$\delta = \begin{pmatrix} d_1 & d \\ d & d_2 \end{pmatrix}$, $c > 0$, $c = 2^{p-1} \cdot q$, q is odd then

$$G = \begin{cases} c^3 \left(\frac{\det}{c} \right) (-1)^{(c-1)/2} & \text{if } c \text{ is odd} \\ c^3 \cdot \left(\frac{\det}{q} \right) (-1)^{(q-1)/2} + (d_1/2) \cdot (d_2/2) (p-1) & \text{if } c, d_1, d_2 \\ & \text{are all even} \\ c^3 \cdot \left(\frac{\det}{q} \right) (-1)^{(q-1)/2} \cdot \zeta & \text{if } c \text{ is even,} \\ & d_1 \text{ is even and} \\ & d_2 \text{ is odd.} \\ c^3 \cdot \left(\frac{\det}{q} \right) (-1)^{(q-1)/2} \cdot \xi & \text{if } c \text{ is even,} \\ & d_1 \text{ and } d_2 \text{ are} \\ & \text{odd.} \end{cases}$$

where

$$\zeta = \begin{cases} i^{-(d_2/2)q} & p: \text{odd} \\ 1 & p \text{ and } \frac{d_2}{2} \text{ even} \\ i^{-qd_1} & p > 2 \text{ even, } d_2/2 \text{ odd} \\ -i^{-qd_1} & p = 2, d_2/2 \text{ odd.} \end{cases}$$

$$\xi = \begin{cases} i^{-[(d_1+d_2+2d)/2]q} & p: \text{ odd} \\ 1 & p: \text{ even and } d_1 \not\equiv d_2 \pmod{4} \\ i^{-qd_1} & p > 2, \text{ even, and } d_1 \equiv d_2 \pmod{4} \\ -i^{-qd_1} & p=2, d_1 \equiv d_2 \pmod{4} \end{cases}$$

Proposition 2. Notations as in Proposition 1, we have correspondingly

$$K(\sigma) = \begin{cases} i(\frac{\det \delta}{c}) (-1)^{(c+1)/2} \\ i(\frac{\det \delta}{q}) (-1)^{(q+1)/2} + (d_1/2)(d_2/2) (p-1) \\ i(\frac{\det \delta}{q}) (-1)^{(q+1)/2} + (c/2)(a_1/2) \cdot \zeta^{-1} \\ i(\frac{\det \delta}{q}) (-1)^{(q+1)/2} + (c/2)(a_1+a_2)/2 \cdot \xi^{-1} \end{cases}$$

Reference

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