

# Optimization of threshold memberships over fuzzy decision process

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## 1 Introduction

Since Bellman and Zadeh has proposed three – *deterministic*, *stochastic* and *fuzzy* – systems on multistage decision processes in fuzzy environment [4, §4,5], an intensive study on fuzzy decision making under uncertainty has been developed both in theory and in its wide applications ([1, 5, 15] and others). In Markov decision processes [6, 17], it has been tacitly known that there exists an optimal policy which is *Markov* for the *additive* criterion, where Markov policy takes decision on the basis of only today's state. Recently, from a stochastic control theory, Iwamoto has developed Bellman and Zadeh's fuzzy decision-making on the stochastic system for a *non-additive (minimum)* criterion. He has shown that an optimal policy does not exist in Markov class for minimum criterion but does exist in *general* class, where general policy depends on state-sequence up until today [7, 8, 9, 10, 11, 12, 13, 14, 18]. His tool is identical twins – both dynamic programming [2] and invariant imbedding [3, 16] – for non-additive criterion in stochastic system [8, 9].

In this paper, we consider a “threshold probability” decision-making in fuzzy environment. On the multi-stage stochastic control process, we evaluate the threshold probability that the minimum criterion exceeds a lower membership-degree. The minimum criterion denotes a *total* membership function of the multistage fuzzy decision process with *stage-wise* membership functions and a *goal* membership function. It is the membership function of intersection of the underlying fuzzy sets [4, p.144, §4,5]. Under the controlled Markov chain we optimize the threshold probability not in general class but in Markov class. We show that this choice will be successful; there exists an optimal policy in Markov class. We also derive the recursive relation for the threshold probability. We use the notations and terminology in [4, 8, 9].

## 2 Decision Process with Threshold Probability

Let us consider an  $N$ -stage ( $N \geq 2$ ) stochastic decision process  $\{(X_n, U_n)\}_0^N$  on a finite state space  $X$  and decision space  $U$ , which is governed by a Markov transition law  $p = \{p(\cdot|\cdot, \cdot)\}$  :

$$p(y|x, u) \geq 0, \quad \sum_{y \in X} p(y|x, u) = 1.$$

Thus  $p(y|x, u)$  is a conditional probability that the next state  $X_{n+1}$  will be  $y$  when the current state  $X_n$  is  $x$  and current decision  $U_n$  is  $u$  :

$$P(X_{n+1} = y | X_n = x, U_n = u) = p(y|x, u).$$

This transition is expressed as  $X_{n+1} \sim p(\cdot|x, u)$ .

We begin to introduce a large class of policies, which depend not only on today's state but also on state-to-date. Let  $X^n := X \times X \times \cdots \times X$  be direct product of  $n$  state spaces  $X$ . A mapping  $\sigma_n : X^{n+1} \rightarrow U$  is called  $n$ -th *general decision function*, whose sequence  $\sigma = \{\sigma_0, \sigma_1, \dots, \sigma_{N-1}\}$  constitutes a *general policy*. The set of all general policies  $\Pi_g$  is called *general class*. When each general decision function  $\sigma_n$  depends only on the last (= current) state, the general policy reduces to a *Markov policy*  $\pi = \{\pi_0, \pi_1, \dots, \pi_{N-1}\}$ . Let *Markov class*  $\Pi$  denote the set of all Markov policies. Thus we have an inclusion relation :  $\Pi \subset \Pi_g$ .

Further, given an  $n$ -th *membership function*  $\mu_n : X \times U \rightarrow [0, 1]$  ( $0 \leq n \leq N-1$ ) and a *goal membership function*  $\mu_G : X \rightarrow [0, 1]$ , the random variables  $\mu_n = \mu_n(X_n, U_n)$ ,  $\mu_G = \mu_G(X_N)$  denote the resulting grade of membership [4].

Now we consider the problem of maximizing a threshold probability that *total membership* is greater than or equal to a given *lower grade*  $\alpha \in [0, 1]$  :

$$\begin{aligned} & \text{Maximize } P_{x_0}^\pi(\mu_0 \wedge \mu_1 \wedge \cdots \wedge \mu_{N-1} \wedge \mu_G \geq \alpha) \\ P_0(x_0) \quad & \text{subject to (i)}_n X_{n+1} \sim p(\cdot | x_n, u_n) \quad 0 \leq n \leq N-1 \\ & \text{(ii)}_n u_n \in U \end{aligned}$$

where  $P_{x_0}^\pi$  is the (discrete) probability measure on history space

$$H_N := X \times U \times X \times U \times \cdots \times U \times X \quad (2N+1)\text{-factors}$$

induced through an initial state  $x_0$ , the Markov transition law  $p$  and a Markov policy  $\pi (\in \Pi)$ .

We dare to maximize the threshold probability over Markov class  $\Pi$ . We do not optimize it over general class  $\Pi_g$ . This choice will be turned to generate a valid recursive equation. Any Markov policy  $\pi (\in \Pi)$  determines the threshold probability in  $P_0(x_0)$ , which is a "partial" multiple sum :

$$P_{x_0}^\pi(\mu_0 \wedge \mu_1 \wedge \cdots \wedge \mu_{N-1} \wedge \mu_G \geq \alpha) = \sum_{(x_1, x_2, \dots, x_N) \in (*)} \sum \cdots \sum p_0 p_1 \cdots p_{N-1} \quad (1)$$

$$(p_n = p(x_{n+1} | x_n, u_n))$$

where the domain  $(*)$  is the set of all  $(x_1, x_2, \dots, x_N) \in X^N$  satisfying

$$\mu_0(x_0, u_0) \wedge \mu_1(x_1, u_1) \wedge \cdots \wedge \mu_{N-1}(x_{N-1}, u_{N-1}) \wedge \mu_G(x_N) \geq \alpha. \quad (2)$$

Here the sequence of decisions  $\{u_0, u_1, \dots, u_{N-1}\}$  in (1),(2) is uniquely determined through Markov policy  $\pi = \{\pi_0, \dots, \pi_{N-1}\}$  :

$$u_0 = \pi_0(x_0), u_1 = \pi_1(x_1), \dots, u_{N-1} = \pi_{N-1}(x_{N-1}). \quad (3)$$

As for controlling threshold probability on the Markov chain  $\{(X_n, U_n)\}$  with reward functions  $\{r_n\}_0^{N-1}$ ,  $r_N$  and a lower level value  $c$ , Markov class  $\Pi$  is not enough for *additive* criteria :

$$\begin{aligned} & \text{Maximize } P_{x_0}^\sigma(r_0 + r_1 + \cdots + r_{N-1} + r_N \geq c) \\ Q_0(x_0) \quad & \text{subject to (i)}_n, \text{ (ii)}_n \quad 0 \leq n \leq N-1 \end{aligned}$$

but general class  $\Pi_g$  is enough [8, 18]. However, in this paper, we dare to maximize the threshold probability for *minimum* criteria over *Markov class*.

Thus our problem  $P_0(x_0)$  is to find the *maximum value function*  $v_0 = v_0(x_0)$  and an *optimal policy*  $\pi^* (\in \Pi)$  which attains the maximum :

$$\begin{aligned} v_0(x_0) &= P_{x_0}^{\pi^*}(\mu_0 \wedge \cdots \wedge \mu_{N-1} \wedge \mu_G \geq \alpha) \\ &= \max_{\pi \in \Pi} P_{x_0}^\pi(\mu_0 \wedge \cdots \wedge \mu_{N-1} \wedge \mu_G \geq \alpha). \end{aligned} \quad x_0 \in X \quad (4)$$

### 3 Recursive formula

We consider the subproblem starting at state  $x_n (\in X)$  on  $n$ -th stage and terminating on the final  $N$ -th stage ( $0 \leq n \leq N-1$ ) :

$$P_n(x_n) \quad \text{Max } P_{x_n}^\pi(\mu_n \wedge \cdots \wedge \mu_{N-1} \wedge \mu_G \geq \alpha) \quad \text{s.t. (i)}_m, \text{(ii)}_m \quad n \leq m \leq N-1$$

where  $\pi = \{\pi_n, \dots, \pi_{N-1}\}$  is taken over Markov class from  $n$ -th stage on  $\Pi(n)$ .

Let  $v_n(x_n)$  be the maximum value, where

$$v_N(x_N) \triangleq P(\mu_G \geq \alpha | X_N = x_N) = \begin{cases} 1 & x_N \geq \alpha \\ 0 & \text{otherwise} \end{cases} \quad x_N \in X. \quad (5)$$

**Lemma 3.1** We have for any  $0 \leq n \leq N-1$ ,  $x_n \in X$  and  $\pi = \{\pi_n, \dots, \pi_{N-1}\} \in \Pi(n)$

$$P_{x_n}^\pi(\mu_n \wedge \cdots \wedge \mu_G \geq c) = \begin{cases} \sum_{x_{n+1} \in X} P_{x_{n+1}}^{\pi'}(\mu_{n+1} \wedge \cdots \wedge \mu_G \geq c) p(x_{n+1} | x_n, u_n) & \text{if } \mu_n(x_n, u_n) \geq \alpha \\ 0 & \text{otherwise} \end{cases}$$

where  $u_n = \pi_n(x_n)$ ,  $\pi' = \{\pi_{n+1}, \dots, \pi_{N-1}\}$  and  $P_{x_N}^{\pi'} := P$  in (5) for  $\pi = \{\pi_{N-1}\}$ .

Equivalently, in terms of multiple sum, we get

$$\sum_{(x_{n+1}, x_{n+2}, \dots, x_N) \in (*)} p_{n+1} p_{n+2} \cdots p_N = \begin{cases} \sum_{x_{n+1} \in X} \left[ \sum_{(x_{n+2}, \dots, x_N) \in (*)} p_{n+2} \cdots p_N \right] p(x_{n+1} | x_n, u_n) & \text{if } \mu_n(x_n, u_n) \geq \alpha \\ 0 & \text{otherwise} \end{cases}$$

where  $p_m = p(x_m | x_{m-1}, u_{m-1})$ ,  $u_m = \pi_m(x_m)$ ,  $(*)$  denotes the partial multiple sum over  $(x_{n+1}, \dots, x_N) \in X \times \cdots \times X$  satisfying  $\mu_n(x_n, u_n) \wedge \cdots \wedge \mu_G(x_N) \geq \alpha$ , and  $(*)$  denotes  $(x_{n+2}, \dots, x_N)$  satisfying  $\mu_{n+1}(x_{n+1}, u_{n+1}) \wedge \cdots \wedge \mu_G(x_N) \geq \alpha$ .

Thus we have the backward recursive relation :

**Theorem 3.1** (Recursive Equation)

$$v_N(x) = \begin{cases} 1 & \text{if } \mu_G(x) \geq \alpha \\ 0 & \text{otherwise} \end{cases} \quad x \in X$$

$$v_n(x) = \begin{cases} \text{Max}_{u; \mu_n(x, u) \geq \alpha} \sum_{y \in X} v_{n+1}(y) p(y | x, u) & \text{if } \exists u; \mu_n(x, u) \geq \alpha \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

$$x \in X, \quad 0 \leq n \leq N-1.$$

Now let us take any pair  $(n, x)$ . If it satisfies  $\mu_n(x, u) \geq \alpha$ , then let  $\pi_n^*(x)$  denote a  $u^* \in U$  which attains the maximum in (6). Otherwise, let  $\pi_n^*(x)$  denote any  $u \in U$ . Then we have an optimal  $n$ -th decision function  $\pi_n^* : X \rightarrow U$ . Thus we construct an optimal policy  $\pi^* = \{\pi_0^*, \dots, \pi_{N-1}^*\}$  in Markov class  $\Pi$ .

## 4 Bellman and Zadeh's Model

Let us consider maximizing the threshold probability with lower membership-degree  $\alpha = 0.7$  on Bellman and Zadeh's model [4, pp.B154] :

$$\begin{aligned} \text{Max} \quad & P_{x_0}^\pi (\mu_0(U_0) \wedge \mu_1(U_1) \wedge \mu_G(X_2) \geq 0.7) \\ \text{s.t.} \quad & \text{(i)}_n \quad X_{n+1} \sim p(\cdot | x_n, u_n) \\ & \text{(ii)}_n \quad u_n \in U \end{aligned} \quad n = 0, 1$$

where the numerical data is theirs :

$$\begin{aligned} \mu_0(a_1) &= 0.7 & \mu_0(a_2) &= 1.0 ; \\ \mu_1(a_1) &= 1.0 & \mu_1(a_2) &= 0.6 ; \\ \mu_G(s_1) &= 0.3 & \mu_G(s_2) &= 1.0 & \mu_G(s_3) &= 0.8 \end{aligned}$$

|                          | $p(x_{n+1} x_n, a_1)$ |       |       | $p(x_{n+1} x_n, a_2)$ |       |       |
|--------------------------|-----------------------|-------|-------|-----------------------|-------|-------|
| $x_n \backslash x_{n+1}$ | $s_1$                 | $s_2$ | $s_3$ | $s_1$                 | $s_2$ | $s_3$ |
| $s_1$                    | 0.8                   | 0.1   | 0.1   | 0.1                   | 0.9   | 0.0   |
| $s_2$                    | 0.0                   | 0.1   | 0.9   | 0.8                   | 0.1   | 0.1   |
| $s_3$                    | 0.8                   | 0.1   | 0.1   | 0.1                   | 0.0   | 0.9   |

The backward recursion (6) yields optimal solution in Markov class  $\Pi$  — a pair of a sequence of optimal value fuctions

$$v_0 = v_0(x_0), \quad v_1 = v_1(x_1), \quad v_2 = v_2(x_2)$$

and an optimal policy

$$\pi^* = \{\pi_0^*(x_0), \pi_1^*(x_1)\}.$$

The optimal solution is tabulated as

| $x_n$ | $v_2(x_2)$ | $v_1(x_1)$ | $\pi_1^*(x_1)$ | $v_0(x_0)$ | $\pi_0^*(x_0)$ |
|-------|------------|------------|----------------|------------|----------------|
| $s_1$ | 0          | 0.2        | $a_1$          | 0.92       | $a_2$          |
| $s_2$ | 1          | 1.0        | $a_1$          | 0.28       | $a_1, a_2$     |
| $s_3$ | 1          | 0.2        | $a_1$          | 0.28       | $a_1$          |

Table 1: Optimal Solution

Furthermore, we have another two methods. One is stochastic decision tree-table methods(Figure 1). The other is total enumalation of all Markov policy and related threshold probability vector(Table 2).

So we have three approaches. Through these three approaches, we have obtained optimal solution ; optimal value (0.92, 0.28, 0.28) and optimal policy  $\pi^*$ .

$$w_0(s_1) = \text{Max}_{\pi \in \Pi} P_{x_0}^{\pi}(\mu_0(U_0) \wedge \mu_1(U_1) \wedge \mu_G(X_2) \geq 0.7)$$

Figure 1 : Two-stage stochastic decision tree-table from state  $s_1$ 

| history |             |       |       |             |       |             | mini<br>mum | path<br>prob. | thre-<br>prob. | sub-<br>prob. | total-<br>prob. |      |     |   |
|---------|-------------|-------|-------|-------------|-------|-------------|-------------|---------------|----------------|---------------|-----------------|------|-----|---|
| $x_0$   | $\mu_0/u_0$ | $p_0$ | $x_1$ | $\mu_1/u_1$ | $p_1$ | $x_2 \mu_G$ |             |               |                |               |                 |      |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0              | 0.16          | 0.28            |      |     |   |
|         |             |       |       |             |       |             | 1.0         | 0.7           | 0.08           |               |                 | 0.08 | 0   |   |
|         |             |       |       |             |       |             | 0.8         | 0.7           | 0.08           |               |                 | 0.08 |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0.08           |               |                 | 0    |     | 0 |
|         |             |       |       |             |       |             | 1.0         | 0.6           | 0.72           |               |                 | 0    |     |   |
|         |             |       |       |             |       |             | 0.8         | 0.6           | 0.0            |               |                 | 0    |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0.0            |               |                 | 0    | 0.1 |   |
|         |             |       |       |             |       |             | 1.0         | 0.7           | 0.01           |               |                 | 0.01 |     |   |
|         |             |       |       |             |       |             | 0.8         | 0.7           | 0.09           |               |                 | 0.09 |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0.08           |               |                 | 0    | 0   |   |
|         |             |       |       |             |       |             | 1.0         | 0.6           | 0.01           |               |                 | 0    |     |   |
|         |             |       |       |             |       |             | 0.8         | 0.6           | 0.01           |               |                 | 0    |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0.08           | 0             | 0.02            |      |     |   |
|         |             |       |       |             |       |             | 1.0         | 0.7           | 0.01           | 0.01          |                 |      |     |   |
|         |             |       |       |             |       |             | 0.8         | 0.7           | 0.01           | 0.01          |                 |      |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0.01           | 0             | 0               |      |     |   |
|         |             |       |       |             |       |             | 1.0         | 0.6           | 0.0            | 0             |                 |      |     |   |
|         |             |       |       |             |       |             | 0.8         | 0.6           | 0.09           | 0             |                 |      |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0.08           | 0             | 0.02            |      |     |   |
|         |             |       |       |             |       |             | 1.0         | 1.0           | 0.01           | 0.01          |                 |      |     |   |
|         |             |       |       |             |       |             | 0.8         | 0.8           | 0.01           | 0.01          |                 |      |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0.01           | 0             | 0               |      |     |   |
|         |             |       |       |             |       |             | 1.0         | 0.6           | 0.09           | 0             |                 |      |     |   |
|         |             |       |       |             |       |             | 0.8         | 0.6           | 0.0            | 0             |                 |      |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0.0            | 0             | 0.9             | 0.92 |     |   |
|         |             |       |       |             |       |             | 1.0         | 1.0           | 0.09           | 0.09          |                 |      |     |   |
|         |             |       |       |             |       |             | 0.8         | 0.8           | 0.81           | 0.81          |                 |      |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0.72           | 0             | 0               |      |     |   |
|         |             |       |       |             |       |             | 1.0         | 0.6           | 0.09           | 0             |                 |      |     |   |
|         |             |       |       |             |       |             | 0.8         | 0.6           | 0.09           | 0             |                 |      |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0.0            | 0             | 0               |      |     |   |
|         |             |       |       |             |       |             | 1.0         | 1.0           | 0.0            | 0.0           |                 |      |     |   |
|         |             |       |       |             |       |             | 0.8         | 0.8           | 0.0            | 0.0           |                 |      |     |   |
|         |             |       |       |             |       |             | 0.3         | 0.3           | 0.0            | 0             | 0               |      |     |   |
|         |             |       |       |             |       |             | 1.0         | 0.6           | 0.0            | 0.0           |                 |      |     |   |
|         |             |       |       |             |       |             | 0.8         | 0.6           | 0.0            | 0             |                 |      |     |   |

$$J(x_0; \pi) = P_{x_0}^\pi(\mu_0 \wedge \mu_1 \wedge \mu_G \geq 0.7)$$

Table 2 : all threshold-probability vectors  $J(\pi) = \begin{pmatrix} J(s_1; \pi) \\ J(s_2; \pi) \\ J(s_3; \pi) \end{pmatrix}$ , where  $\pi = \{\pi_0, \pi_1\}$  is Ma

| $\pi_1 \backslash \pi_0$                          | $\begin{pmatrix} a_1 \\ a_1 \\ a_1 \end{pmatrix}$    | $\begin{pmatrix} a_1 \\ a_1 \\ a_2 \end{pmatrix}$    | $\begin{pmatrix} a_1 \\ a_2 \\ a_1 \end{pmatrix}$    | $\begin{pmatrix} a_1 \\ a_2 \\ a_2 \end{pmatrix}$    | $\begin{pmatrix} a_2 \\ a_1 \\ a_1 \end{pmatrix}$    | $\begin{pmatrix} a_2 \\ a_1 \\ a_2 \end{pmatrix}$ | $\begin{pmatrix} a_2 \\ a_2 \\ a_1 \end{pmatrix}$    | $\begin{pmatrix} a_2 \\ a_2 \\ a_2 \end{pmatrix}$ |
|---------------------------------------------------|------------------------------------------------------|------------------------------------------------------|------------------------------------------------------|------------------------------------------------------|------------------------------------------------------|---------------------------------------------------|------------------------------------------------------|---------------------------------------------------|
| $\begin{pmatrix} a_1 \\ a_1 \\ a_1 \end{pmatrix}$ | $\begin{pmatrix} 0.28 \\ 0.28 \\ 0.28 \end{pmatrix}$ | $\begin{pmatrix} 0.26 \\ 0.1 \\ 0.26 \end{pmatrix}$  | $\begin{pmatrix} 0.18 \\ 0.18 \\ 0.18 \end{pmatrix}$ | $\begin{pmatrix} 0.16 \\ 0.0 \\ 0.16 \end{pmatrix}$  | $\begin{pmatrix} 0.12 \\ 0.28 \\ 0.12 \end{pmatrix}$ | $\begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix}$ | $\begin{pmatrix} 0.02 \\ 0.18 \\ 0.02 \end{pmatrix}$ | $\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$ |
| $\begin{pmatrix} a_1 \\ a_1 \\ a_2 \end{pmatrix}$ | $\begin{pmatrix} 0.28 \\ 0.28 \\ 0.2 \end{pmatrix}$  | $\begin{pmatrix} 0.26 \\ 0.1 \\ 0.02 \end{pmatrix}$  | $\begin{pmatrix} 0.18 \\ 0.18 \\ 0.2 \end{pmatrix}$  | $\begin{pmatrix} 0.16 \\ 0.0 \\ 0.02 \end{pmatrix}$  | $\begin{pmatrix} 0.12 \\ 0.28 \\ 0.18 \end{pmatrix}$ | $\begin{pmatrix} 0.1 \\ 0.1 \\ 0.0 \end{pmatrix}$ | $\begin{pmatrix} 0.02 \\ 0.18 \\ 0.18 \end{pmatrix}$ | $\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$ |
| $\begin{pmatrix} a_1 \\ a_2 \\ a_1 \end{pmatrix}$ | $\begin{pmatrix} 0.28 \\ 0.28 \\ 0.28 \end{pmatrix}$ | $\begin{pmatrix} 0.26 \\ 0.26 \\ 0.26 \end{pmatrix}$ | $\begin{pmatrix} 0.18 \\ 0.18 \\ 0.18 \end{pmatrix}$ | $\begin{pmatrix} 0.16 \\ 0.16 \\ 0.16 \end{pmatrix}$ | $\begin{pmatrix} 0.12 \\ 0.12 \\ 0.12 \end{pmatrix}$ | $\begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix}$ | $\begin{pmatrix} 0.02 \\ 0.02 \\ 0.02 \end{pmatrix}$ | $\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$ |
| $\begin{pmatrix} a_1 \\ a_2 \\ a_2 \end{pmatrix}$ | $\begin{pmatrix} 0.28 \\ 0.28 \\ 0.2 \end{pmatrix}$  | $\begin{pmatrix} 0.26 \\ 0.26 \\ 0.02 \end{pmatrix}$ | $\begin{pmatrix} 0.18 \\ 0.18 \\ 0.2 \end{pmatrix}$  | $\begin{pmatrix} 0.16 \\ 0.16 \\ 0.02 \end{pmatrix}$ | $\begin{pmatrix} 0.12 \\ 0.12 \\ 0.18 \end{pmatrix}$ | $\begin{pmatrix} 0.1 \\ 0.1 \\ 0.0 \end{pmatrix}$ | $\begin{pmatrix} 0.02 \\ 0.02 \\ 0.18 \end{pmatrix}$ | $\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$ |
| $\begin{pmatrix} a_2 \\ a_1 \\ a_1 \end{pmatrix}$ | $\begin{pmatrix} 0.92 \\ 0.28 \\ 0.28 \end{pmatrix}$ | $\begin{pmatrix} 0.92 \\ 0.1 \\ 0.26 \end{pmatrix}$  | $\begin{pmatrix} 0.02 \\ 0.18 \\ 0.18 \end{pmatrix}$ | $\begin{pmatrix} 0.02 \\ 0.0 \\ 0.16 \end{pmatrix}$  | $\begin{pmatrix} 0.9 \\ 0.28 \\ 0.12 \end{pmatrix}$  | $\begin{pmatrix} 0.9 \\ 0.1 \\ 0.1 \end{pmatrix}$ | $\begin{pmatrix} 0.0 \\ 0.18 \\ 0.02 \end{pmatrix}$  | $\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$ |
| $\begin{pmatrix} a_2 \\ a_1 \\ a_2 \end{pmatrix}$ | $\begin{pmatrix} 0.92 \\ 0.28 \\ 0.2 \end{pmatrix}$  | $\begin{pmatrix} 0.92 \\ 0.1 \\ 0.26 \end{pmatrix}$  | $\begin{pmatrix} 0.02 \\ 0.18 \\ 0.2 \end{pmatrix}$  | $\begin{pmatrix} 0.02 \\ 0.0 \\ 0.02 \end{pmatrix}$  | $\begin{pmatrix} 0.9 \\ 0.28 \\ 0.18 \end{pmatrix}$  | $\begin{pmatrix} 0.9 \\ 0.1 \\ 0.0 \end{pmatrix}$ | $\begin{pmatrix} 0.0 \\ 0.18 \\ 0.18 \end{pmatrix}$  | $\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$ |
| $\begin{pmatrix} a_2 \\ a_2 \\ a_1 \end{pmatrix}$ | $\begin{pmatrix} 0.92 \\ 0.28 \\ 0.28 \end{pmatrix}$ | $\begin{pmatrix} 0.92 \\ 0.26 \\ 0.26 \end{pmatrix}$ | $\begin{pmatrix} 0.02 \\ 0.18 \\ 0.18 \end{pmatrix}$ | $\begin{pmatrix} 0.02 \\ 0.16 \\ 0.16 \end{pmatrix}$ | $\begin{pmatrix} 0.9 \\ 0.12 \\ 0.12 \end{pmatrix}$  | $\begin{pmatrix} 0.9 \\ 0.1 \\ 0.1 \end{pmatrix}$ | $\begin{pmatrix} 0.0 \\ 0.02 \\ 0.02 \end{pmatrix}$  | $\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$ |
| $\begin{pmatrix} a_2 \\ a_2 \\ a_2 \end{pmatrix}$ | $\begin{pmatrix} 0.92 \\ 0.28 \\ 0.2 \end{pmatrix}$  | $\begin{pmatrix} 0.92 \\ 0.26 \\ 0.26 \end{pmatrix}$ | $\begin{pmatrix} 0.02 \\ 0.18 \\ 0.2 \end{pmatrix}$  | $\begin{pmatrix} 0.02 \\ 0.16 \\ 0.02 \end{pmatrix}$ | $\begin{pmatrix} 0.9 \\ 0.12 \\ 0.18 \end{pmatrix}$  | $\begin{pmatrix} 0.9 \\ 0.1 \\ 0.0 \end{pmatrix}$ | $\begin{pmatrix} 0.0 \\ 0.02 \\ 0.18 \end{pmatrix}$  | $\begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \end{pmatrix}$ |

## References

- [1] J.F. Baldwin and B.W. Pilsworth, "Dynamic programming for fuzzy systems with fuzzy environment," *J. Math. Anal. Appl.*, vol.85,no.1, pp.1-23, 1982.
- [2] R.E. Bellman, *Dynamic Programming*, NJ: Princeton Univ. Press, 1957.
- [3] R.E. Bellman and E.D. Denman, "Invariant Imbedding," *Lect. Notes in Operation Research and Mathematical Systems*, vol.52, Berlin: Springer-Verlag, 1971.
- [4] R.E. Bellman and L.A. Zadeh, "Decision-making in a fuzzy environment," *Management Science*, vol.17, no.4, pp.B141-B164, 1970.
- [5] A.O. Esogbue and R.E. Bellman, "Fuzzy dynamic programming and its extensions," *TIMS/Studies in the Management Sciences*, vol.20, pp.147-167, 1984.
- [6] R.A. Howard, *Dynamic Programming and Markov Processes*, Mass.: MIT Press, 1960.
- [7] S. Iwamoto, "Decison-making in fuzzy environment: a survey from stochastic decision process," *Proc. 2<sup>nd</sup> Intl Conf. Knowledge-based Intelligent Electronics Systems (KES '98)*, Adelaide, AUS, Apr. 1998, pp.542-546.
- [8] S. Iwamoto, "Maximizing threshold probability through invariant imbedding," *Proc. Intl Workshop on Intelligent Systems Resolutions; The 8-th Bellman Continuum*, Hsinchu, ROC, Dec. 2000, pp.17-22.
- [9] S. Iwamoto, "Fuzzy decision-making through three dynamic programming approaches," *Proc. Intl Workshop on Intelligent Systems Resolutions; The 8-th Bellman Continuum*, Hsinchu, ROC, Dec. 2000, pp.23-28.
- [10] S. Iwamoto and T. Fujita, "Stochastic decision-making in a fuzzy environment," *J. Oper. Res. Soc. Japan*, vol.38, no.4, pp.467-482, 1995.
- [11] S. Iwamoto and M. Sniedovich, "Sequential decision making in fuzzy environment," *J. Math. Anal. Appl.*, vol.222, no.2, pp.208-224, 1998.
- [12] S. Iwamoto, K. Tsurusaki and T. Fujita, "Conditional decision-making in a fuzzy environment," *J. Oper. Res. Soc. Japan*, vol.42, no.2, pp.198-218, 1999.
- [13] S. Iwamoto, K. Tsurusaki and T. Fujita, "On Markov policies for minimax decision processes," *J. Math. Anal. Appl.*, vol.253, no.1, 58-78, 2001.
- [14] S. Iwamoto, T. Ueno and T. Fujita, "Controlled Markov chains with utility functions," *Proc. of Intl Workshop on Markov Processes and Controlled Markov Chains*, Changsha, China, Aug. 1999.
- [15] J. Kacprzyk and A.O. Esogbue, "Fuzzy dynamic programming: Main developments and applications," *Fuzzy Sets and Systems*, vol.81, no.1, pp.31-45, 1996.
- [16] E.S. Lee, *Quasilinearization and Invariant Imbedding*, New York: Academic Press, 1968.
- [17] M.L. Puterman, *Markov Decision Processes : discrete stochastic dynamic programming*, New York: Wiley & Sons, 1994.
- [18] T. Ueno and S. Iwamoto, "A dual approach for optimizing threshold probabilities," under consideration.