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Unramified extensions and geometric $\mathbb{Z}_p$-extensions of global function fields

By

Tsuyoshi Itoh*

Abstract

We study on finite unramified extensions of global function fields (that is, function fields of one variable over a finite field). We show two results. One is an extension of Perret's result about the ideal class group problem. Another is a construction of a geometric $\mathbb{Z}_p$-extension which has a certain property.

§ 1. Main theorems

Throughout the present paper, we fix a prime number p and a finite field $\mathbb{F}$ of characteristic p. Let q be the number of elements of $\mathbb{F}$. Recall that a global function field is a function field of one variable over a finite field. Let k be a global function field with full constant field $\mathbb{F}$. We also recall that a finite algebraic extension K/k is geometric if and only if the constant field of K is also $\mathbb{F}$.

It is known that there is a finite abelian group G which is not isomorphic to the divisor class group of degree 0 of any global function field (Stichtenoth [20]). On the other hand, Perret [16] showed the following:

Theorem 1.1 ([16]). *For any given finite abelian group G, there is a finite separable geometric extension $k/\mathbb{F}(T)$ such that $\mathrm{Cl}(\mathcal{O}) \cong G$, where $\mathcal{O}$ is the integral closure of $\mathbb{F}[T]$ in k and $\mathrm{Cl}(\mathcal{O})$ is the ideal class group of $\mathcal{O}$.*

This theorem is shown by using the following:

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*College of Science and Engineering, Ritsumeikan University, 1-1-1 Noji Higashi, Kusatsu, Shiga 525-8577, Japan. (Current Address : Division of Mathematics, Education Center, Faculty of Social Systems Science, Chiba Institute of Technology, 2-1-1 Shibazono, Narashino, Chiba 275-0023, Japan. e-mail: tsuyoshi.itoh@it-chiba.ac.jp)

Theorem 1.2 ([16]). *For any given finite abelian group G, there is a global function field k with full constant field $\mathbb{F}$ and a non-empty finite set S of places of k such that $\mathrm{Cl}_S(k) \cong G$, where $\mathrm{Cl}_S(k)$ is the S-class group of k.*

Let S be a non-empty finite set of places of k, and $H_S(k)$ the S-Hilbert class field of k, that is, the maximal unramified abelian extension field of k in which all places of S split completely (see [17]). We note that $\mathrm{Cl}_S(k) \cong \mathrm{Gal}(H_S(k)/k)$ by class field theory. Hence Theorem 1.2 also implies the existence of k and S which satisfy $\mathrm{Gal}(H_S(k)/k) \cong G$. (More precisely, we can take k and S such that $H_S(k)/k$ is a geometric extension. See [16].)

In the present paper, we extend the above result to non-abelian finite groups. We will show the following:

Theorem 1.3. *For any given finite group G, there is a global function field k with full constant field $\mathbb{F}$ and a non-empty finite set S of places of k such that $\mathrm{Gal}(\tilde{H}_S(k)/k) \cong G$, where $\tilde{H}_S(k)$ denotes the maximal unramified Galois extension field of k in which all places of S split completely. Moreover, we can take k and S such that $\tilde{H}_S(k)/k$ is a geometric extension.*

See Ozaki [15] for the number field case.

We will prove Theorem 1.3 in section 2. Our proof is due to Perret's idea (see [16]). That is, we will construct an unramified G-extension, and take a sufficiently large set S of places such that $\mathrm{Gal}(\tilde{H}_S(k)/k) \cong G$. (We use the term "$G$-extension" as a Galois extension whose Galois group is isomorphic to G.) To construct an unramified G-extension, we shall show an analog (Theorem 2.2) of Fröhlich's classical result [4] for number fields.

In section 3, we shall apply Perret's idea to Iwasawa theory. Let k be a global function field with full constant field $\mathbb{F}$, S a non-empty finite set of places of k. We recall that a $\mathbb{Z}_p$-extension is an infinite Galois extension whose Galois group is topologically isomorphic to the additive group of the ring $\mathbb{Z}_p$ of p-adic integers. Let k_∞/k be a geometric $\mathbb{Z}_p$-extension, that is, k_∞/k is a $\mathbb{Z}_p$-extension which satisfies that every finite subextension over k is a geometric extension (see, e.g., [7]). (Recall that p is the characteristic of $\mathbb{F}$.) We assume that

(A) only finitely many places of k ramify in k_∞/k, and

(B) all places of S split completely in k_∞/k.

Under these assumptions, we can treat Iwasawa theory for the S-class group (see [17]). For a non-negative integer n, let k_n be the nth layer of k_∞/k. That is, k_n is the unique subfield of k_∞ which is a cyclic extension over k of degree p^n. Moreover, let A_n be the

Sylow p-subgroup of the S-class group of k_n. (Here we use the same symbol S as the set of places of k_n lying above S.) We put $X_S = \varprojlim A_n$, where the projective limit is taken with respect to the norm maps. We call X_S the Iwasawa module of k_∞/k for the S-class group. We put $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(k_\infty/k)]]$. Note that $\Lambda \cong \mathbb{Z}_p[[T]]$. It is known that X_S is a finitely generated torsion Λ-module, and the "Iwasawa type formula" holds for A_n (see [17]). That is, there are non-negative integers λ, μ, and an integer ν such that $|A_n| = p^{\lambda n + \mu p^n + \nu}$ for all sufficiently large n. Aiba [1] studied these invariants λ, μ, and ν for certain geometric $\mathbb{Z}_p$-extensions.

There is a natural problem: characterize the Λ-modules which appear as X_S. (For the number field case, the same problem is dealt in, e.g., [14], [5].) Concerning this problem, we shall give the following result including "non-abelian" cases.

Theorem 1.4. *For any given finite p-group G, there exist a global function field k with full constant field $\mathbb{F}$, a non-empty finite set S of places of k, and a geometric $\mathbb{Z}_p$-extension k_∞/k satisfying the above assumptions* (A) *and* (B) *such that* $\mathrm{Gal}(\tilde{L}_S(k_n)/k_n) \cong G$ *(as groups) for all $n \geq 0$, where $\tilde{L}_S(k_n)$ is the maximal unramified Galois pro-p-extension field of k_n in which all places lying above S split completely.*

For the number field case, Ozaki [14] showed that every "finite Λ-module" appears as the Iwasawa module of a $\mathbb{Z}_p$-extension. Theorem 1.4 for G abelian gives a weak analog of Ozaki's result. That is, every finite Λ-module on which $\mathrm{Gal}(k_\infty/k)$ acts trivially appears as X_S. We will prove Theorem 1.4 in section 3.

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§ 2. Proof of Theorem 1.3

§ 2.1. Function field analog of Fröhlich's result

At first, we shall show that for any finite group G, there is an unramified geometric extension K/k of global function fields such that $\mathrm{Gal}(K/k) \cong G$. Recall that any finite group can be embedded into a finite symmetric group. Hence it is sufficient to consider the case that G is a finite symmetric group. For the number field case, Fröhlich already showed the following result.

Theorem 2.1 ([4]). *For every positive integer n, there is an unramified Galois extension K/k of algebraic number fields such that* $\mathrm{Gal}(K/k) \cong \mathfrak{S}_n$*, where $\mathfrak{S}_n$ denotes the symmetric group of degree n.*

We will show the following:

Theorem 2.2. *For every positive integer n, there is a global function field k with full constant field $\mathbb{F}$ and an unramified geometric Galois extension K/k such that $\mathrm{Gal}(K/k) \cong \mathfrak{S}_n$. More precisely, there exist a geometric Galois extension $K/\mathbb{F}(T)$ and a subextension $k/\mathbb{F}(T)$ of $K/\mathbb{F}(T)$ such that K/k is unramified and that $\mathrm{Gal}(K/k) \cong \mathfrak{S}_n$.*

To prove this, we follow Fröhlich's original argument (see also Malinin [10]). That is, we construct a certain (ramified) $\mathfrak{S}_n$-extension over $\mathbb{F}(T)$ and then we take a certain base change of this extension. Let ∞ be the infinite place of $\mathbb{F}(T)$.

Lemma 2.3. *There is a Galois extension k' over $\mathbb{F}(T)$ which satisfies all of the following properties.*

- *$k'/\mathbb{F}(T)$ is a geometric extension,*
- *$\mathrm{Gal}(k'/\mathbb{F}(T)) \cong \mathfrak{S}_n$, and*
- *∞ is unramified in $k'/\mathbb{F}(T)$.*

Proof. At first, we must see that there is an $\mathfrak{S}_n$-extension over $\mathbb{F}(T)$. This follows from the fact that $\mathbb{F}(T)$ is a Hilbertian field (see, e.g, [3, Corollary 16.2.7]). We put $A = \mathbb{F}[T]$. For an element r of A, let $\deg(r)$ be the degree of r as a polynomial of T. Fix a monic separable polynomial $F(X) \in A[X]$ of degree n such that the splitting field of $F(X)$ over $\mathbb{F}(T)$ is an $\mathfrak{S}_n$-extension.

We claim that there is an element $N_F \in A$ which satisfies the following property: if a monic polynomial $G(X) \in A[X]$ of degree n satisfies $G(X) \equiv F(X) \pmod{N_F}$, then the splitting field of $G(X)$ over $\mathbb{F}(T)$ is also an $\mathfrak{S}_n$-extension. We shall show this claim. By using the Chebotarev density theorem, we can take an irreducible monic polynomial p_1 such that if $G(X) \equiv F(X) \pmod{p_1}$ then $G(X)$ is irreducible and separable. Similarly, we can take distinct irreducible monic polynomials p_2, p_3 of $A = \mathbb{F}(T)$ which are distinct from p_1 and satisfy the following properties: (i) if $G(X) \equiv F(X) \pmod{p_2}$ then the Galois group of $G(X)$ contains a cycle of length $n-1$ (as a subgroup of $\mathfrak{S}_n$), and (ii) if $G(X) \equiv F(X) \pmod{p_3}$ then the Galois group of $G(X)$ contains a transposition. We put $N_F = p_1p_2p_3$. This N_F satisfies the above claim. Moreover, we can take N_F which is prime to T by the Chebotarev density theorem. We also fix such N_F.

To construct a geometric $\mathfrak{S}_n$-extension which is unramified at the infinite place, we take $G(X)$ as follows:

$$\begin{aligned} &G(X) \equiv F(X) && \pmod{N_F}, \\ &G(X) \equiv \text{(a product of distinct monic polynomials of degree 1)} && \pmod{r}, \textit{and} \\ &G(X) \equiv \text{(a separable polynomial)} && \pmod{T}, \end{aligned}$$

where r is a monic irreducible polynomial of $A = \mathbb{F}[T]$ such that $n < q^{\deg(r)}$, $\deg(r)$ is odd, and r is prime to TN_F. By the first congruence, we see that the splitting field k'

of $G(X)$ is an $\mathfrak{S}_n$-extension. We shall show that the constant field of k' is $\mathbb{F}$. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. We note that $M := k' \cap \overline{\mathbb{F}}(T)$ is a finite cyclic extension over $\mathbb{F}(T)$. Since $\mathrm{Gal}(k'/\mathbb{F}(T)) \cong \mathfrak{S}_n$, M must be $\mathbb{F}(T)$ or the unique quadratic subfield in $k'/\mathbb{F}(T)$. If $M \neq \mathbb{F}(T)$, then no odd degree place of $\mathbb{F}(T)$ splits in M. However, we see that the place of $\mathbb{F}(T)$ corresponding to r splits completely in k' by the second congruence. It is a contradiction.

By the third congruence, we see that the place of $\mathbb{F}(T)$ corresponding to T is unramified in k'. We replace the indeterminate T by $U = 1/T$, then the infinite place of $\mathbb{F}(U)$ is unramified in k' (and the former two conditions are also satisfied). $\square$

We shall prove Theorem 2.2. We may assume that $n \geq 2$. Fix a geometric $\mathfrak{S}_n$-extension $k'/\mathbb{F}(T)$ satisfying the properties of Lemma 2.3. We put $m = n!$. We can take a separable monic polynomial $F(X) \in A[X]$ of degree m (as a polynomial of X) whose splitting field over $\mathbb{F}(T)$ is k'. Let M' be the unique quadratic subextension field of $\mathbb{F}(T)$ contained in k'.

We define the following notation.

- $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$: the set of distinct places of $\mathbb{F}(T)$ which ramify in k' (hence are distinct from ∞).
- $\mathfrak{p}_{t+1}$: a place $\neq \infty, \mathfrak{p}_1, \ldots, \mathfrak{p}_t$ of $\mathbb{F}(T)$ which is inert in M' and has degree $> \frac{\log(m)}{\log(q)}$.
- $\mathfrak{p}_{t+2}$: a place $\neq \infty$ of $\mathbb{F}(T)$ which splits completely in k' and has **odd** degree $> \frac{\log(m)}{\log(q)}$ (hence is distinct from $\mathfrak{p}_1, \ldots, \mathfrak{p}_t, \mathfrak{p}_{t+1}$).
- $p_1, \ldots, p_{t+2}$: irreducible monic polynomials of $A = \mathbb{F}[T]$ corresponding to $\mathfrak{p}_1, \ldots, \mathfrak{p}_{t+2}$, respectively.

Note that we can take $\mathfrak{p}_{t+1}$ (resp. $\mathfrak{p}_{t+2}$) by using Theorem 9.13B of [18], which is an effective version of the Chebotarev density theorem for global function fields. (See also [12], etc.) Indeed, by this theorem, there is a place of $\mathbb{F}(T)$ of arbitrary sufficiently large degree which is inert in M' (resp. splits completely in k'), as $M'/\mathbb{F}(T)$ is a geometric cyclic extension (resp. $k'/\mathbb{F}(T)$ is a geometric Galois extension).

By using Lemma 2.3, we can also construct an $\mathfrak{S}_m$-extension over $\mathbb{F}(T)$. Let $H(X)$ be a monic polynomial in $A[X]$ of degree m which gives an $\mathfrak{S}_m$-extension. Then there is an element N_H of A having the following property: if a monic polynomial $G(X) \in A[X]$ of degree m satisfies $G(X) \equiv H(X) \pmod{N_H}$, then the splitting field of $G(X)$ over $\mathbb{F}(T)$ is also an $\mathfrak{S}_m$-extension (see the proof of Lemma 2.3). We can also take N_H such that it is prime to $p_1, \ldots, p_{t+2}$.

We take a monic polynomial $G(X)$ of $A[X]$ (having degree m) which satisfies the following conditions (2.1)–(2.4).

$$G(X) \equiv H(X) \pmod{N_H}. \tag{2.1}$$

If $G(X)$ satisfies (2.1), then $G(X)$ gives an $\mathfrak{S}_m$-extension. Let L be the splitting field of $G(X)$ over $\mathbb{F}(T)$.

(2.2) $\quad G(X) \equiv (\text{a product of distinct monic polynomials of degree } 1) \pmod{p_{t+1}}.$

If $G(X)$ satisfies (2.1) and (2.2), then we see that $\mathfrak{p}_{t+1}$ splits in the unique quadratic subextension, say M_L, over $\mathbb{F}(T)$ contained in L. On the other hand, $\mathfrak{p}_{t+1}$ is inert in the unique quadratic subextension M' over $\mathbb{F}(T)$ contained in k'. We claim that $k' \cap L = \mathbb{F}(T)$. Indeed, suppose that $k' \cap L \neq \mathbb{F}(T)$. Then $k' \cap L$ is a quadratic extension over $\mathbb{F}(T)$. If $n = 2$, this is clear. For $n \geq 3$, we have $\mathrm{Gal}(L/\mathbb{F}(T)) \cong \mathfrak{S}_m$, where $m = n! \geq 5$. Observe also that $k' \cap L \neq L$, as $m > n$. Now, since the alternating group $\mathfrak{A}_m$ is the unique nontrivial proper normal subgroup of $\mathfrak{S}_m$ when $m \geq 5$ (see, e.g., [19]), $k' \cap L$ is a quadratic extension over $\mathbb{F}(T)$. Since this quadratic extension is contained in both k' and L, it must coincide with both M' and M_L at a time. This contradicts the above observation on the behavior of $\mathfrak{p}_{t+1}$ in M' and M_L. Thus, we have proved the claim. Then we see $\mathrm{Gal}(Lk'/L) \cong \mathfrak{S}_n$.

(2.3) $\quad G(X) \equiv (\text{a product of distinct monic polynomials of degree } 1) \pmod{p_{t+2}}.$

If $G(X)$ satisfies (2.1)–(2.3), then the odd degree place $\mathfrak{p}_{t+2}$ splits completely in $Lk'/\mathbb{F}(T)$. We claim that $Lk'/\mathbb{F}(T)$ is a geometric extension. Note that the degree of a place of k' lying above $\mathfrak{p}_{t+2}$ is also odd because $\mathfrak{p}_{t+2}$ splits completely in k'. Since $\mathrm{Gal}(Lk'/k') \cong \mathfrak{S}_m$ and an odd degree place splits completely in Lk'/k', we see that Lk'/k' is also a geometric extension. Hence the claim follows. By using Krasner's lemma, we can see that there is a positive integer s_i for each $i = 1, \ldots, t$ depending only on $F(X)$ such that if $G(X) \equiv F(X) \pmod{p_i^{s_i}}$ then $L\mathbb{F}(T)_{\mathfrak{p}_i} = k'\mathbb{F}(T)_{\mathfrak{p}_i}$, where $\mathbb{F}(T)_{\mathfrak{p}_i}$ is the completion of $\mathbb{F}(T)$ at $\mathfrak{p}_i$ (see, e.g., [13]). Hence if we take $G(X)$ satisfying (2.1)–(2.3) and

$$G(X) \equiv F(X) \pmod{p_i^{s_i}} \text{ for } i = 1, \ldots, t, \tag{2.4}$$

then we can see that Lk'/L is unramified at all places.

We can take $G(X)$ satisfying (2.1)–(2.4). By the above arguments, the extension Lk'/L satisfies the assertion of Theorem 2.2. □

Remark. When G is abelian, an unramified geometric G-extension was constructed by Angles [2]. Moret-Bailly [11] also gives a result which is very close to ours. Probably, it seems that one can prove our main theorems by using the result given in [11] instead of Theorem 2.2.

§ 2.2. Proof of Theorem 1.3

Since G is embedded into $\mathfrak{S}_n$ for some $n > 0$, Theorem 2.2 implies that there exists a global function field k with full constant field $\mathbb{F}$ and an unramified geometric Galois extension K/k such that $\mathrm{Gal}(K/k) \cong G$.

Proposition 2.4. *There is a non-empty finite set S of places of k such that* (i) *all places of S split completely in K, and* (ii) *$\tilde{H}_S(k)/k$ is a finite extension.*

Proof. The crucial point of this proposition is choosing a set S to satisfy (ii). For a positive integer N, we put

$$B_N = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a place of } k \text{ having degree } N,\ \mathfrak{p} \text{ splits completely in } K/k.\}.$$

Since K/k is a geometric extension, Theorem 9.13B of [18] implies that

$$|B_N| = \frac{q^N}{|G|N} + O\left(\frac{q^{N/2}}{N}\right)$$

(recall that q is the number of elements of $\mathbb{F}$). In particular, if N is sufficiently large, then we obtain the inequality

$$|B_N| > \frac{q^{N/2}-1}{N}\mathrm{Max}(g-1,0),$$

where g is the genus of k. We fix an integer N which satisfies the above inequality. According to Ihara's theorem [8, Theorem 1(FF)], if $S \supset B_N$, then $\tilde{H}_S(k)/k$ is a finite extension. Hence we can take S to satisfy the conditions (i) and (ii). □

The rest of the proof of Theorem 1.3 is quite similar to Perret's argument given in [16]. We choose a set S of places which satisfies the conditions of Proposition 2.4. We remark that K is contained in $\tilde{H}_S(k)$. For a nontrivial element σ of $\mathrm{Gal}(\tilde{H}_S(k)/K)$, we can take a place $\mathfrak{P}$ of $\tilde{H}_S(k)$ corresponding to σ by the Chebotarev density theorem. We can take $\mathfrak{P}$ which is unramified in $\tilde{H}_S(k)/K$. Let $\mathfrak{p}$ be the place of k which is lying below $\mathfrak{P}$. Since the decomposition field of $\mathfrak{P}$ in $\tilde{H}_S(k)/k$ contains K and K/k is a Galois extension, we see that $\mathfrak{p}$ splits completely in K/k. Then we see $\tilde{H}_S(k) \supsetneq \tilde{H}_{S\cup\{\mathfrak{p}\}}(k) \supset K$. Replacing $S \cup \{\mathfrak{p}\}$ by S and repeating the above operation, we can see that $\tilde{H}_S(k) = K$ for some finite set S. This implies $\mathrm{Gal}(\tilde{H}_S(k)/K) \cong G$.

We recall that K/k is a geometric extension. Hence the final part of the theorem follows. □

§ 3. Proof of Theorem 1.4

Firstly, we shall show the following:

Theorem 3.1. *Let k be a finite Galois extension over $\mathbb{F}(T)$. Then, there exist a non-empty finite set S of places of $\mathbb{F}(T)$ and a geometric $\mathbb{Z}_p$-extension $F_\infty/\mathbb{F}(T)$ which satisfy the following properties.*

- $F_\infty \cap k = \mathbb{F}(T)$,
- *all places of S split completely in k,*
- *both of $F_\infty/\mathbb{F}(T)$ and $F_\infty k/k$ satisfy the assumptions (A) and (B) in section 1, and*
- *the Sylow p-subgroup of $\mathrm{Cl}_S(F_n k)$ is trivial for all $n \geq 0$,*

where F_n is the nth layer of $F_\infty/\mathbb{F}(T)$. (*We use the same symbol S as the set of places lying above S.*)

Proof. We take a place $\mathfrak{p}_0$ of $\mathbb{F}(T)$ which splits completely in k. We also take a place $\mathfrak{r}$ of $\mathbb{F}(T)$ which is distinct from $\mathfrak{p}_0$ and unramified in k. We claim that there is a geometric $\mathbb{Z}_p$-extension $F_\infty/\mathbb{F}(T)$ unramified outside $\mathfrak{r}$ which satisfies that

- $\mathfrak{r}$ is totally ramified, and
- $\mathfrak{p}_0$ splits completely.

We shall show this claim. Let M be the maximal pro-p abelian extension over $\mathbb{F}(T)$ which is unramified outside $\mathfrak{r}$. We know that $\mathrm{Gal}(M/\mathbb{F}(T))$ is isomorphic to a countable infinite product of the additive group of $\mathbb{Z}_p$ (see [21], [9]). Hence there are infinitely many geometric $\mathbb{Z}_p$-extensions which satisfy the above conditions.

By the above choice of F_∞, we see $F_1 \cap k = \mathbb{F}(T)$. We put $k_1 = F_1 k$. Then $k_1/\mathbb{F}(T)$ is a Galois extension, and $\mathfrak{p}_0$ splits completely in k_1. We set $S_0 = \{\mathfrak{p}_0\}$, and we use the same symbol to denote the set of places lying above $\mathfrak{p}_0$. We can see that $H_{S_0}(k_1)$ is a finite Galois extension over $\mathbb{F}(T)$. We take a nontrivial element σ_1 of $\mathrm{Gal}(H_{S_0}(k_1)/k_1)$.

By using the above argument, we can take a geometric $\mathbb{Z}_p$-extension $F'_\infty/\mathbb{F}(T)$ unramified outside $\mathfrak{r}$ which satisfies

- $F'_\infty \cap F_\infty = \mathbb{F}(T)$,
- $\mathfrak{r}$ is totally ramified in $F'_\infty F_\infty$, and
- $\mathfrak{p}_0$ splits completely in F'_∞.

Let F'_1 be the initial layer of $F'_\infty/\mathbb{F}(T)$. Then we see that $F'_1 \cap k_1 = \mathbb{F}(T)$ and $k_1 F'_1 \cap H_{S_0}(k_1) = k_1$. We note that

$$\mathrm{Gal}(F'_1 H_{S_0}(k_1)/k_1) \cong \mathrm{Gal}(F'_1 k_1/k_1) \times \mathrm{Gal}(H_{S_0}(k_1)/k_1), \quad \mathrm{Gal}(F'_1 k_1/k_1) \cong \mathrm{Gal}(F'_1/\mathbb{F}(T)).$$

Hence there is an isomorphism

$$\mathrm{Gal}(F'_1/\mathbb{F}(T)) \times \mathrm{Gal}(H_{S_0}(k_1)/k_1) \xrightarrow{\sim} \mathrm{Gal}(F'_1 H_{S_0}(k_1)/k_1).$$

Let τ be a generator of the cyclic group $\mathrm{Gal}(F_1'/\mathbb{F}(T))$, and τ_1 an element of $\mathrm{Gal}(F_1'H_{S_0}(k_1)/k_1)$ which is the image of (τ, σ_1) under the above isomorphism. We can regard τ as an element of $\mathrm{Gal}(F_1'H_{S_0}(k_1)/\mathbb{F}(T))$. By the Chebotarev density theorem, there is a place $\mathfrak{P}_1$ of $F_1'H_{S_0}(k_1)$ which corresponds to τ_1. Let $\mathfrak{p}_1$ be the place of $\mathbb{F}(T)$ lying below $\mathfrak{P}_1$. We can take $\mathfrak{P}_1$ such that $\mathfrak{p}_1$ is not ramified in $F_1'H_{S_0}(k_1)$. Then we see that $\mathfrak{p}_1$ splits completely in k_1 and is inert in F_1'. We put $S_1 = S_0 \cup \{\mathfrak{p}_1\}$.

In general, $\mathfrak{p}_1$ may not split completely in F_∞. This is a problem because we need the assumption (B). We remark that $F_\infty F_\infty'/\mathbb{F}(T)$ is a $\mathbb{Z}_p^2$-extension unramified outside $\mathfrak{r}$. We recall that $\mathfrak{p}_1$ does not split in F_1'. Hence the decomposition field of $F_\infty F_\infty'/\mathbb{F}(T)$ for $\mathfrak{p}_1$ is a $\mathbb{Z}_p$-extension over $\mathbb{F}(T)$. We denote it by F_∞''. We also note that $F_\infty''/\mathbb{F}(T)$ is the unique $\mathbb{Z}_p$-extension contained in $F_\infty F_\infty'$ such that $\mathfrak{p}_1$ splits completely. Then the initial layer of $F_\infty''/\mathbb{F}(T)$ must coincide with F_1. We replace F_∞ by F_∞''.

We note that $H_{S_0}(k_1) \supsetneq H_{S_1}(k_1)$ by the definition of $\mathfrak{p}_1$. Similarly, we can choose a place $\mathfrak{p}_2$, put $S_2 = S_1 \cup \{\mathfrak{p}_2\}$, and modify the $\mathbb{Z}_p$-extension such that all places of S_2 splits completely. Repeating this operation, we see that $H_{S_t}(k_1) = k_1$ for some finite set S_t. From the above construction, we see that $F_\infty \cap k = \mathbb{F}(T)$ and that $F_\infty k/k$ satisfies the assumptions (A) and (B).

Finally, we shall give an Iwasawa-theoretic argument. In $F_\infty k/k$, all ramified places (which are lying above $\mathfrak{r}$) are totally ramified. From this, we also see $H_{S_t}(k) = k$. Let A_n be the Sylow p-subgroup of $\mathrm{Cl}_{S_t}(kF_n)$. By the above results, we see that both of A_0 and A_1 are trivial. In this situation, we can use the method given in Fukuda [6]. Namely, if all places which ramify in $F_\infty k/k$ are totally ramified and both of A_0 and A_1 are trivial, then A_n is trivial for all $n \geq 0$. (See [6, Theorem 1]. We note that the same method is also applicable for our situation.) Hence we see that A_n is trivial for all $n \geq 0$. $\square$

We shall show Theorem 1.4. We fix a finite p-group G. By using Theorem 2.2, we can take a geometric Galois extension $K/\mathbb{F}(T)$ and a subextension $k/\mathbb{F}(T)$ of $K/\mathbb{F}(T)$ such that K/k is unramified and $\mathrm{Gal}(K/k) \cong G$. By Theorem 3.1, we can take a geometric $\mathbb{Z}_p$-extension $F_\infty/\mathbb{F}(T)$ and a set S of places of $\mathbb{F}(T)$ such that $F_\infty \cap K = \mathbb{F}(T)$, all places of S split completely in K, both of $F_\infty/\mathbb{F}(T)$ and $F_\infty K/K$ satisfy the assumptions (A) and (B), and A_n is trivial for all $n \geq 0$ (where A_n is the Sylow p-subgroup of $\mathrm{Cl}_S(F_nK)$, and F_n is the nth layer of $F_\infty/\mathbb{F}(T)$). We note that $F_\infty k/k$ also satisfies the assumptions (A) and (B). We claim that $\tilde{L}_S(F_nK) = F_nK$ for all $n \geq 0$. Indeed, if $\tilde{L}_S(F_nK)/F_nK$ is nontrivial, then there is a nontrivial finite Galois p-subextension over F_nK. Moreover, there is a nontrivial finite abelian p-subextension over F_nK because every p-group is solvable. Since A_n is trivial, it is a contradiction. We have shown the above claim. This implies that $\tilde{L}_S(F_nk) = F_nK$ because F_nK/F_nk is unramified and all places of F_nk lying above S split completely in F_nK. Hence

$\mathrm{Gal}(\tilde{L}_S(F_n k)/F_n k) \cong G$ for all $n \geq 0$. Then the theorem follows. □

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