

The universal cover of hypertoric varieties and Bogomolov's decomposition

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§1 Background and Motivation

Definition ((conical) symplectic variety)

(Y, ω) is a *(conical) symplectic variety* if Y is normal and satisfies
 (1) $\omega \in \Omega^2_{Y_{\text{reg}}}$ is symplectic and extends to a 2-form on any resolution.
 (2) Y is affine, and $\exists \mathbb{C}^* \curvearrowright Y$ s.t. positive weight and $s^* \omega = s^\ell \omega$.)

Example

(1) ADE type surface singularity (3) (affine) Nakajima quiver variety
 (2) Nilpotent orbit closures $\overline{\mathcal{O}} \subset \mathcal{N} \subset \mathfrak{g}$ (4) (affine) Hypertoric variety

Consider $\pi_1(Y_{\text{reg}})$ of conical symplectic varieties (Y, ω) .

Theorem (Namikawa)

For a conical symplectic variety (Y, ω) , $|\pi_1^{\text{alg}}(Y_{\text{reg}})| < \infty$.

It is expected but not known that $|\pi_1(Y_{\text{reg}})| < \infty$.

Example

(1) $Y = \mathbb{C}^n / \Gamma$: isolated quotient singularity, $\pi_1(Y_{\text{reg}}) = \Gamma$.
 (2) (Beauville) $Y = \overline{\mathcal{O}}_{A_n}^{\text{min}} \subset \mathfrak{sl}_{n+1}$: min nilpotent orbit closure, $\pi_1(Y_{\text{reg}}) = 0$.

Proposition (The universal coverings of conical symplectic varieties)

Assume $|\pi_1(Y_{\text{reg}})| < \infty$. Then, $\exists!$ a conical symplectic variety $(\overline{Y}, \overline{\omega})$ s.t.

$$\begin{array}{ccc} (\overline{Y}, \overline{\omega}) & \supset & \varphi^{-1}(Y_{\text{reg}}) \\ \downarrow \exists! \varphi : \text{finite} & & \downarrow \varphi_0 := \varphi : \text{universal cover} \\ (Y, \omega) & \supset & Y_{\text{reg}} \end{array}$$

Problem (Namikawa, Analogue of Bogomolov's decomposition)

Assume $|\pi_1(Y_{\text{reg}})| < \infty$. Is $\overline{Y}$ (uniquely) decomposed into *irreducible* ones as

$$(\overline{Y}, \overline{\omega}) \cong (Y_1, \omega_1) \times \cdots \times (Y_r, \omega_r)?$$

(That is, any conical symplectic form Y_m is ω_m up to scalar.)

Goal

- Describe $\pi_1(Y_A(0)_{\text{reg}})$ and $\overline{Y_A(0)}$ for affine hypertoric varieties $Y_A(0)$.
- Establish Bogomolov's decomposition of $Y_A(0)$.

§2 Hypertoric variety $Y_A(\alpha)$

[Principle in (hyper)toric geometry : Combinatorics $\leftrightarrow$ Geometry]

Polytope (eg. $\triangle$) $\leftrightarrow$ (projective) Toric variety (eg. $\mathbb{P}^2$)

Hyperplane arrangement $\mathcal{H}_B^\alpha \leftrightarrow$ Hypertoric variety $Y_A(\alpha)$

Ingredients of hypertoric variety $Y_A(\alpha)$

$$\begin{array}{ccc} 0 \rightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \rightarrow 0 & & B^T = (\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n) \}_{n-d} \\ \downarrow \varpi & & \downarrow \varpi \\ \tilde{\alpha} \mapsto \alpha & & A = \left(\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n \right) \}_d \end{array}$$

Construction of $Y_A(\alpha) = \mu_A^{-1}(0)^{\alpha-ss} // \mathbb{T}_{\mathbb{C}}^d$

$$\begin{array}{ccc} \mathbb{T}_{\mathbb{C}}^d \xrightarrow{A^T} \mathbb{T}_{\mathbb{C}}^n & \hookrightarrow & (\mathbb{C}^{2n}, \omega_{\mathbb{C}}) \\ \downarrow \mu_A : \text{moment map} & & \mathbf{t} \cdot (z_i, w_i) := (\mathbf{t}^{a_i} z_i, \mathbf{t}^{-a_i} w_i) \\ \mathbb{C}^d = \text{Lie}(\mathbb{T}_{\mathbb{C}}^d)^* & & \end{array}$$

Definition (Hypertoric variety)

Hypertoric variety is defined as the GIT quotient $\mathbf{Y}_A(\alpha) := \mu_A^{-1}(0)^{\alpha-ss} // \mathbb{T}_{\mathbb{C}}^d$.

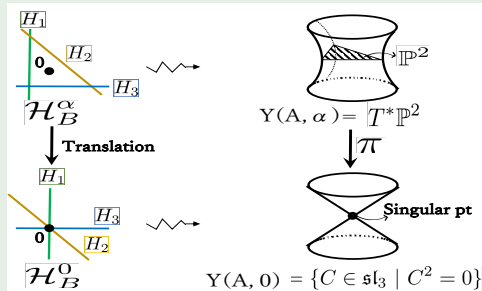
Define the ass. hyperplane arrangement $\mathcal{H}_B^\alpha := \{H_i : \langle \mathbf{b}_i, - \rangle = \tilde{\alpha}_i\} \subset \mathbb{R}^{n-d}$.

By definition, $\mu_A^{-1}(0)^{\alpha-ss} \subset \mu_A^{-1}(0)$ induces a projective morphism

$$\pi_\alpha : Y_A(\alpha) \rightarrow Y_A(0).$$

Example (Springer resolution $T^*\mathbb{P}^2 \rightarrow \overline{\mathcal{O}}_{A_2}^{\text{min}}$)

Let $A = (111)$, $B^T = \begin{pmatrix} 10 & -1 \\ 01 & -1 \end{pmatrix} = (\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3)$. $\tilde{\alpha} = (1, 1, 1)$



Basic Properties of $Y_A(\alpha)$

Let α be generic. Then, $Y_A(0)$ and $Y_A(\alpha)$ are simpl. var. of dim $2(n-d)$, and
 (1) $\pi_\alpha : Y_A(\alpha) \rightarrow Y_A(0)$ gives a crepant (symplectic) resolution.
 (2) $\text{Sing}(Y_A(0))$ corresponds to the “**non-general**” locus of $\mathcal{H}_B^0$.

translation of the arrangement $\mathcal{H}_B^0 \leftrightarrow$ taking the *crepant resolution*

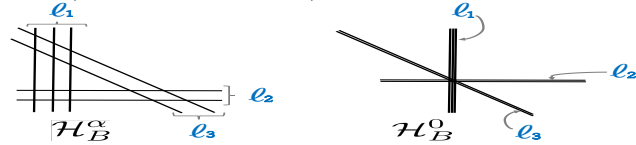
Theorem (Classification of 4-dimensional $Y_A(0)$)

Every 4-dim $Y_A(0)$ is isomorphic to one of the following:

(1) $S_{A_{\ell_1-1}} \times S_{A_{\ell_2-1}}$: the product of A type surface singularities.
 (2)

$$\overline{\mathcal{O}}^{\text{min}}(\ell_1, \ell_2, \ell_3) := \left\{ \begin{pmatrix} u_1 x_1 x_3 \\ y_1 u_2 x_2 \\ y_3 y_2 u_3 \end{pmatrix} \in \mathfrak{sl}_3 \mid \text{All } 2 \times 2\text{-minors of } \begin{pmatrix} u_1^{\ell_1} x_1 & x_3 \\ y_1 & u_2^{\ell_2} x_2 \\ y_3 & y_2 & u_3^{\ell_3} \end{pmatrix} = 0 \right\}.$$

$$\text{Case (2)} \quad B^T = \begin{pmatrix} \overbrace{1 \cdots 1}^{\ell_1} & \overbrace{0 \cdots 0}^{\ell_2} & \overbrace{-1 \cdots -1}^{\ell_3} \\ 0 \cdots 0 & 1 \cdots 1 & -1 \cdots -1 \end{pmatrix}$$



Remark

In general, isomorphism classes of $Y_A(0)$ as conical symplectic varieties are classified by the combinatorial str. of $\mathcal{H}_B^0$, i.e., “matroid”.

§3 Main result 1 : $\pi_1(Y_A(0)_{\text{reg}})$ and the universal cover

In general, we can assume as $B^T = (\overbrace{\mathbf{b}^{(1)} \cdots \mathbf{b}^{(1)}}^{\ell_1} \cdots \overbrace{\mathbf{b}^{(s)} \cdots \mathbf{b}^{(s)}}^{\ell_s})$

It is natural to ask whether the simplification $Y_{\underline{A}}(0)$ gives univ. cover of $Y_A(0)$.

$$\begin{array}{ccc} B^T = (\mathbf{b}^{(1)} \cdots \mathbf{b}^{(1)} \cdots \mathbf{b}^{(s)} \cdots \mathbf{b}^{(s)}) & \dashrightarrow & Y_A(0) \\ \downarrow \text{simplification} & & \uparrow \exists? \text{ finite} \\ \overline{B}^T = (\mathbf{b}^{(1)} \cdots \mathbf{b}^{(s)}) & \dashrightarrow & Y_{\underline{A}}(0) \end{array}$$

,

$$0 \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{\overline{B}} \mathbb{Z}^s \xrightarrow{A} \mathbb{Z}^{s-(n-d)} \longrightarrow 0$$

Theorem (N.)

$\exists$ finite $\varphi : Y_{\underline{A}}(0) \rightarrow Y_A(0) \cong Y_{\underline{A}}(0)/G$: universal cover, and we have

$$G \cong \pi_1(Y_A(0)_{\text{reg}}) \cong \Gamma / \Gamma \cap \mathbb{T}_{\mathbb{C}}^{s-(n-d)} = \Gamma / (\text{Ker } \tilde{B}^T) \Gamma,$$

where $\Gamma := \prod_{k=1}^s \mathbb{Z} / \ell_k \mathbb{Z} \hookrightarrow \mathbb{T}_{\mathbb{C}}^s \xrightarrow{A^T} \mathbb{T}_{\mathbb{C}}^{d-(n-s)}$ and $\tilde{B}^T := (m_1 \mathbf{b}^{(1)} \cdots m_s \mathbf{b}^{(s)})$
 $(m_k := \ell_1 \cdots \ell_s / \ell_k)$.

simplification of the arrangement $\mathcal{H}_B^0 \leftrightarrow$ taking the *universal covering*

Corollary

$$\pi_1(Y_A(0)_{\text{reg}}) = 0 \Leftrightarrow \ell_1 = \cdots = \ell_s = 1, \text{ i.e., } \mathcal{H}_B^0 \text{ is simple.}$$

Example ($\overline{\mathcal{O}}^{\text{min}}(\ell_1, \ell_2, \ell_3)$)

$\tilde{B}^T = \begin{pmatrix} \ell_2 \ell_3 & 0 & -\ell_1 \ell_2 \\ 0 & \ell_1 \ell_3 & -\ell_1 \ell_2 \end{pmatrix}$, and $g := \gcd(\ell_1, \ell_2, \ell_3)$. Then,

$$\pi_1(\overline{\mathcal{O}}^{\text{min}}(\ell_1, \ell_2, \ell_3)_{\text{reg}}) \cong \frac{\prod_{k=1}^3 \mathbb{Z} / \ell_k \mathbb{Z}}{\mathbb{Z} \left(\left\langle \frac{\ell_1}{g}, \frac{\ell_2}{g}, \frac{\ell_3}{g} \right\rangle \right)}$$

§4 Main result 2 : Bogomolov's decomposition for $Y_A(0)$

If $A = A_1 \oplus \cdots \oplus A_r$, then $Y_A(0) \cong \prod_{m=1}^r Y_{A_m}(0)$. So, we can expect as follows.

Theorem (N.)

$Y_A(0)$ is irreducible $\Leftrightarrow A$ is indecomposable as a matrix.

【Key of the proof】

- Reduce to the simple and smooth case $Y_{\underline{A}}(\alpha)$.
- $\mu_{\underline{A}}^{-1}(0)$ is C.I. & codim 3 smooth. $\rightsquigarrow \Omega^2_{\mu_{\underline{A}}^{-1}(0)}$ is reflexive.
- Consider which $\sigma \in \Omega^2_{\mu_{\underline{A}}^{-1}(0)}$ descends to a $\bar{\sigma} \in \Omega^2_{Y_{\underline{A}}(\alpha)}$.

Corollary

For any two smooth hypertoric varieties $Y_A(\alpha)$ and $Y_{A'}(\alpha')$, TFAE:

- $Y_A(\alpha) \cong Y_{A'}(\alpha') : \mathbb{C}^*$ -equivariant isomorphic.
- $Y_A(\alpha) \cong Y_{A'}(\alpha') : \mathbb{C}^* \times \mathbb{T}_{\mathbb{C}}^{n-d}$ -equivariant isomorphic as symplectic varieties.
- $\mathcal{H}_B^\alpha \cong \mathcal{H}_{B'}^{\alpha'}$.

The same statement will hold for any two affine hypertoric varieties.

【Reference】

[Nag] T. Nagaoka, The universal covers of hypertoric varieties and Bogomolov's decomposition, arXiv.

[Nam1] Y. Namikawa, Equivalence of symplectic singularities. Kyoto J.Math. 53 (2013), no. 2, 483–514.

[Nam2] Y. Namikawa, Fundamental groups of symplectic singularities. Higher dimensional algebraic geometry, 321–334, Adv. Stud. Pure Math., 74, Math. Soc. Japan, Tokyo, 2017.