

FEIGIN AND ODESSKII'S ELLIPTIC ALGEBRAS

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ABSTRACT. This article summarizes some of the results in joint papers [CKS19, CKS21, CKS23] with Alex Chirvasitu and S. Paul Smith. We studied elliptic algebras introduced by Feigin and Odesskii in 1989, which are noncommutative graded algebras $Q_{n,k}(E, \eta)$ parametrized by an elliptic curve E , a point $\eta \in E$, and coprime positive integers $n > k$. These algebras are a generalization of Sklyanin algebras, recognized as important examples of Artin-Schelter regular algebras. One of our main results is that $Q_{n,k}(E, \eta)$ has the same Hilbert series as the polynomial ring in n variables when η is not a torsion point.

ACKNOWLEDGMENTS

The author would like to express his gratitude to the organizers of Kinoshita Algebraic Geometry Symposium 2023 for providing him with the opportunity to deliver a talk.

The author was supported by JSPS KAKENHI Grant Numbers JP16H06337, JP17K14164, JP20K14288, and JP21H04994, Leading Initiative for Excellent Young Researchers, MEXT, Japan, and Osaka Central Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JP-MXP0619217849.

1. INTRODUCTION

This article summarizes some of the results in joint papers [CKS19, CKS21, CKS23] with Alex Chirvasitu and S. Paul Smith.

In 1989, Feigin and Odesskii [FO89, OF89] defined a family of graded algebras $Q_{n,k}(E, \eta)$ called *elliptic algebras*. These algebras have played an important role in noncommutative algebraic geometry, especially in the study of Artin-Schelter regular algebras.

Let $\tau \in \mathbb{H}$ be a complex number that is in the upper half plane, and define the lattice $\Lambda := \mathbb{Z} + \mathbb{Z}\tau$. Consider the elliptic curve $E := \mathbb{C}/\Lambda$ and fix a closed point $\eta \in E$. Let $n > k \geq 1$ be coprime integers. The algebra $Q_{n,k}(E, \eta)$ is defined to be the graded $\mathbb{C}$ -algebra generated by n variables x_i in degree one, indexed by $i \in \mathbb{Z}/n\mathbb{Z}$, with n^2 quadratic relations

$$(1.1) \quad \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+(k-1)r}(0)}{\theta_{j-i-r}(-\eta)\theta_{kr}(\eta)} x_{j-r}x_{i+r} = 0 \quad (i, j \in \mathbb{Z}/n\mathbb{Z}),$$

where θ_α ($\alpha \in \mathbb{Z}/n\mathbb{Z}$) are certain theta functions of order n that are quasi-periodic with respect to the lattice Λ (see [CKS21, §2.2.5]). More precisely, they are holomorphic functions on $\mathbb{C}$ characterized by the following properties (up to common

scalar multiple):

$$\begin{aligned}\theta_\alpha(z + \tfrac{1}{n}) &= e^{\frac{2\pi\sqrt{-1}\alpha}{n}} \theta_\alpha(z), \\ \theta_\alpha(z + \tfrac{1}{n}\tau) &= e^{2\pi\sqrt{-1}(-z - \frac{1}{2n} - \frac{n-1}{2n}\tau)} \theta_{\alpha+1}(z).\end{aligned}$$

The zeros of θ_α are the points in $-\frac{\alpha}{n} + \frac{1}{n}\mathbb{Z} + \mathbb{Z}\tau$ (all have multiplicity one), so the denominator in the relations (1.1) can be zero when η is in $\frac{1}{n}\Lambda$ (that is, η is an n -torsion point). Nonetheless, there is a way to extend the definition to all $\eta \in E$, and then, $Q_{n,k}(E, 0)$ is the polynomial ring in n variables ([CKS21, Proposition 5.1]). So $\{Q_{n,k}(E, \eta)\}_{\eta \in E}$ can be thought as a family of noncommutative deformations of the polynomial ring.

These algebras have been studied from various perspectives. One significant aspect is that they are a rich source of Artin-Schelter regular algebras.

A graded $\mathbb{C}$ -algebra A generated by finitely many degree-one elements $x_1, \dots, x_r$ over $A_0 = \mathbb{C}$ with relations $f_1, \dots, f_s$ is called *Artin-Schelter regular* (or *AS-regular* for short) of dimension n if

- (1) the global dimension of A is n ;
- (2) $\text{Ext}_A^i(\mathbb{C}, A) \cong \begin{cases} \mathbb{C} & \text{if } i = n, \\ 0 & \text{if } i \neq n; \end{cases}$
- (3) the sequence $\{\dim_{\mathbb{C}} A_i\}_{i=0}^\infty$ has polynomial growth.

Artin-Schelter [AS87] gave a partial classification of 3-dimensional AS-regular algebras. Their first observation was that the 3-dimensional AS-regular algebras are either of *quadratic type* or of *cubic type*, the former has 3 generators (in degree 1) and 3 quadratic relations (as the polynomial ring in 3 variables does), and the latter has 2 generators and 2 cubic relations.

Artin-Tate-Van den Bergh [ATVdB90] showed that one can associate a triple $(X, \sigma, \mathcal{L})$ to a 3-dimensional AS-regular algebra, where X is a scheme (a closed subscheme of $\mathbb{P}^2$ or $\mathbb{P} \times \mathbb{P}$), σ is an automorphism of X , and $\mathcal{L}$ is an invertible sheaf on X , and that the algebra can be recovered from the associated triple. In this way, the classification of 3-dimensional AS-regular algebras can be reduced to the classification of such triples. In the recent works by Itaba-Matsuno [IM21] and Matsuno [Mat21], the classification of quadratic 3-dimensional algebras has been completed, in the sense that they wrote down the relations for each isomorphism classes explicitly. We may associate a triple to a higher-dimensional AS-regular algebra, but there is no method to recover the algebra from the triple in general. Thus the classification for AS-regular algebras of dimension ≥ 4 is not in sight.

When $k = 1$, $Q_{n,1}(E, \eta)$ are known as *Sklyanin algebras*, and have been considered as important examples of quadratic Artin-Schelter regular algebras. The name comes from the appearance of $Q_{4,1}(E, \eta)$ in Sklyanin's papers on the quantum scattering inverse method [Skl82]. The list of 3-dimensional quadratic Artin-Schelter regular algebras of Artin-Schelter [AS87] or Artin-Tate-Van den Bergh [ATVdB90] suggests that $Q_{3,1}(E, \eta)$ are the most "standard" family among those algebras.

In 1992, Smith-Stafford [SS92] showed that $Q_{4,1}(E, \eta)$ is a 4-dimensional AS-regular algebra, and that it is a noetherian domain that has the same Hilbert series

as the polynomial ring in 4 variables. This result was generalized by Tate and Van den Bergh [TVdB96] to $Q_{n,1}(E, \eta)$ for arbitrary n ; they showed that $Q_{n,1}(E, \eta)$ is n -dimensional AS-regular and has many other good homological properties shared with a polynomial ring in n variables.

When $k > 1$, not much was known about $Q_{n,k}(E, \eta)$. Thus, Alex Chirvasitu, S. Paul Smith, and the author initiated a project to investigate these algebras. This leads to the results that we will review in later sections.

2. HILBERT SERIES AND AS-REGULARITY

Feigin-Odesskii ([FO98, Ode02], for example) claimed that the algebras $Q_{n,k}(E, \eta)$ have the same Hilbert series as the polynomial ring in n variables (namely $(1-t)^{-n}$) for generic η , and provided some ideas for the proof. We made the statement more precise, and gave a complete proof for that.

Theorem 2.1 ([CKS23, Theorem 1.1]). *Assume that $\eta \in E$ is not a torsion point.*

- (1) *The Hilbert series of $Q_{n,k}(E, \eta)$ is the polynomial ring in n variables.*
- (2) *$Q_{n,k}(E, \eta)$ is a Koszul algebra whose global dimension is n .*

We will briefly sketch the proof for the first statement. The key fact is that the defining relations for $Q_{n,k}(E, \eta)$ come from a certain solution to the quantum Yang-Baxter equation. Let V be the degree-one part of the algebra $Q_{n,k}(E, \eta)$, namely the vector space with basis $\{x_i\}_{i \in \mathbb{Z}/n\mathbb{Z}}$. Define the linear operator $R_\eta(z): V \otimes_{\mathbb{C}} V \rightarrow V \otimes_{\mathbb{C}} V$ by

$$R_\eta(z)(x_i \otimes x_j) := \frac{\theta_0(-z) \cdots \theta_{n-1}(-z)}{\theta_1(0) \cdots \theta_{n-1}(0)} \sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{\theta_{j-i+r(k-1)}(-z+\eta)}{\theta_{j-i-r}(-z)\theta_{kr}(\eta)} x_{j-r} \otimes x_{i+r}.$$

The family of operators $\{R_\eta(z)\}_{z \in \mathbb{C}}$ satisfies the *quantum Yang-Baxter equation with one spectral parameter*:

$$R_\eta(u)_{12} R_\eta(u+v)_{23} R_\eta(v)_{12} = R_\eta(v)_{23} R_\eta(u+v)_{12} R_\eta(u)_{23}$$

on $V \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V$, where $A_{12} := A \otimes \text{id}_V$ and $A_{23} := \text{id}_V \otimes A$ for an operator A on $V \otimes_{\mathbb{C}} V$. This solution is essentially Belavin's elliptic solution ([Bel80]; see [CKS23, §1.3]).

It is easy to see that the space for quadratic relations for $Q_{n,k}(E, \eta)$ is the image of $R_\eta(\eta)$, that is, $Q_{n,k}(E, \eta)$ is the tensor algebra $T_{\mathbb{C}} V$ modulo the ideal generated by $\text{Im } R_\eta(\eta)$.

We construct an operator $F_d(-\eta)$ on $V^{\otimes d}$ by composing some operators of the form $\text{id}_V^{\otimes s} \otimes R_\eta(z) \otimes \text{id}_V^{\otimes t}$ for various z and $s+t+2=d$, and show that

$$(2.1) \quad \sum_{s+t+2=d} V^{\otimes s} \otimes \text{Im } R_\eta(\eta) \otimes V^{\otimes t} \subseteq \text{Ker } F_d(-\eta),$$

where the left-hand side is the space of degree- d relations for $Q_{n,k}(E, \eta)$. We do not present the construction of $F_d(-\eta)$ here but $F_2(-\eta) = R_\eta(-\eta)$ in the case of $d=2$. (2.1) can be deduced using the fact that $R_\eta(z)$ satisfies the quantum Yang-Baxter equation, with some additional observation on $R_\eta(z)$ combined.

We will consider the limit $\eta \rightarrow 0$. It is easy to see that the map $\eta \mapsto R_\eta(\eta)$ is holomorphic on $\mathbb{C}$ so taking the limit is essentially substitution $\eta = 0$, but we should note that the limit of $R_\eta(\eta)$ may be, and actually is, different from the limit of $R_\eta(-\eta)$. This is because the map $(\eta, z) \mapsto R_\eta(z)$ is not holomorphic on $\mathbb{C}^2$. Taking the limit $\eta \rightarrow 0$, the image of the operator $R_\eta(\eta)$ becomes the space of quadratic relations for $Q_{n,k}(E, 0)$, which is the polynomial ring in n variables, so the dimension of the left-hand side of (2.1) becomes the dimension of the space of degree- d relations for the polynomial ring, which is $n^d - \binom{n+d-1}{d}$. On the other hand, it can be observed by a direct computation that

$$\lim_{\eta \rightarrow 0} F_d(-\eta) = \prod_{m=1}^{d-1} m! \cdot \sum_{\sigma \in S_d} \sigma,$$

where the symmetric group S_d acts on $V^{\otimes d}$ by permuting tensorands. So the dimension of the right-hand side of (2.1) also becomes $n^d - \binom{n+d-1}{d}$.

For a continuous family of operators, the dimension of the image attains the largest value on a Zariski-open subset, so the dimension of the left-hand side of (2.1) has the same property, and hence it is $\geq n^d - \binom{n+d-1}{d}$ for a general point η . Similarly, the dimension of the kernel attains the smallest value on a Zariski-open subset, so the dimension of the right-hand side of (2.1) is $\leq n^d - \binom{n+d-1}{d}$ for general η . Since we have the inclusion (2.1) for all η , it follows that (2.1) is an equality for general η and, in that case, they have dimension $n^d - \binom{n+d-1}{d}$. Moreover,

$$(2.2) \quad \dim \sum_{s+t+2=d} V^{\otimes s} \otimes \operatorname{Im} R_\eta(\eta) \otimes V^{\otimes t} \leq n^d - \binom{n+d-1}{d} \leq \dim \operatorname{Ker} F_d(-\eta)$$

for all η . We have to show that the inequalities in (2.2) are equalities whenever η is not a torsion point. Since the treatment for general d is complicated [CKS23, §6.4], we will focus on the case $d = 2$ here. In this case, (2.1) is

$$\operatorname{Im} R_\eta(\eta) \subseteq \operatorname{Ker} R_\eta(-\eta)$$

and we already know from (2.2) that

$$(2.3) \quad \dim \operatorname{Im} R_\eta(\eta) \leq \binom{n}{2} \leq \dim \operatorname{Ker} R_\eta(-\eta).$$

Thus it suffices to compute the dimension of $\operatorname{Ker} R_\eta(z)$ for all $z \in \mathbb{C}$ (by the rank-nullity theorem). The key observation for this is that, for all $p \in \mathbb{C}$,

$$(2.4) \quad \dim \operatorname{Ker} R_\eta(p) \leq \operatorname{mult}_p \det R_\eta(z),$$

where the right-hand side is the multiplicity of the function $z \mapsto \det R_\eta(z)$ at $z = p$. (2.4) is not specific to $R_\eta(z)$ and can be shown elementarily ([CKS23, Lemma 4.1]). By looking at the definition of $R_\eta(z)$, we notice that the function $z \mapsto \det R_\eta(z)$ is a theta function of order n^4 with respect to Λ , and hence has exactly n^4 zeros in each fundamental parallelogram for Λ . So we obtain

$$(2.5) \quad \sum_p \dim \operatorname{Ker} R_\eta(p) \leq \sum_p \operatorname{mult}_p \det R_\eta(z) = n^4,$$

where p runs over a fundamental parallelogram. The final step is to show that this is an equality when τ is not a torsion point. By (2.3), we already know that $\dim \operatorname{Ker} R_\eta(-\eta) \geq \binom{n}{2}$, and a similar argument shows that $\dim \operatorname{Ker} R_\eta(\eta) \geq \binom{n+1}{2}$. We observe that $R_\eta(z+\zeta)$ has the same rank as that of $R_\eta(z)$ for all $\zeta \in \frac{1}{n}\Lambda$ ([CKS23, Proposition 2.6]), so $\dim \operatorname{Ker} R_\eta(-\eta + \zeta) \geq \binom{n}{2}$ and $\dim \operatorname{Ker} R_\eta(\eta + \zeta) \geq \binom{n+1}{2}$ for all $\zeta \in \frac{1}{n}\Lambda$. There are n^2 such ζ in the fundamental parallelogram for Λ . If $\eta \notin \frac{1}{2n}\Lambda$, then $\{\eta + \zeta \mid \zeta \in \frac{1}{n}\Lambda\}$ and $\{-\eta + \zeta \mid \zeta \in \frac{1}{n}\Lambda\}$ have no intersection, so the dimensions of the kernels add up to (at least)

$$\binom{n}{2} \cdot n^2 + \binom{n+1}{2} \cdot n^2 = n^4.$$

Therefore (2.5) is an equality and so is (2.4).

We also have a result on the AS-regularity:

Theorem 2.2 ([CKS23, Theorem 1.2]). *$Q_{n,k}(E, \eta)$ is n -dimensional AS-regular for all but countably many $\eta \in E$.*

This result is of weaker form than Theorem 2.1 in the sense that we cannot detect the points where $Q_{n,k}(E, \eta)$ is (possibly) not AS-regular. Nevertheless, this fact tells us that elliptic algebras provide a large family of AS-regular algebras.

3. POINT MODULES

The statements of Theorem 2.1 and Theorem 2.2 do not involve the number k at all, so the reader may wonder how different $Q_{n,k}(E, \eta)$ are from $Q_{n,1}(E, \eta)$. In fact, they are quite different, and looking at the structure of point modules is one way to observe this.

For a graded $\mathbb{C}$ -algebra A generated by finitely many degree-one elements $x_1, \dots, x_r$ over $A_0 = \mathbb{C}$, a *point module* is a graded (right) A -module M such that

- (1) $\dim_{\mathbb{C}} M_i = \begin{cases} 1 & \text{if } i \geq 0, \\ 0 & \text{if } i < 0; \end{cases}$
- (2) M is generated in degree zero, that is, $M = M_0 A$.

Considering point modules for a given algebra was essential in the work of Artin-Tate-Van den Bergh [ATVdB90]. Indeed, in the triple $(X, \sigma, \mathcal{L})$ constructed from a 3-dimensional AS-regular algebra A , the closed points of the scheme X parametrizes the isomorphism classes of point modules over A . Such X is called the *point scheme* of A . The point scheme can be defined for A that is not necessarily 3-dimensional AS-regular. It is defined to be an inverse limit of schemes (more precisely, an inverse limit of sets, each of which has a natural scheme structure), but the inverse system often stabilizes, which makes the inverse limit an actual scheme.

The point scheme of the Sklyanin algebras $Q_{n,1}(E, \eta)$ are understood when η is a general point. For all $n \geq 3$, except for $n = 4$, the point scheme is the elliptic curve E ([Smi94]). When $n = 4$, the point scheme is the union of E and 4 additional points. If we embed E into $\mathbb{P}^3$ as an elliptic normal curve of degree 4, those four

points are realized as the vertices (singular points) of the four singular quadrics that contain E ([SS92]).

The point scheme for $Q_{n,k}(E, \eta)$ for $k > 1$ is not well-understood. Feigin-Odesskii [OF89, FO98] claimed that the point scheme of $Q_{n,k}(E, \eta)$ is a finite product E^g modulo an action of a finite group, where the number g is the length of the negative continued fraction for n/k :

$$(3.1) \quad \frac{n}{k} = n_1 - \frac{1}{n_2 - \frac{1}{\dots - \frac{1}{n_g}}} \quad (n_1, \dots, n_g \text{ are integers } \geq 2).$$

However, if $k = 1$, then $g = 1$, so according to this claim the point scheme of $Q_{n,1}(E, \eta)$ should be a quotient of E , but this is not the case when $n = 4$, because the point scheme has 4 additional points.

In [CKS19], we defined the action of a finite group explicitly and concluded that the point scheme always *contains* the scheme described by Feigin-Odesskii. To describe the action, let $\varepsilon: E^g \rightarrow E^{g+1}$ be the injection defined by

$$\varepsilon(z_1, \dots, z_g) := (z_1, z_2 - z_1, \dots, z_g - z_{g-1}, -z_g).$$

and identify E^g with $\text{Im } \varepsilon$, which consists of those points in E^{g+1} whose sum of coordinates is zero. The symmetric group S_{g+1} acts on E^{g+1} by permutation, and it induces the action on $\text{Im } \varepsilon$, hence on E^g . Define Σ to be the subgroup of S_{g+1} generated by transpositions $(i, i+1)$ for all $1 \leq i \leq g$ satisfying $n_i = 2$.

Theorem 3.1 ([CKS19, Theorem 1.2, Proposition 5.12]). *For all $\eta \in E$, the point scheme of $Q_{n,k}(E, \eta)$ contains $X_{n/k} := E^g / \Sigma$.*

This reveals a significant difference between $Q_{n,1}(E, \eta)$ and $Q_{n,k}(E, \eta)$ for $k > 1$. For example, if $n = 5$ and $k = 2$, then the negative continued fraction is

$$\frac{5}{2} = 3 - \frac{1}{2},$$

so $g = 2$ and $(n_1, n_2) = (3, 2)$. A direct computation shows that $X_{5/2} = E^2 / \Sigma$ is the symmetric product $S^2 E$. So the point scheme of $Q_{5,2}(E, \eta)$ contains $X_{5/2}$. In contrast, when $\eta \in E$ is a general point, the point scheme of $Q_{5,1}(E, \eta)$ is E , which cannot contain $S^2 E$.

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