

On the Oseen semigroup with rotating effect

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1 Introduction

Consider a rigid body $\mathcal{R}$ moving through an incompressible viscous fluid $\mathcal{L}$ that fills the whole three-dimensional space exterior to $\mathcal{R}$. We assume that with respect to a frame attached to $\mathcal{R}$, the translational velocity u_∞ and the angular velocity ω of $\mathcal{R}$ are both constant vectors. Without loss of generality we may assume that $\omega = {}^T(0, 0, a)$ ¹. If the flow is non-slip at the boundary, then the motion is described by the following equation:

$$(1) \quad \begin{aligned} v_t + v \cdot \nabla v - \Delta v + \nabla \pi &= g && \text{in } \Omega(t), \quad t > 0 \\ \operatorname{div} v &= 0 && \text{in } \Omega(t), \quad t > 0 \\ v(y, t) &= \omega \times y && \text{on } \partial\Omega(t), \quad t > 0 \\ v(y, t) &\rightarrow u_\infty \neq 0 && \text{as } |y| \rightarrow \infty, \quad t > 0 \\ v(y, 0) &= v_0(y) \end{aligned}$$

in the time-dependent exterior domain

$$\Omega(t) = O(at)\Omega$$

where $O(t)$ denotes the orthogonal matrix

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and Ω is a fixed exterior domain in $\mathbb{R}^3$ with $C^{1,1}$ boundary $\partial\Omega$.

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^{1T} M denotes the transposed M .

Over the last several years, the study of well-posedness of the initial boundary value problem of (1) and related topics has attracted the attention of several authors; see, e.g. [1]–[36]. Besides the intrinsic mathematical interest, this is probably also due to the fact that problem (1) is at the foundation of several important engineering applications. The interested reader is referred to [11] and to the literature cited therein.

To treat (1) in the time-independent domain Ω , we introduce

$$(2) \quad \begin{aligned} x &= O(t)^T y, \\ u(x, t) &= O(t)^T (v(y, t) - u_\infty), \\ p(x, t) &= \pi(y, t) \end{aligned}$$

Then, we see that (u, p) satisfies the modified Navier-Stokes equations:

$$(3) \quad \begin{aligned} u_t + u \cdot \nabla u - \Delta u + (O(t)^T u_\infty) \cdot \nabla u & \\ -(\omega \times x) \cdot \nabla u + \omega \times u + \nabla \pi &= f && \text{in } \Omega \times (0, \infty) \\ \operatorname{div} u &= 0 && \text{in } \Omega \times (0, \infty) \\ u(x, t) &= \omega \times x - O(t)u_\infty && \text{on } \partial\Omega \times (0, \infty) \\ u(x, t) &\rightarrow 0 && \text{as } |x| \rightarrow \infty, t > 0 \\ u(x, 0) &= v_0(x) \end{aligned}$$

In this paper, we consider only the case where

$$u_\infty = k e_3$$

with $e_3 = {}^T(0, 0, 1)$ so that

$$O(t)^T u_\infty = k e_3 \quad \text{for all } t > 0.$$

Therefore, the equation (1) leads to the system:

$$(4) \quad \begin{aligned} u_t + u \cdot \nabla u - \Delta u + k \partial_3 u & \\ -(\omega \times x) \cdot \nabla u + \omega \times u + \nabla \pi &= f && \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u &= 0 && \text{in } \Omega \times (0, \infty), \\ u(x, t) &= \omega \times x - k e_3 && \text{on } \partial\Omega \times (0, \infty), \\ u(x, t) &\rightarrow 0 && \text{as } |x| \rightarrow \infty, t > 0 \\ u(x, 0) &= v_0(x) \end{aligned}$$

One of the interesting questions is to find the unique existence of physically reasonable stationary solutions to (4) and to show their stability by initial disturbance in the $L_3(\Omega)$ framework when the external force f is independent of time variable, which is an extension of results obtained in [47] to the case where $\omega \neq 0$. But, so far we could not show the existence of stationary solutions of (4) which behave like

$O((|x|s(x)^\epsilon)^{-1})$ and the gradient of which behave like $O((|x|^{\frac{1}{2}}s(x)^{\frac{1}{2}+\epsilon})^{-1})$ with some small positive constant ϵ as $|x| \rightarrow \infty$ (cf. [47], [48] in the case where $a = 0$ and also Theorem 4 below), where $s(x) = |x| + x_3$. Therefore, in this paper we shall only give the existence of solutions and their decay properties of the linearized equations, and assuming the existence of physically reasonable stationary solutions of (4), we shall state their stability theorem.

The crucial step in proving the stability theorem of stationary solutions is to show the L_p - L_q decay estimate of the solutions to the following linearized equations of (4):

$$(5) \quad \begin{aligned} u_t + L_{k,a}u + \nabla\pi &= 0 \quad \text{in } \Omega, \quad t > 0, \\ \nabla \cdot u &= 0 \quad \text{in } \Omega, \quad t > 0, \\ u &= 0 \quad \text{on } \Gamma, \\ u|_{t=0} &= f \quad \text{in } \Omega, \end{aligned}$$

Here,

$$L_{k,a}u = -\Delta u + k\partial_3 u - (\omega \times x) \cdot \nabla u + \omega \times u$$

$\omega = (0, 0, a)$, k and a are real constants, and $u = (u_1, u_2, u_3)$ and π are unknown velocity field and pressure, respectively. The equation in (5) is written component-wise as follows:

$$\begin{aligned} \frac{\partial u_1}{\partial t} - \Delta u_1 + k \frac{\partial u_1}{\partial x_3} - a \left(x_1 \frac{\partial u_1}{\partial x_2} - x_2 \frac{\partial u_1}{\partial x_1} \right) - u_2 + \frac{\partial \pi}{\partial x_1} &= 0 \\ \frac{\partial u_2}{\partial t} - \Delta u_2 + k \frac{\partial u_2}{\partial x_3} - a \left(x_1 \frac{\partial u_2}{\partial x_2} - x_2 \frac{\partial u_2}{\partial x_1} \right) + u_1 + \frac{\partial \pi}{\partial x_2} &= 0 \\ \frac{\partial u_3}{\partial t} - \Delta u_3 + k \frac{\partial u_3}{\partial x_3} - a \left(x_1 \frac{\partial u_3}{\partial x_2} - x_2 \frac{\partial u_3}{\partial x_1} \right) + \frac{\partial \pi}{\partial x_3} &= 0 \end{aligned}$$

To show the existence of solutions to (5), we use the semigroup approach. Since the pressure term π has no time evolution, we have to eliminate π by using the Helmholtz decomposition. Therefore, at this stage we shall introduce the Helmholtz decomposition. Let D be one of $\mathbb{R}^3$, Ω and $\Omega_R = \Omega \cap B_R$ ($B_R = \{x \in \mathbb{R}^3 \mid |x| < R\}$). Let R be a large number such that $B_{R-5} \supset \mathbb{R}^3 \setminus \Omega$ and $1 < q < \infty$. Set

$$\begin{aligned} L_q(D)^3 &= \{f = (f_1, f_2, f_3) \mid f_i \in L_q(D) \ (i = 1, 2, 3)\}, \\ J_q(D) &= \overline{C_{0,\sigma}^\infty(D)}^{L_q(D)} \\ G_q(D) &= \{\nabla\pi \mid \pi \in \hat{W}_q^1(D)\} \\ C_{0,\sigma}^\infty(D) &= \{u \in C_0^\infty(D)^3 \mid \operatorname{div} u = 0 \text{ in } D\} \\ \hat{W}_q^1(D) &= \{\pi \in L_{q,\operatorname{loc}}(\bar{D}) \mid \nabla\pi \in L_q(D)^3, \int_{\Omega_R} \pi \, dx = 0\} \end{aligned}$$

Then, we have the Helmholtz decomposition:

$$L_q(D)^3 = J_q(D) \oplus G_q(D), \quad \oplus : \text{direct sum}$$

(cf. [39], [41], [46], [49]). In fact, given $f = (f_1, f_2, f_3) \in L_q(D)^3$ let π be a weak solution to the Laplace equation:

$$(6) \quad \Delta \pi = \operatorname{div} f \text{ in } D, \quad \partial_\nu \pi = \nu \cdot f$$

Here, $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit outer normal to ∂D and $\partial_\nu = \nu \cdot \nabla$. Setting $g = f - \nabla \pi$, we have the decomposition: $f = g + \nabla \pi$, which is required one. In particular, we know that

$$J_q(D) = \{g \in L_q(D) \mid \operatorname{div} g = 0 \text{ in } D, \nu \cdot g|_{\partial D} = 0\}$$

When $f = g + \nabla \pi$ with $g \in J_q(D)$ and $\pi \in \hat{W}_q^1(D)$, we set

$$(7) \quad P_D f = g, \quad Q_D f = \pi$$

and therefore $P_D : L_q(D)^3 \rightarrow J_q(D)$ and $Q_D : L_q(D)^3 \rightarrow \hat{W}_q^1(D)$ are both bounded linear operators.

Now, using the Helmholtz decomposition we shall formulate (5) in the semigroup setting. Let us define the operator $\mathcal{L}_D$ by the formula:

$$\mathcal{L}_D u = P_D L_{k,a} u = P_D (-\Delta u + k \partial_3 u - (\omega \times x) \cdot \nabla u + \omega \times u)$$

with domain:

$$\mathcal{D}_q(D) = \{u \in J_q(D) \cap W_q^2(D) \mid u|_{\partial D} = 0, (\omega \times x) \cdot \nabla u \in L_q(D)\}$$

And then, problem (5) is written as follows:

$$(8) \quad \begin{aligned} u_t + \mathcal{L}_\Omega u &= 0 \quad \text{in } J_q(\Omega) \quad \text{for } t > 0 \\ u|_{t=0} &= f \\ u(t) &\in \mathcal{D}_q(\Omega) \quad \text{for } t > 0 \end{aligned}$$

One of the significant characters of the operator $\mathcal{L}_\Omega$ is that the crucial drift operator $(\omega \times x) \cdot \nabla$ is never subordinate to the viscous term Δ and equation (8) provides both parabolic and hyperbolic features. In fact, the following theorem was proved by Farwig, Nečasová and Neustupa: [9], [10].

Theorem 1. *The essential spectrum of $\mathcal{L}_\Omega$ coincides with*

$$\bigcup_{j=-\infty}^{\infty} \{ \sqrt{-1} a_j + \{ \lambda \in \mathbb{C} \mid k^2 \operatorname{Re} \lambda + (\operatorname{Im} \lambda)^2 > 0 \} \}$$

From Theorem 1, we know that the operator $\mathcal{L}_\Omega$ does not generate an analytic semigroup. But, we can show the generation of the continuous semigroup and its decay estimate. Namely, the following two theorems are main results which the author would like to report in this paper.

Theorem 2. *Let $1 < q < \infty$. Then, $\mathcal{L}_\Omega$ generates a C^0 semigroup $\{T(t)\}_{t \geq 0}$ on $J_q(\Omega)$.*

Theorem 3. *There hold the following estimates for $f \in J_q(\Omega)$ and $t > 0$:*

$$(9) \quad \|T(t)f\|_{L_r(\Omega)} \leq C_{q,r} t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})} \|f\|_{L_q(\Omega)} \quad (1 < q \leq r \leq \infty, q \neq \infty)$$

$$(10) \quad \|\nabla T(t)f\|_{L_r(\Omega)} \leq C_{q,r} t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})} \|f\|_{L_q(\Omega)} \quad (1 < q \leq r \leq 3)$$

Here, the constant $C_{q,r}$ depends on a_0 and k_0 whenever $|a| \leq a_0$ and $|k| \leq k_0$ but is independent of a, k, t and f .

Remark 1. Theorem 2 was first proved by Hishida [19] when $q = 2$ and $k = 0$. Later on, Geissert-Heck-Hieber [16] proved Theorem 2 when $1 < q < \infty$ and $k = 0$. Our proof is different from [19] and [16] and based on some new consideration on the pressure terms.

Theorem 3 plays an essential role in proving the stability theorem. In fact, the estimates (7) and (10) were proved by Iwashita [42] when $a = k = 0$, by Kobayashi and Shibata [44] when $a = 0$ and $k \neq 0$ and by Hishida and Shibata [24] when $a \neq 0$ and $k = 0$. The restriction: $1 < q \leq r \leq 3$ in (10) is unavoidable at least in the case where $a = k = 0$, which was proved by Maremonti-Solonnikov [45].

2 Remark on the stability theorem

Consider the original non-linear problem (4) in the case where $f = f(x)$. And then, the corresponding stationary problem is given as

$$(11) \quad \begin{aligned} w \cdot \nabla w - \Delta w + k \partial_3 w & \\ -(\omega \times x) \cdot \nabla w + \omega \times w + \nabla \theta &= f & \text{in } \Omega \\ \operatorname{div} w &= 0 & \text{in } \Omega \\ w &= \omega \times x - k e_3 & \text{on } \partial\Omega \\ w(x) &\rightarrow 0 & \text{as } |x| \rightarrow \infty \end{aligned}$$

Setting $u(x, t) = w(x) + z(x, t)$ and $\pi(x, t) = \theta(x) + \kappa(x, t)$ in (4), we have the equation for z and κ as follows:

$$\begin{aligned}
 (12) \quad & z_t + z \cdot \nabla z + z \cdot \nabla w + w \cdot \nabla z - \Delta z + k \partial_3 z \\
 & -(\omega \times x) \cdot \nabla z + \omega \times z + \nabla \kappa = 0 \quad \text{in } \Omega \times (0, \infty) \\
 & \operatorname{div} z = 0 \quad \text{in } \Omega \times (0, \infty) \\
 & z = \omega \times x - k e_3 \quad \text{on } \partial\Omega \times (0, \infty) \\
 & z(t, x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty \quad t > 0 \\
 & z(0, x) = v_0(x) - w(x)
 \end{aligned}$$

Following Kato [43], instead of (12) we consider the integral equation:

$$(13) \quad z(t) = T(t)(v_0 - w) - \int_0^t T(t-s)[z(s) \cdot \nabla z(s) + z(s) \cdot \nabla w + w \cdot \nabla z(s)] ds$$

Then, using Theorem 2 and employing the same argument as in Shibata [47] we have the following theorem.

Theorem 4 (Stability Theorem). *Assume that problem (11) admits solutions $w(x)$ and $\theta(x)$. Let σ be a small positive number and $3 < q < \infty$. Then, there exists a small positive number ϵ depending on σ and q such that if $v_0 - w \in J_3(\Omega)$ and*

$$(14) \quad \|w\|_{L_{3-\sigma}(\Omega) \cap L_{3+\sigma}(\Omega)} + \|\nabla w\|_{L_{(3/2)-\sigma}(\Omega) \cap L_{(3/2)+\sigma}(\Omega)} + \|v_0 - w\|_{L_3(\Omega)} \leq \epsilon$$

then problem (13) admits a unique solution

$$z(t) \in C([0, \infty), J_3(\Omega)) \cap C((0, \infty), L_q(\Omega) \cap W_3^1(\Omega))$$

such that

$$\begin{aligned}
 & [z]_{3,0,t} + [z]_{q,\mu(q),t} + [\nabla z]_{3,1/2,t} \leq \sqrt{\epsilon} \\
 & \lim_{t \rightarrow 0+} \|z(t) - (v_0 - w)\|_{L_3(\Omega)} = 0 \\
 & \lim_{t \rightarrow 0+} ([z]_{q,\mu(q),t} + [\nabla z]_{3,1/2,t}) = 0
 \end{aligned}$$

Here, we have set

$$[z]_{p,\varphi,t} = \sup_{0 < s < t} s^\varphi \|z(\cdot, s)\|_{L_p(\Omega)}, \quad \mu(q) = \frac{3}{2} \left(\frac{1}{3} - \frac{1}{q} \right) = \frac{1}{2} - \frac{3}{2q}$$

Remark 2. Galdi and Silvestre [14] proved the existence of solutions to (11), but the velocity fields in [14] behave $O(|x|^{-1})$ when $|x| \rightarrow \infty$ and they did not show the asymptotic behaviour of the gradient of the velocity fields. We do not know any existence theorem of solutions to (11) which satisfy (14) as far as the author checked. But, the author believes the existence of stationary solutions satisfying (14), because the motion is stabilized by translation.

3 Analysis of the whole space problem

To show theorems 2 and 3, we start with the analysis of the whole space problem:

$$(15) \quad \begin{aligned} u_t + L_{k,a}u + \nabla\pi &= 0 \quad \text{in } \mathbb{R}^3, \quad t > 0, \\ \nabla \cdot u &= 0 \quad \text{in } \mathbb{R}^3, \quad t > 0, \\ u|_{t=0} &= f \quad \text{in } \mathbb{R}^3, \end{aligned}$$

with initial data $f \in J_q(\mathbb{R}^3)$. We know that the solution u is given by the following formula:

$$\begin{aligned} u(t) &= S_{\mathbb{R}^3}(t)f = \int_{\mathbb{R}^3} \frac{\exp\left(-\frac{|O(at)x-y-ke_3t|^2}{4t}\right)}{(4\pi t)^{3/2}} O(at)^T P_{\mathbb{R}^3} f(y) dy \\ &= \mathcal{F}^{-1} \left[e^{-(|\xi|^2 + ik\xi_3)t} O(at)^T \mathcal{F}[P_{\mathbb{R}^3} f](O(at)\xi) \right] (x) \end{aligned}$$

Here, $P_{\mathbb{R}^3}$ and $Q_{\mathbb{R}^3}$ are defined as follows:

$$P_{\mathbb{R}^3} f = \mathcal{F}^{-1} \left[\left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) \hat{f}(\xi) \right] (x), \quad Q_{\mathbb{R}^3} f = \mathcal{F}^{-1} \left[\frac{\xi \cdot \hat{f}(\xi)}{i|\xi|^2} \right] (x) + c(f)$$

and $c(f)$ is a constant such that $\int_{\Omega_R} Q_{\mathbb{R}^3} f dx = 0$. Set

$$\begin{aligned} \mathcal{A}_{\mathbb{R}^3,a,k}(\lambda)f &= \mathcal{A}(\lambda)f = \int_0^\infty e^{-\lambda t} S_{\mathbb{R}^3}(t) P_{\mathbb{R}^3} f dt \\ &= \mathcal{F}^{-1} \left[\int_0^\infty e^{-(\lambda + |\xi|^2 + i\xi_3 k)t} O(at)^T \widehat{P_{\mathbb{R}^3} f}(O(at)\xi) dt \right] (x) \end{aligned}$$

Then, we have the following two theorems:

Theorem 5. Let $1 < q < \infty$, $k_0 > 0$ and $a_0 > 0$. Assume that $|k| \leq k_0$ and $|a| \leq a_0$.

(1) Let $\gamma > 0$, $0 < \epsilon < \pi/2$ and $N \in \mathbb{N}$ with $N \geq 4$. Set

$$\begin{aligned} \mathbb{C}_\gamma &= \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq \gamma\}, \quad \mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > 0\}, \\ \Sigma_\epsilon &= \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}, \\ L_{q,R+2}(\mathbb{R}^3) &= \{f \in L_q(\mathbb{R}^3)^3 \mid f(x) = 0 \text{ for } x \notin B_{R+2}\}, \\ \mathcal{L}_R(\mathbb{R}^3) &= \mathcal{L}(L_{q,R+2}(\mathbb{R}^3), W_q^2(\mathbb{R}^3)^3) \end{aligned}$$

Then, $R(\lambda)P_{\mathbb{R}^3} \in \operatorname{Anal}(\mathbb{C}_+, \mathcal{L}_R(\mathbb{R}^3))$ and there exist three operators:

$$\mathcal{A}_{1,a}^N(\lambda), \tilde{\mathcal{A}}_{1,a}^N(\lambda) \in \operatorname{Anal}(\mathbb{C} \setminus (-\infty, 0], \mathcal{L}_R(\mathbb{R}^3)), \quad \mathcal{A}_{2,a}^N(\lambda) \in \operatorname{Anal}(\mathbb{C}_+, \mathcal{L}_R(\mathbb{R}^3))$$

such that

$$\begin{aligned} \mathcal{A}_{\mathbb{R}^3,a,k}(\lambda)f &= \mathcal{A}_{1,a}^N(\lambda) + \mathcal{A}_{2,a}^N(\lambda) \\ \mathcal{A}_{1,a}^N(\lambda) &= (\lambda - \Delta_{\mathbb{R}^3} + k\partial_3)^{-1} P_{\mathbb{R}^3} + \tilde{\mathcal{A}}_{1,a}^N(\lambda) \\ \|\partial_x^\beta \mathcal{A}_{1,a}^N(\lambda)f\|_{L_q(\mathbb{R}^3)} &\leq C|\lambda|^{-(1-(|\beta|/2))} \|f\|_{L_q(\mathbb{R}^3)} \\ \|\partial_x^\beta \tilde{\mathcal{A}}_{1,a}^N(\lambda)f\|_{L_q(\mathbb{R}^3)} &\leq C|\lambda|^{-((3/2)-(|\beta|/2))} \|f\|_{L_q(\mathbb{R}^3)} \end{aligned}$$

for any $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geq c_\epsilon > 0$ (c_ϵ being some constant depending on ϵ) and β with $|\beta| \leq 2$, and

$$\|\partial_x^\beta \mathcal{A}_{2,a}^N(\lambda)f\|_{L_q(\mathbb{R}^3)} \leq C \gamma^{-1} |\lambda|^{-(N/2)+(|\beta|/2)} \|f\|_{L_q(\mathbb{R}^3)}$$

for any $\lambda \in \mathbb{C}_\gamma$ with $|\lambda| \geq 1$ and β with $|\beta| \leq 2$ provided that $f \in L_{q,R+2}(\mathbb{R}^3)$. Here, the constant C depends on a_0 and k_0 but is independent of a and k , and we have set

$$(\lambda - \Delta_{\mathbb{R}^3} + k\partial_3)^{-1}g = \mathcal{F}_\xi^{-1}[(\lambda + |\xi|^2 + ik\xi_3)^{-1}\hat{g}(\xi)](x)$$

Theorem 6. Let $1 < q < \infty$, $k_0 > 0$, $a_0 > 0$, γ_0 and $K \geq 10a_0 + 2$. Assume that $|k| \leq k_0$, $|a| \leq a_0$ and $0 \leq \gamma \leq \gamma_0$. Set

$$\mathcal{L}_{R,\text{comp}}(\mathbb{R}^3) = \mathcal{L}(L_{q,R+2}(\mathbb{R}^3), W_q^2(B_{R+3})^3), \quad [\cdot]_{\mathbb{R}^3,R} = \|\cdot\|_{\mathcal{L}(L_{q,R+2}(\mathbb{R}^3), W_q^2(B_{R+3})^3)}$$

$$\overline{\mathbb{C}_+} = \{\lambda \in \mathbb{C} \mid \text{Re } \lambda \geq 0\}$$

Then, $\mathcal{A}(\lambda) = \mathcal{A}_{\mathbb{R}^3,a,k}(\lambda) \in C(\overline{\mathbb{C}_+}, \mathcal{L}_{R,\text{comp}}(\mathbb{R}^3))$ and satisfies the following conditions:

$$\sup_{|s| \leq K} [\mathcal{A}(\gamma + is)]_{\mathbb{R}^3,R} \leq C_{\gamma_0,a_0,K}$$

$$\int_{-K}^K [(\partial_\lambda \mathcal{A})(\gamma + is)]_{\mathbb{R}^3,R}^p ds \leq C_{\gamma_0,a_0,K} \quad (1 \leq p < 2)$$

$$\sup_{0 < |h| \leq 1} |h|^{-1/2} \left(\int_{-K}^K [\mathcal{A}(\gamma + i(s+h)) - \mathcal{A}(\gamma + is)]_{\mathbb{R}^3,R}^p ds \right)^{1/p} \leq C_{\gamma_0,a_0,K} \quad (1 \leq p < 4)$$

$$\sup_{0 < |h| \leq 1} |h|^{-1/2} \int_{-K}^K [(\partial_\lambda \mathcal{A})(\gamma + i(s+h)) - (\partial_\lambda \mathcal{A})(\gamma + is)]_{\mathbb{R}^3,R} ds \leq C_{\gamma_0,a_0,K}$$

$$\|(\partial_\lambda^m \mathcal{A}_{\mathbb{R}^3,a})(\gamma + is)\|_{\mathcal{L}(L_{q,R+2}(\mathbb{R}^3), W_q^j(B_{R+3})^3)} \leq C_{\gamma_0,a_0,K} |s|^{-m-(1-(j/2))}$$

for $m = 0, 1, 2, 3$, $j = 0, 1, 2$ and $s \in \mathbb{R}$ with $|s| \geq K - 2$, and

$$\lim_{\gamma \rightarrow 0^+} \sup_{s \in \mathbb{R}} [\mathcal{A}(\gamma + is) - \mathcal{A}(is)]_{\mathbb{R}^3,R} = 0$$

$$\lim_{\gamma \rightarrow 0^+} \int_{-\infty}^{\infty} [(\partial_\lambda \mathcal{A})(\gamma + is) - (\partial_\lambda \mathcal{A})(is)]_{\mathbb{R}^3,R} = 0$$

$$\lim_{R \rightarrow \infty} R^{-1} \int_{R \leq |x| \leq 2R} |[\mathcal{A}(\lambda)f](x)|^2 dx = 0 \quad (k \neq 0)$$

$$\lim_{R \rightarrow \infty} R^{-2} \int_{R \leq |x| \leq 2R} |[\mathcal{A}(\lambda)f](x)|^2 dx = 0 \quad (k = 0)$$

$$\|\mathcal{A}(\lambda_1)f - \mathcal{A}(\lambda_2)f\|_{W_q^2(B_{R+3})} \leq C_{q,R} |\lambda_1 - \lambda_2|^{1/4} \|f\|_{L_q(\mathbb{R}^3)}$$

for any λ, λ_1 and $\lambda_2 \in \overline{\mathbb{C}_+}$ provided that $f \in L_{q,R+2}(\mathbb{R}^3)$.

4 An idea of Theorem 2

Since the whole space problem is solvable, to show the existence of solutions to the equations:

$$(16) \quad \begin{aligned} u_t - \Delta u + M_{k,a}u + \nabla \pi &= 0 & \text{in } \Omega, \quad t > 0, \\ \nabla \cdot u &= 0 & \text{in } \Omega, \quad t > 0, \\ u &= 0 & \text{on } \Gamma, \\ u|_{t=0} &= f & \text{in } \Omega. \end{aligned}$$

with $M_{k,a}u = k\partial_3 u - (\omega \times x) \cdot \nabla u + \omega \times u$, the following theorem plays an essential role.

Theorem 7. *Let $1 < q < \infty$ and set*

$$L_{q,R-1}(\Omega) = \{f \in L_q(\Omega)^3 \mid f(x) = 0 \text{ } (|x| > R-1)\}$$

For every $f \in L_{q,R-1}(\Omega)$, problem (16) admits a unique solution (u, π) having the following regularity properties:

$$\begin{aligned} u &\in C^0([0, \infty), J_q(\Omega)) \cap C^1((0, \infty), L_q(\Omega)) \cap C^0((0, \infty), W_q^2(\Omega)) \\ \pi &\in C^0((0, \infty), \dot{W}_q^1(\Omega)) \end{aligned}$$

and satisfying the following estimates:

$$\begin{aligned} \|u(t)\|_{L_q(\Omega)} + t^{1/2} \|\nabla u(t)\|_{L_q(\Omega)} + t(\|u_t(t)\|_{L_q(\Omega)} + \|u(t)\|_{W_q^2(\Omega)} + \|\nabla \pi(t)\|_{L_q(\Omega)}) \\ \leq C_\gamma e^{\gamma t} \|f\|_{L_q(\Omega)} \\ t^{(1/2)(1+(1/q))} (\|u_t(t)\|_{L_q(\Omega_b)} + \|\pi(t)\|_{L_q(\Omega_b)}) \leq C_{\gamma,b} e^{\gamma t} \|f\|_{L_q(\Omega)} \end{aligned}$$

for any $t > 0$. Here, $\Omega_b = B_b \cap \Omega$ ($b > R$), $\gamma > 0$ is any real number, and C_γ and $C_{\gamma,b}$ are constants depending on a_0 and k_0 whenever $|a| \leq a_0$ and $|k| \leq k_0$ but are independent of a, k, t and f .

Moreover, if $f \in L_{q,R-1}(\Omega) \cap \mathcal{D}_q(\Omega)$, then we have

$$\begin{aligned} u &\in C^0([0, \infty), W_q^2(\Omega)) \cap C^1([0, \infty), L_q(\Omega)) \\ \|u(t)\|_{W_q^2(\Omega)} + \|u_t(t)\|_{L_q(\Omega)} &\leq C_\gamma e^{\gamma t} \|f\|_{W_q^2(\Omega)} \end{aligned}$$

Now, we shall show Theorem 2 by using Theorem 7.

1 st step

Given $f \in \mathcal{D}_q(\Omega)$, let $\tilde{f} \in \mathcal{D}_q(\mathbb{R}^3)$ be an extension of f to the whole space such that $\tilde{f} = f$ on Ω and $\|\tilde{f}\|_{\mathcal{D}_q(\mathbb{R}^3)} \leq C_q \|f\|_{\mathcal{D}_q(\Omega)}$ where we have set

$$\|f\|_{\mathcal{D}_q(D)} = \|f\|_{W_q^2(D)} + \|(\omega \times x) \cdot \nabla f\|_{L_q(D)}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| \leq R - 2$ and $\varphi(x) = 0$ for $|x| \geq R - 1$ and set

$$v(t) = (1 - \varphi)S_{\mathbb{R}^3}(t)\tilde{f} + \mathbb{B}[(\nabla\varphi) \cdot S_{\mathbb{R}^3}(t)\tilde{f}]$$

where $\mathbb{B}$ denotes the Bogovski operator satisfying the estimates:

$$\begin{aligned} \|\mathbb{B}[(\nabla\varphi) \cdot v]\|_{W_q^j(\mathbb{R}^3)} &\leq C\|v\|_{W_q^{j-1}(\text{supp}(\nabla\varphi))}, \quad j = 1, 2 \\ \|\mathbb{B}[(\nabla\varphi) \cdot \nabla v]\|_{W_q^j(\mathbb{R}^3)} &\leq C\|v\|_{W_q^j(\text{supp}(\nabla\varphi))}, \quad j = 0, 1, 2 \end{aligned}$$

(cf. [37], [38], [41], [16]).

2nd step

To obtain the solution $u(t)$ of (16), we set $u(t) = v(t) + w(t)$, and then $w(t)$ and $\pi(t)$ satisfy the equations:

$$\begin{aligned} w_t + L_{k,a}w + \nabla\pi &= F, \quad \text{div } w = 0 \quad \text{in } \Omega \times (0, \infty) \\ w|_{\partial\Omega} &= 0, \quad w|_{t=0} = \varphi f - \mathbb{B}[(\nabla\varphi) \cdot f] = g \end{aligned}$$

Here

$$\begin{aligned} F &= -2(\nabla\varphi) \cdot \nabla S_{\mathbb{R}^3}(t)\tilde{f} - (\Delta\varphi)S_{\mathbb{R}^3}(t)\tilde{f} + k(\partial_3\varphi)S_{\mathbb{R}^3}(t)\tilde{f} - ((\omega \times x) \cdot \nabla\varphi)S_{\mathbb{R}^3}(t)\tilde{f} \\ &\quad + (\partial_t + L_{k,a})\mathbb{B}[(\nabla\varphi) \cdot S_{\mathbb{R}^3}(t)\tilde{f}] \end{aligned}$$

Observe that

$$\begin{aligned} \partial_t \mathbb{B}[(\nabla\varphi) \cdot S_{\mathbb{R}^3}(t)\tilde{f}] &= \mathbb{B}[(\nabla\varphi) \cdot \partial_t S_{\mathbb{R}^3}(t)\tilde{f}] \\ &= \mathbb{B}[(\nabla\varphi) \cdot \Delta S_{\mathbb{R}^3}(t)\tilde{f}] + \mathbb{B}[(\nabla\varphi) \cdot (-k\partial_3 + ((\omega \times x) \cdot \nabla - \omega \times)S_{\mathbb{R}^3}(t)\tilde{f})] \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|F(t)\|_{W_q^2(\Omega)} &\leq C_\gamma t^{-1/2} e^{\gamma t} \|f\|_{\mathcal{D}_q(\Omega)} \\ \|F(t)\|_{L_q(\Omega)} &\leq C_\gamma t^{-1/2} e^{\gamma t} \|f\|_{L_q(\Omega)} \end{aligned}$$

If we write

$$w(t) = S_\Omega(t)g + \int_0^t S_\Omega(t-s)F(s)ds$$

then by Theorem 7 we have

$$\begin{aligned} w(t) &\in C^0([0, \infty), W_q^2(\Omega)) \cap C^1([0, \infty), L_q(\Omega)) \\ \|w(t)\|_{W_q^2(\Omega)} + \|w_t(t)\|_{L_q(\Omega)} &\leq C_\gamma e^{\gamma t} \|f\|_{\mathcal{D}_q(\Omega)} \\ \|w(t)\|_{L_q(\Omega)} &\leq C_\gamma e^{\gamma t} \|f\|_{L_q(\Omega)} \end{aligned}$$

Therefore, we can construct a solution

$$u(t) \in C^0([0, \infty), W_q^2(\Omega)) \cap C^1([0, \infty), L_q(\Omega))$$

which satisfies the estimate:

$$\begin{aligned} \|u(t)\|_{W_q^2(\Omega)} + \|u_t(t)\|_{L_q(\Omega)} &\leq C_\gamma e^{\gamma t} \|f\|_{\mathcal{D}_q(\Omega)} \\ \|u(t)\|_{L_q(\Omega)} &\leq C_\gamma e^{\gamma t} \|f\|_{L_q(\Omega)} \end{aligned}$$

If we define $\{T(t)\}_{t \geq 0}$ by the formula: $T(t)f = u(t)$, then the uniqueness of solutions and the denseness of $\mathcal{D}_q(\Omega)$ in $J_q(\Omega)$ imply that $\{T(t)\}_{t \geq 0}$ is a C^0 semigroup on $J_q(\Omega)$. Since we know that the resolvent set of $\mathcal{L}_q$ contains the complex plane with positive real part (cf. Theorem 8 below), we see that the generator of $\{T(t)\}_{t \geq 0}$ is $\mathcal{L}_q$, which completes the proof of Theorem 2.

5 An idea of a proof of Theorem 7

To prove Theorem 7, we consider the corresponding resolvent problem:

$$(17) \quad \lambda u - \Delta u + M_{k,a}u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

We construct the parametrix of the form:

$$\begin{aligned} \Phi(\lambda)f &= (1 - \varphi)\mathcal{A}_{\mathbb{R}^3, a, k}(\lambda)f_0 + \varphi R_{\Omega_R}(\lambda)P_{\Omega_R}f|_{\Omega_R} \\ &\quad + \mathbb{B}[(\nabla\varphi) \cdot (\mathcal{A}_{\mathbb{R}^3, a, k}(\lambda)f_0 - R_{\Omega_R}(\lambda)P_{\Omega_R}f|_{\Omega_R})] \\ \Psi(\lambda)f &= (1 - \varphi)\mathcal{Q}_{\mathbb{R}^3}f_0 + \varphi(\mathcal{Q}_{\Omega_R}f|_{\Omega_R} + p_{\Omega_R}(\lambda)P_{\Omega_R}f|_{\Omega_R}) \end{aligned}$$

for $\lambda \in \mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > 0\}$ and $f \in L_{q, R-1}(\Omega)$. Here, f_0 denotes the zero extension of f to the whole space, $f|_{\Omega_R}$ the restriction of f on Ω_R ,

$$\mathcal{A}_{\mathbb{R}^3, a, k}(\lambda)f_0 = \int_0^\infty e^{-\lambda t} S_{\mathbb{R}^3}(t)f_0 dt$$

and $u = R_{\Omega_R}(\lambda)g$ and $\theta = p_{\Omega_R}(\lambda)g$ are solutions to the equation:

$$\begin{aligned} \lambda u - \Delta u + M_{k,a}u + \nabla \theta &= P_{\Omega_R}g \quad \text{in } \Omega_R \\ \operatorname{div} u &= 0 \quad \text{in } \Omega_R \\ u|_{\partial\Omega_R} &= 0 \end{aligned}$$

We see that

$$\begin{aligned} (\lambda - \Delta + M_{k,a})\Phi(\lambda)f + \nabla \Psi(\lambda)f &= (I + T)f + S(\lambda)f \quad \text{in } \Omega \\ \operatorname{div} \Phi(\lambda)f &= 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

Here T is the operator defined by the formula:

$$(18) \quad Tf = -(\nabla\varphi)(\mathcal{Q}_{\mathbb{R}^3}f_0 - \mathcal{Q}_{\Omega_R}f|_{\Omega_R}) - \mathbb{B}[(\nabla\varphi) \cdot \nabla(\mathcal{Q}_{\mathbb{R}^3}f_0 - \mathcal{Q}_{\Omega_R}f|_{\Omega_R})]$$

and $S(\lambda)$ is a linear operator is defined as follows:

$$\begin{aligned} S(\lambda)f &= 2(\nabla\varphi) : \nabla(C_a(\lambda)f) + (\Delta\varphi)C_a(\lambda)f - k(\partial_3\varphi)C_a(\lambda)f \\ &\quad + [(\omega \times x) \cdot \nabla\varphi]C_a(\lambda)f + \mathbb{B}[(\nabla\varphi) \cdot \Delta C_a(\lambda)f] \\ &\quad - \mathbb{B}[(\nabla\varphi) \cdot M_{k,a}C_a(\lambda)f] + \mathbb{B}[(\nabla\varphi) \cdot \nabla\mathcal{B}_{\Omega_R,a}(\lambda)\gamma_{\Omega_R}f] \\ &\quad - \Delta\mathbb{B}[(\nabla\varphi) \cdot C_a(\lambda)f] + M_{k,a}\mathbb{B}[(\nabla\varphi) \cdot C_a(\lambda)f] + (\nabla\varphi)\mathcal{B}_{\Omega_R,a}(\lambda)\gamma_{\Omega_R}f \\ C_a(\lambda)f &= \mathcal{A}_{\mathbb{R}^3,a}(\lambda)f_0 - \mathcal{A}_{\Omega_R,a}(\lambda)\gamma_{\Omega_R}f \end{aligned}$$

We can show the following lemma which is one of the points in our argument.

Lemma 1. *Let $1 < q < \infty$. Then, there exists the inverse operator $(I + T)^{-1} \in \mathcal{L}(L_{q,R-1}(\Omega))$.*

Moreover, we know that

$$(19) \quad \|S(\lambda)f\|_{L_q(\Omega)} \leq C_q |\lambda|^{-\frac{1}{2}(1-\frac{1}{q})}, \quad |\lambda| \geq 1, f \in L_{q,R-1}(\Omega)$$

We can define the operators:

$$T_1(\lambda) \in \text{Anal}(\tilde{\Sigma}_\epsilon, \mathcal{L}(L_{q,R-1}(\Omega))), \quad T_2(\lambda) \in \text{Anal}(\mathbb{C}_+, \mathcal{L}(L_{q,R-1}(\Omega)))$$

such that

$$\begin{aligned} (I + T + S(\lambda))^{-1} &= (I + \sum_{j=1}^{\infty} ((I + T)^{-1} S(\lambda))^j) (I + T)^{-1} \\ &= I + T_1(\lambda) + T_2(\lambda) \end{aligned}$$

for $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq c$ and

$$\begin{aligned} \|T_1(\lambda)f\|_{L_q(\Omega)} &\leq C |\lambda|^{-(1/2)(1-(1/q))} \|f\|_{L_q(\Omega)} \quad (\lambda \in \tilde{\Sigma}_\epsilon, |\lambda| \geq 1) \\ \|T_2(\lambda)f\|_{L_q(\Omega)} &\leq C_\gamma |\lambda|^{-3} \|f\|_{L_q(\Omega)} \quad (\lambda \in \mathbb{C}_+ : \text{Re } \lambda \geq \gamma > 0, |\lambda| \geq 1) \end{aligned}$$

Here, c is some large positive number, $0 < \epsilon < \pi/2$,

$$\tilde{\Sigma}_\epsilon = \mathbb{C}_+ \cup \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon, |\lambda| \geq c(\epsilon)\}$$

and $c(\epsilon)$ is some large positive number depending on ϵ , a_0 and k_0 whenever $|a| \leq a_0$ and $|k| \leq k_0$.

We can also show that $(I + T + S(\lambda))^{-1}$ exists for λ with $\text{Re } \lambda > 0$. And therefore, we set $(R(\lambda), \Xi(\lambda)) = (\Phi(\lambda), \Psi(\lambda))(I + T + S(\lambda))^{-1}$. Then, $(R(\lambda), \Xi(\lambda))$ is the solution operator to the equation:

$$(\lambda - \Delta + M_{k,a})u = f, \quad \text{div } u = 0 \quad \text{in } \Omega \quad u|_{\partial\Omega} = 0$$

Moreover, using Theorem 5, we have

$$\begin{aligned}
 (20) \quad & R(\lambda) = R_1(\lambda) + R_2(\lambda), \quad \Xi(\lambda) = \Xi_0 + \Xi_1(\lambda) + \Xi_2(\lambda) \\
 & R_1(\lambda) \in \text{Anal}(\tilde{\Sigma}_\epsilon, \mathcal{L}(L_{q,R-1}(\Omega), W_q^2(\Omega))) \\
 & R_2(\lambda) \in \text{Anal}(\mathbb{C}_+, \mathcal{L}(L_{q,R-1}(\Omega), W_q^2(\Omega))) \\
 & \Xi_0 \in \mathcal{L}(L_{q,R-1}(\Omega), \hat{W}_q^1(\Omega)) \\
 & \Xi_1(\lambda) \in \text{Anal}(\tilde{\Sigma}_\epsilon, \mathcal{L}(L_{q,R-1}(\Omega), \hat{W}_q^1(\Omega))) \\
 & \Xi_2(\lambda) \in \text{Anal}(\mathbb{C}_+, \mathcal{L}(L_{q,R-1}(\Omega), \hat{W}_q^1(\Omega))) \\
 & \|\nabla^j R_1(\lambda)f\|_{L_q(\Omega)} \leq C|\lambda|^{-1+(j/2)}\|f\|_{L_q(\Omega)} \\
 & |\lambda|^{(1/2)(1-(1/q))}\|\Xi_1(\lambda)f\|_{L_q(\Omega_b)} + \|\nabla \Xi_1(\lambda)f\|_{L_q(\Omega)} \leq C_b\|f\|_{L_q(\Omega)}
 \end{aligned}$$

for $\lambda \in \tilde{\Sigma}_\epsilon$ with $|\lambda| \geq 1$

$$\begin{aligned}
 \|R_2(\lambda)f\|_{W_q^2(\Omega)} &\leq C_\gamma|\lambda|^{-3}\|f\|_{L_q(\Omega)} \\
 \|\Xi_2(\lambda)f\|_{L_q(\Omega_b)} + \|\nabla \Xi_2(\lambda)f\|_{L_q(\Omega)} &\leq C_b|\lambda|^{-3}\|f\|_{L_q(\Omega)}
 \end{aligned}$$

for $\lambda \in \mathbb{C}_+$ with $|\lambda| \geq 1$ and $\text{Re } \lambda \geq \gamma > 0$.

Using the above solution formula of (17), we have

Theorem 8. *Let $1 < q < \infty$. Then, the resolvent set of $\mathcal{L}_q$ contains $\mathbb{C}_+$.*

Moreover, if we define $u(t)$ and $\pi(t)$ by the formula:

$$\begin{aligned}
 (21) \quad u(t) &= \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} R_1(\lambda) f d\lambda + \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} R_2(\lambda) f d\lambda \\
 \theta(t) &= \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (\Xi_0 f + \Xi_1(\lambda) f) d\lambda + \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \Xi_2(\lambda) f d\lambda
 \end{aligned}$$

for $f \in L_{q,R-1}(\Omega)$ where

$$\Gamma = \bigcup_{\pm} \{\sigma + se^{\pm i\kappa} \mid s \geq 0\}$$

with large $\sigma > 0$ and $\pi/2 < \kappa < \pi$, then we can show Theorem 7 by (20).

6 An idea of Proof of Theorem 3

To show Theorem 3, the main step is to show the following theorem.

Theorem 9 (Local Energy Decay). *Let $1 < q < \infty$. Then, we have*

$$\|\partial_t^j T(t) P_{\Omega} f\|_{W_q^2(\Omega_R)} \leq C t^{-\frac{3}{2}} \|f\|_{L_q(\Omega)}, \quad t > 1$$

for any $f \in L_{q,R-1}(\Omega)$ and $j = 0, 1$.

We can show Theorem 9 by shifting the contour in the definition of $u(t)$ in (21) to the imaginary axis, applying Theorem 6 and using the following lemma:

Lemma 2. *Let $0 < \kappa < 1$. Let X and $\|\cdot\|_X$ be a Banach space and its norm, respectively. Let $f(s)$ be a function in $L_1(\mathbb{R}, X)$, which satisfies the condition:*

$$(22) \quad \sup_{0 < |h| \leq 1} |h|^{-\kappa} \int_{\mathbb{R}} \|f(s+h) - f(s)\|_X ds \leq C_\kappa M$$

for some $M > 0$. Set $g(t) = \int_{\mathbb{R}} e^{its} f(s) ds$. Then, we have

$$\|g(t)\|_X \leq |e^{-i} - 1|^{-1} M t^{-\kappa}$$

for any $t \geq 1$.

By Young's inequality, we see easily that

$$(23) \quad \|\nabla^j S_{\mathbb{R}^3}(t)f\|_{L_r(\mathbb{R}^3)} \leq C_{q,r} t^{-\frac{j}{2} - \frac{3}{2}(\frac{1}{q} - \frac{1}{r})} \|f\|_{L_q(\mathbb{R}^3)}$$

for $1 < q \leq r \leq \infty$ with $q \neq \infty$ and $t > 0$. Combining (23) with Theorem 9 by cut-off technique, we have Theorem 3.

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