

OBSERVATION ON VARIOUS CONJUGATES OF QUASICONVEX FUNCTIONS

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Abstract

We observe various conjugates and their biconjugacies of quasiconvex functions. Especially, we give a sufficient condition which assures biconjugacy is satisfied for 0-quasiconjugate.

1 Introduction

Throughout this paper, let f be a function from $\mathbb{R}^n$ to $(-\infty, \infty]$, and assume f is proper, that is, its domain $\text{dom} f = \{x \in \mathbb{R}^n \mid f(x) < \infty\}$ is not empty. Remember that f is said to be convex if for all $x_1, x_2 \in \text{dom} f$ and $\alpha \in (0, 1)$,

$$f((1 - \alpha)x_1 + \alpha x_2) \leq (1 - \alpha)f(x_1) + \alpha f(x_2),$$

and its Fenchel conjugate function f^* is defined as follows: for any $\xi \in \mathbb{R}^n$,

$$f^*(\xi) = \sup\{\langle \xi, x \rangle - f(x) \mid x \in \text{dom} f\}.$$

We know that $f^* : \mathbb{R}^n \rightarrow (-\infty, \infty]$ is proper convex lower semicontinuous, and also if f is lower semicontinuous, then we have

$$f = f^{**},$$

that is, lower semicontinuity and the biconjugacy are equivalent for any proper convex function. It is well-known that this property plays very important roles to consider dual problems of convex minimization problem.

Similar researches of conjugates of quasiconvex functions, have been observed, see [1, 2, 3, 4, 5]. Various types of conjugates are introduced, and biconjugacies of functions are investigated. In this paper, we give a sufficient condition which assures biconjugacy is satisfied for a notion of conjugate, called 0-quasiconjugate.

2 Conjugates of quasiconvex functions

Remember that f is said to be quasiconvex if, for all $x_1, x_2 \in \text{dom} f$ and $\alpha \in (0, 1)$,

$$f((1 - \alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\},$$

or equivalently, for any $\alpha \in \mathbb{R}$, its level set

$$L_\alpha(f) = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$$

is a convex set. Clearly, the notion of quasiconvexity is a generalization of convexity. For quasiconvex functions, various conjugates have been defined. In this paper, we treat λ -quasiconjugate, quasiconjugate, and R -quasiconjugate. At first, we mention about λ -quasiconjugate.

Definition 1. For any $\lambda \in \mathbb{R}$, the λ -quasiconjugate of f is the functional $f_\lambda^\nu : \mathbb{R}^n \rightarrow (\infty, \infty]$ defined as follows: for any $\xi \in \mathbb{R}^n$,

$$f_\lambda^\nu(x^*) = \lambda - \inf\{f(x) \mid \langle x^*, x \rangle \geq \lambda\}.$$

By Greenberg and Pierskalle, the normalized second quasiconjugate is introduced. Note that this notion is not given by two-times iteration of the same operation.

Definition 2. The normalized second quasiconjugate of f is the functional $f^{\nu\nu} : \mathbb{R}^n \rightarrow (-\infty, \infty]$ defined as follows: for any $x \in \mathbb{R}^n$,

$$f^{\nu\nu}(x) = \sup_{\lambda \in \mathbb{R}} (f_\lambda^\nu)_\lambda^\nu(x).$$

Evenly quasiconvexity, defined as follows, assures biconjugacy.

Definition 3. A subset A of $\mathbb{R}^n$ is evenly convex if there exists a family of open half space such that A is equal to the intersection of the family of open half space.

Definition 4. A function f is evenly quasiconvex if for all $\alpha \in (-\infty, \infty]$, $L_\alpha(f)$ is evenly convex.

Theorem 1. If a function f is evenly quasiconvex, then $f^{\nu\nu} = f$.

Theorem 2. The following formula holds:

$$f^{\nu\nu} = \max\{(f_{-1}^\nu)_{-1}^\nu, (f_0^\nu)_0^\nu, (f_1^\nu)_1^\nu\}.$$

Next, we define notions of quasiconjugate and R -quasiconjugate, which are closely concerned with $(f_{-1}^\nu)_{-1}^\nu$, $(f_1^\nu)_1^\nu$, and we state sufficient conditions to obtain that each biconjugates are equal to f .

Definition 5 ([4]). *Quasiconjugate of f is the functional $f^H : \mathbb{R}^n \rightarrow (-\infty, \infty]$ defined by*

$$f^H(\xi) = \begin{cases} -\inf\{f(x) \mid \langle x, \xi \rangle \geq 1\} & \text{if } \xi \neq 0 \\ -\sup\{f(x) \mid x \in \mathbb{R}^n\} & \text{if } \xi = 0. \end{cases}$$

The quasiconjugate of the function f^H is called the biquasiconjugate of f and denoted by f^{HH} .

Note that $f_1^\nu = 1 - f^H$ on $\mathbb{R}^n \setminus \{0\}$.

Definition 6. *We say that f achieves the maximum value at the infinite if $f(x_n) \rightarrow \sup\{f(x) \mid x \in \mathbb{R}^n\}$ for any sequence $\{x_n\}$ such that $\|x_k\| \rightarrow \infty$.*

Theorem 3. *Let f be a lower semicontinuous quasiconvex function satisfying*

$$f(0) = \inf\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}.$$

If f achieves the maximum value at the infinite, then $f^{HH} = f$.

Definition 7. *R -quasiconjugate of f is the functional $f^R : \mathbb{R}^n \rightarrow (-\infty, \infty]$ defined by*

$$f^R(\xi) = -\inf\{f(x) \mid \langle \xi, x \rangle \geq -1\}.$$

The R -quasiconjugate of the function f^R is called the R -biquasiconjugate of f and denoted by f^{RR} .

Note that $f_{-1}^\nu = -1 - f^R$ on $\mathbb{R}^n$.

Definition 8. *A subset A of $\mathbb{R}^n$ is R -evenly convex if the intersection of a family of open half spaces whose closure do not contain 0.*

Definition 9. *A function f is R -evenly quasiconvex if, $L_\alpha(f)$ is R -evenly convex for all $\alpha \in (-\infty, \infty]$.*

Theorem 4. *If a function f is R -evenly quasiconvex, then $f^{RR} = f$.*

3 Main theorem

Motivated by Theorems 1, 2, 3, and 4 in the previous section, we consider biconjugacy for 0-quasiconjugate.

Example 1. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = (x - 1)^2 + y^2$. Then we can calculate conjugate*

$$f_0^\nu(a, b) = \begin{cases} -\frac{a^2}{a^2 + b^2} & \text{if } a < 0 \\ 0 & \text{if } a \geq 0. \end{cases}$$

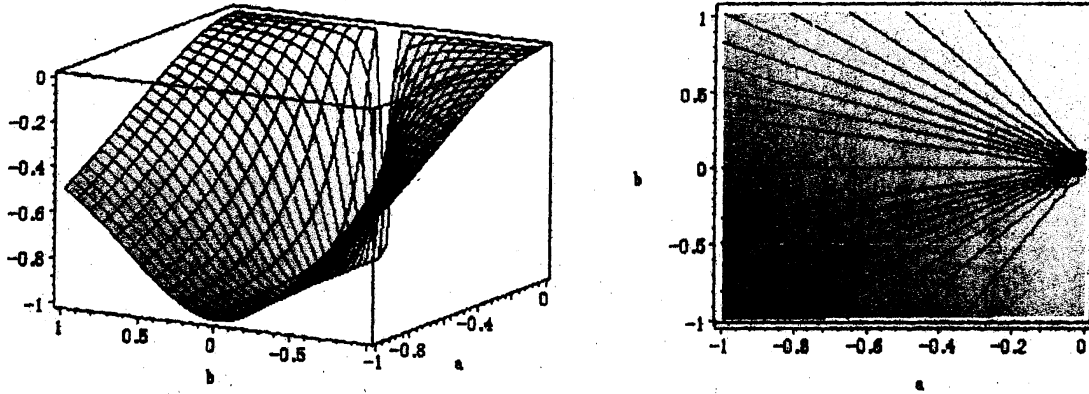


Figure: graph and contour graph of function f'_0 on $(-\infty, 0) \times \mathbb{R}$.

Let $g = f'_0$, then we have the conjugate g'_0 as follows:

$$g'_0(x, y) = \begin{cases} \frac{y^2}{x^2 + y^2} & \text{if } x > 0 \\ 1 & \text{if } x \leq 0. \end{cases}$$

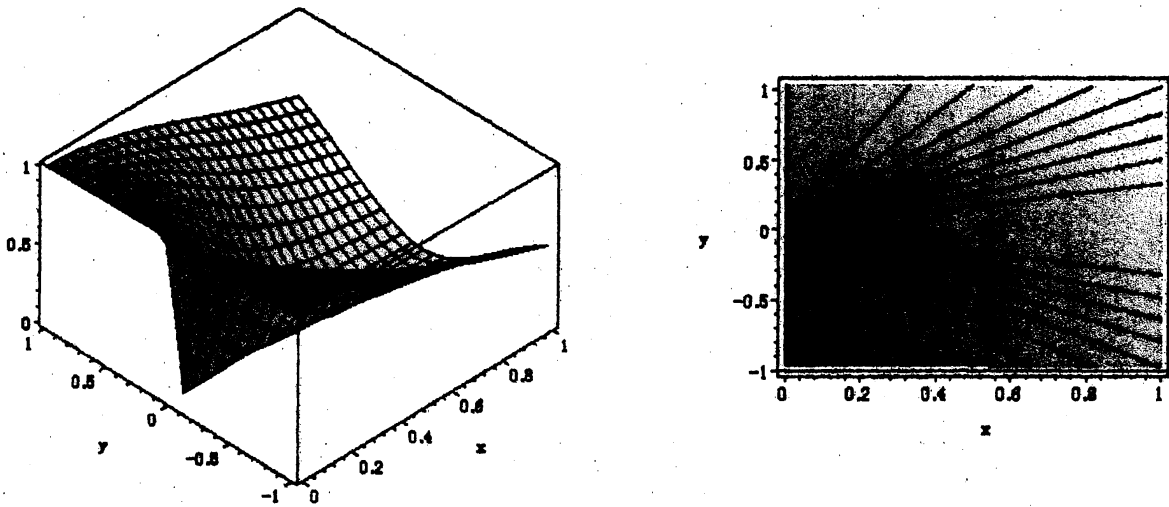


Figure: graph and contour graph of function g'_0 on $(0, \infty) \times \mathbb{R}$.

From this, we have $(f'_0)'_0 \neq f$. However, we can show $(g'_0)'_0 = g$.

Inspired the example, we give a sufficient condition for biconjugacy. To the purpose, we show the following properties concerned with *convex cone*.

Lemma 1. If K be a nonempty closed convex pointed cone in $\mathbb{R}^n$, then $\text{int}K^*$ is not empty, where $K^* = \{x^* \in \mathbb{R}^n \mid \langle x^*, x \rangle \leq 0, \forall x \in K\}$.

This proof is omitted. The following lemma is important to show our main result.

Lemma 2. Let K be a nonempty closed convex pointed cone in $\mathbb{R}^n$. If $x_0 \notin K \setminus \{0\}$, then there exists $a \in \mathbb{R}^n$ such that $\langle a, x_0 \rangle \geq 0 > \langle a, x \rangle$ for all $x \in K \setminus \{0\}$.

Proof. From the assumption, we have $\text{int}K^* \neq \emptyset$ from Lemma 1 and $K = K^{**}$. Since $x_0 \notin K^{**}$, there exists $x^* \in K^*$ such that $\langle x_0, x^* \rangle > 0$. By continuity of the inner product, we can choose $r > 0$ such that $y^* \in B(x^*, r)$ implies $\langle x_0, y^* \rangle > 0$. Choose $z^* \in \text{int}K^*$ such that $z^* \neq x^*$, and let

$$a = \frac{\|x^* - z^*\|}{\|x^* - z^*\| + r} x^* + \frac{r}{\|x^* - z^*\| + r} z^*,$$

then we can check $a \in B(x^*, r)$ and $a \in \text{int}K^*$. Hence we have $\langle a, x_0 \rangle > 0$ and $\langle a, x \rangle < 0$ for all $x \in K \setminus \{0\}$. $\square$

Lemma 3. The following formula holds

$$f_0^\nu(0) = \max\{f_0^\nu(x^*) \mid x^* \in \mathbb{R}^n\}.$$

Proof. For any $x^* \in \mathbb{R}^n$,

$$f_0^\nu(x^*) = -\inf\{f(x) \mid \langle x^*, x \rangle \geq 0\},$$

and also

$$f_0^\nu(0) = -\inf\{f(x) \mid \langle 0, x \rangle \geq 0\} = -\inf\{f(x) \mid x \in \mathbb{R}\},$$

then we have $f_0^\nu(x^*) \leq f_0^\nu(0)$. $\square$

Theorem 5. Assume that $L_\alpha(f) \cup \{0\}$ is a closed convex pointed cone, or $\mathbb{R}^n$, for all $\alpha \in \mathbb{R}$. If $f(0) = \sup\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$, then $f = (f_0^\nu)_0^\nu$.

Proof. It is clear that $f \geq (f_0^\nu)_0^\nu$. Assume that there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) > (f_0^\nu)_0^\nu(x_0)$. By using Lemma 3 and $f(0) = \sup\{f(x) \mid x \in \mathbb{R}^n \setminus \{0\}\}$, we may assume $x_0 \neq 0$. Choose $\alpha \in \mathbb{R}^n$ satisfying

$$f(x_0) > \alpha > (f_0^\nu)_0^\nu(x_0)$$

Since $x_0 \notin L_\alpha(f) \cup \{0\}$, and $L_\alpha(f) \cup \{0\}$ is a closed convex pointed cone, then

$$\exists a \in \mathbb{R}^n \text{ s.t. } \langle a, x_0 \rangle \geq 0 > \langle a, x \rangle, \forall x \in L_\alpha(f)$$

by using Lemma 2. This shows

$$x \in L_\alpha(f) \implies \langle a, x \rangle < 0,$$

or equivalently,

$$\langle a, x \rangle \geq 0 \implies f(x) > \alpha.$$

Hence

$$(f_0^\nu)_0^\nu(x_0) = -\inf\{f_0^\nu(x^*) \mid \langle x^*, x_0 \rangle \geq 0\} \geq -f_0^\nu(a) = \inf\{f(x) \mid \langle x, a \rangle \geq 0\} \geq \alpha,$$

this is contraction. $\square$

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