

THE TENSOR PRODUCT OF WEIGHTS

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1. Introduction

Let φ (resp. ψ) be a normal semi-finite weight on a von Neumann algebra M (resp. N). There exists the maximal weight $\varphi \otimes \psi$ on $M \otimes N$ such that $\varphi \otimes \psi (x \otimes y) = \varphi(x)\psi(y)$ for each x in $(m_\varphi)_+$ and y in $(m_\psi)_+$. Furthermore if φ and ψ are faithful in addition, $\varphi \otimes \psi$ is a faithful semi-finite weight on $M \otimes N$, and its one-parameter modular automorphism group is the tensor product of one-parameter modular automorphism groups Σ and Σ^ψ . Let φ_1 (resp. ψ_1) be a normal semi-finite, Σ -invariant weight on M (resp. Σ^ψ , N). By [5] Theorem 5.12 there is a unique positive self-adjoint operator h affiliated with the sub-algebra of fix-points for Σ (resp. k, Σ^ψ) such that $\varphi_1 = \varphi(h \cdot)$ (resp. $\psi_1 = \psi(k \cdot)$). We get $\varphi_1 \otimes \psi_1 = \varphi \otimes \psi (h \otimes k \cdot)$.

2. The Tensor Product of Unbounded Self-Adjoint Operators

Theorem 2.1. Let H_1 and H_2 be Hilbert spaces, K_1 and K_2 self-adjoint operators on H_1 and H_2 respectively. Then there exists a unique self-adjoint operator $K_1 \otimes K_2$ on the Hilbert space $H_1 \otimes H_2$ such that $D(K_1 \otimes K_2) \supset D(K_1) \otimes_a D(K_2)$ and $K_1 \otimes K_2(\xi_1 \otimes \xi_2) = K_1 \xi_1 \otimes K_2 \xi_2$ for all $\xi_1 \in D(K_1)$ and $\xi_2 \in D(K_2)$, where $D(K_1) \otimes_a D(K_2) = \{ \sum_{k=1}^n \xi_k^1 \otimes \xi_k^2 \in H_1 \otimes H_2 : \xi_k^1 \in D(K_1), \xi_k^2 \in D(K_2) \text{ for } k = 1, \dots, n \}$. Moreover if K_1 and K_2 are positive, $K_1 \otimes K_2$ is positive.

Proof. Let $K_1 = \int_{-\infty}^{\infty} \lambda dE(\lambda)$ and $K_2 = \int_{-\infty}^{\infty} \nu dE(\nu)$ be the spectral decompositions of K_1 and K_2 respectively.

Put $D = \bigcup_{n,m=1}^{\infty} R(e_n \otimes E_m)$ where $e_n = e(n) - e(-n)$ and $E_m = E(m) - E(-m)$.

Define an operator $K_1 \otimes K_2$ on D by ;

$$(K_1 \otimes K_2)\xi = (K_1 e_n \otimes K_2 E_m)\xi, \quad \text{where } \xi \text{ in } R(e_n \otimes E_m).$$

Then $K_1 \otimes K_2$ is a well-defined and densely defined symmetric operator. Furthermore, it is essentially self-adjoint.

[1] $K_1 \otimes K_2$ is well defined. Suppose that ξ in $R(e_n \otimes E_m)$ and ξ in $R(e_{n_1} \otimes E_{m_1})$.

We may assume $n \leq n_1$ and $m \leq m_1$ without the loss of generality.

Then we have

$$\begin{aligned} (K_1 e_{n_1} \otimes K_2 E_{m_1})\xi &= (K_1 e_{n_1} \otimes K_2 E_{m_1})(e_n \otimes E_m)\xi \\ &= (K_1 e_{n_1} e_n \otimes K_2 E_{m_1} E_m)\xi \\ &= (K_1 e_n \otimes K_2 E_m)\xi. \end{aligned}$$

[ii] $K_1 \otimes K_2$ is densely defined and symmetric

$D = \bigcup_{n,m=1}^{\infty} R(e_n \otimes E_m)$ is dense in $H_1 \otimes H_2$ since $s - \lim e_n = 1$ and $s - \lim E_m = 1$.

For all ξ in D and η in D we have

$$((K_1 \otimes K_2)\xi \mid \eta) = ((K_1 e_n \otimes K_2 E_m)\xi \mid \eta) \quad \text{for sufficient large } n, m.$$

Since $K_1 e_n \otimes K_2 E_m$ is bounded and self-adjoint, we have

$$\begin{aligned} ((K_1 \otimes K_2)\xi \mid \eta) &= (\xi \mid (K_1 e_n \otimes K_2 E_m)\eta) \\ &= (\xi \mid (K_1 \otimes K_2)\eta). \end{aligned}$$

$K_1 \otimes K_2$ is densely defined and symmetric

[iii] $K_1 \otimes K_2$ is essentially self-adjoint. Suppose that there exists a constant C such that $|((K_1 \otimes K_2)\xi \mid \eta)| \leq C \|\xi\|$, for all ξ in D

Take $\xi = (K_1 e_n \otimes K_2 E_m)\eta = (e_n \otimes E_m)(K_1 e_n \otimes K_2 E_m)\eta$ in D
then we have

$$\|(K_1 e_n \otimes K_2 E_m)\eta\| \leq C \quad \text{for all } n, m.$$

Since $\|(K_1 e_n \otimes K_2 E_m)\eta\|^2 = ((K_1^2 e_n \otimes K_2^2 E_m)\eta \mid \eta)$ is monotone increasing with respect to (n, m) , there exists $\lim_{(n,m)} \|(K_1 e_n \otimes K_2 E_m)\eta\|^2$.

We have, for $n \leq n_1, m \leq m_1$,

$$\begin{aligned} &\|(K_1 \otimes K_2)(e_{n_1} \otimes E_{m_1} - e_n \otimes E_m)\eta\|^2 \\ &= \|(K_1 e_{n_1} \otimes K_1 E_{m_1})(e_{n_1} \otimes E_{m_1} - e_n \otimes E_m)\eta\|^2 \\ &= \|(K_1 e_{n_1} \otimes K_2 E_{n_1})\eta\|^2 - \|(K_1 e_n \otimes K_2 E_m)\eta\|^2 \\ &\leq \epsilon \quad \text{for sufficient large } (n_1, m_1) \geq (n, m). \end{aligned}$$

Since $s\text{-}\lim_{(n,m)} (e_n \otimes E_m)_n = 1$, and $s\text{-}\lim_{(n,m)} (K_1 \otimes K_2)(e_n \otimes E_m)_n$ exists, we get 1 in $D((K_1 \otimes K_2)^{**})$, therefore $(K_1 \otimes K_2)^{**}$ is equal to $(K_1 \otimes K_2)^*$. We denote the closed extension of $K_1 \otimes K_2$ defined above again by $K_1 \otimes K_2$. It is noticed that $(K_1 \otimes K_2)e_n \otimes E_m = K_1 e_n \otimes K_2 E_m$ for all $n, m \in \mathbb{N}$.

For each ξ_1 in $D(K_1)$ and ξ_2 in $D(K_2)$,

$$\begin{aligned} & \| (K_1 \otimes K_2)(e_{n_1} \otimes E_{m_1} - e_n \otimes E_m)(\xi_1 \otimes \xi_2) \|^2 \\ &= \| K_1 e_{n_1} \xi_1 \otimes K_2 E_{m_1} \xi_2 - K_1 e_n \xi_1 \otimes K_2 E_m \xi_2 \|^2 \\ &\leq 2\{ \| (K_1 e_{n_1} \xi_1 - K_1 e_n \xi_1) \otimes K_2 E_{m_1} \xi_2 \|^2 + \| K_1 e_n \xi_1 \otimes (K_2 E_{m_1} \xi_2 - K_2 E_m \xi_2) \|^2 \} \\ &\leq 2\{ \| (K_1 e_{n_1} - K_1 e_n) \xi_1 \|^2 \cdot \| K_2 E_{m_1} \xi_2 \|^2 + \| K_1 \xi_1 \|^2 \cdot \| (K_2 E_{m_1} - K_2 E_m) \xi_2 \|^2 \}. \end{aligned}$$

We get $\xi_1 \otimes \xi_2$ in $D(K_1 \otimes K_2)$ by the closedness of $K_1 \otimes K_2$, which means that $D(K_1 \otimes K_2) \supset D(K_1) \otimes_a D(K_2)$ and $K_1 \otimes K_2(\xi_1 \otimes \xi_2) = K_1 \xi_1 \otimes K_2 \xi_2$ for all ξ_1 in $D(K_1)$ and ξ_2 in $D(K_2)$.

Let T be another self-adjoint operator on $H_1 \otimes H_2$ with the above properties. By $T(e_n \otimes E_m)(\xi_1 \otimes \xi_2) = K_1 e_n \xi_1 \otimes K_2 E_m \xi_2 = (K_1 \otimes K_2)(e_n \otimes E_m)(\xi_1 \otimes \xi_2)$ for all ξ_1 in $D(K_1)$ and ξ_2 in $D(K_2)$, and the closedness of T , we get $T(e_n \otimes E_m) = (K_1 \otimes K_2)e_n \otimes E_m$. Using the closedness of T again and $s\text{-}\lim_{(n,m)} e_n \otimes E_m = 1$, we have $T \supset K$, therefore $T = K$ by the self-adjointness of T and K , then $K_1 \otimes K_2$ is determined uniquely. If K_1 and K_2 are positive, $K_1 \otimes K_2$ is positive since $(K_1 \otimes K_2)e_n \otimes E_m = K_1 e_n \otimes K_2 E_m$ is a positive bounded operator.

Notice 2.2. Let K_1 and K_2 be bounded positive operators on H_1 and H_2 respectively, $K_1 \otimes K_2$ is a positive (bounded) operator on $H_1 \otimes H_2$.

Remark 2.3. In the Theorem 2.1 if K_1 and K_2 are affiliated with von Neumann algebras M and N respectively, then $K_1 \otimes K_2$ is affiliated with the von Neumann algebra $M \otimes N$.

Definition 2.4. If h and k are positive self-adjoint operators on Hilbert space H and $\varepsilon > 0$ we put $h_\varepsilon = h(1 + \varepsilon h)^{-1}$. We write $h \leq k$ if $h_\varepsilon \leq k_\varepsilon$ for some (and hence any) $\varepsilon > 0$. This is equivalent to the two conditions

$$D(h^{\frac{1}{2}}) \supset D(k^{\frac{1}{2}}) \quad \text{and} \quad \|h^{\frac{1}{2}} \xi\|^2 \leq \|k^{\frac{1}{2}} \xi\|^2$$

for each ξ in $D(k^{\frac{1}{2}})$. We say that a net $\{h_i\}$ of positive self-adjoint operators increases to the self-adjoint operator h , and write $h_i \nearrow h$ if $h_{i_\xi} \nearrow h_\varepsilon$. Thus $h_\varepsilon \nearrow h$ when $\varepsilon \searrow 0$.

Lemma 2.5. $K_{1_\delta} \otimes K_{2_\varepsilon} \nearrow K_1 \otimes K_2$ when K_1 and K_2 are positive self-adjoint operators on H_1 and H_2 respectively, $\delta \searrow 0$, $\varepsilon \searrow 0$.

Proof.

$$(K_{1_\delta} \otimes K_{2_\varepsilon})(e_n \otimes E_m) = K_{1_\delta} e_n \otimes K_{2_\varepsilon} E_m, \quad \text{for each } n, m \text{ in } \mathbb{N}$$

$$K_{1_\delta} e_n \leq K_{1_{\delta'}} e_n \leq K_1 e_n, \quad K_{2_\varepsilon} E_m \leq K_{2_{\varepsilon'}} E_m \leq K_2 E_m$$

$$\text{where } \delta \geq \delta' \text{ and } \varepsilon \geq \varepsilon',$$

By Notice 2.2, we get

$$(K_{1_\delta} \otimes K_{2_\varepsilon})(e_n \otimes E_m) \leq (K_{1_{\delta'}} \otimes K_{2_{\varepsilon'}})(e_n \otimes E_m) \leq (K_1 \otimes K_2)(e_n \otimes E_m)$$

moreover

$$K_{1_\delta} e_n \otimes K_{2_\varepsilon} E_m \nearrow (K_1 \otimes K_2) e_n \otimes E_m.$$

Then

$$(1 + (K_{1_\delta} \otimes K_{2_\varepsilon}))^{-1} e_n \otimes E_m \searrow (1 + (K_1 \otimes K_2))^{-1} e_n \otimes E_m.$$

Since the operator norms of $(1 + K_{1_\delta} \otimes K_{2_\varepsilon})^{-1}$ and $(1 + K_1 \otimes K_2)^{-1}$ are smaller than 1, $s - \lim_{(n,m)} e_n \otimes E_m = 1$, we get

$$(1 + K_{1_\delta} \otimes K_{2_\epsilon})^{-1} \nearrow (1 + K_1 \otimes K_2)^{-1}.$$

Then we get

$$(K_{1_\delta} \otimes K_{2_\epsilon})_1 = 1 - (1 + K_{1_\delta} \otimes K_{2_\epsilon})^{-1} \nearrow 1 - (1 + K_1 \otimes K_2)^{-1} = (K_1 \otimes K_2)_1.$$

Hence

$$K_{1_\delta} \otimes K_{2_\epsilon} \nearrow K_1 \otimes K_2.$$

3. The Tensor Product of Normal Semi-Finite Weights.

In this chapter, we often refer to [5] The Radon-Nikodym theorem for von Neumann algebra, and let φ be a faithful normal semi-finite weight on von Neumann algebra M , which gives rise to a one-parameter group Σ of automorphisms of M . The proof of Lemma 3.1 is almost similar to [5] Lemma 5.2.

Lemma 3.1. Let ψ be a normal semi-finite weight on M , if there exists a σ -weakly dense $*$ -subalgebra B in m_φ , invariant under Σ such that $\varphi = \psi$ on B , then we have $\psi \leq \varphi$, $\dot{\psi}|_{m_\varphi} = \dot{\varphi}$, and ψ is faithful.

Proof. If x and y are in B , then $\varphi(x \cdot y)$ and $\psi(x \cdot y)$ are normal functionals on M which agree on B , since B is an algebra. Therefore $\varphi(x \cdot y) = \psi(x \cdot y)$. Since B is a dense $*$ -algebra there is a net $\{u_\lambda\}$ in B_+ such that u_λ converges σ^* -strongly to 1 and $\|u_\lambda\| \leq 1$. Put

$$h_\lambda = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \exp -t^2 \sigma_t(u_\lambda) dt.$$

Since B is invariant under Σ we have

$$\varphi(\sigma_t(u_\lambda) \times \sigma_s(u_\lambda)) = \psi(\sigma_t(u_\lambda) \times \sigma_s(u_\lambda))$$

for all s and t and each x in M . It follows from [5] Lemma 3.1, by the polarization identity, that $\varphi(h_\lambda \times h_\lambda) = \psi(h_\lambda \times h_\lambda)$.

Each h_λ is an analytic element with

$$\begin{aligned}\sigma(h_\lambda) &= \frac{1}{\pi^2} \int \exp(-(t-\alpha)^2) \sigma_t(u_\lambda) dt \\ \| (1-\sigma_\alpha(h_\lambda)) \xi \| &= \| (1-\frac{1}{\pi^2} \int \exp(-(t-\alpha)^2) \sigma_t(u_\lambda) dt) \xi \| \\ &= \| \frac{1}{\pi^2} \int \exp(-(t-\alpha)^2) \sigma_t(1-u_\lambda) \xi dt \| \\ &\leq \frac{1}{\pi^2} \int |\exp(-(t-\alpha)^2)| \| \sigma_t(1-u_\lambda) \xi \| dt \\ &= \frac{1}{\pi^2} \exp(\operatorname{Im} \alpha)^2 \int \exp(-(t-\operatorname{Re} \alpha)^2) \| \sigma_t(1-u_\lambda) \xi \| dt\end{aligned}$$

$\lim_\lambda \| \sigma_t(1-u_\lambda) \xi \| = 0$ and $\| \sigma_t(1-u_\lambda) \xi \| \leq 2 \| \xi \|$, for all λ in $\mathbb{C}$,
and so by Lebesgue dominated convergence theorem we have

$$\lim_\lambda \| (1-\sigma_\alpha(h_\lambda)) \xi \| = 0 \text{ i.e. } \lim_\alpha \sigma_\alpha(h_\lambda) = 1 \text{ for all } \alpha \text{ in } \mathbb{C}.$$

Take now x in m_+ . Using the σ -weakly lower semi-continuity of ψ and

$$\frac{1}{\Delta^2} h_\lambda \Delta^{-\frac{1}{2}} = \sigma_{-1/2}(h_\lambda) \text{ on } D(\Delta^{-\frac{1}{2}}) \text{ by [5] Lemma 3.5 we get}$$

$$\begin{aligned}\psi(x) &\leq \liminf \psi(h_\lambda x h_\lambda) = \liminf \varphi(h_\lambda x h_\lambda) = \liminf \| \eta(x^{\frac{1}{2}} h_\lambda) \|^2 \\ &= \liminf \| \operatorname{Sh} S_\eta(x^{\frac{1}{2}}) \|^2 = \liminf \| J_{\Delta^{\frac{1}{2}}} h_\lambda \Delta^{-\frac{1}{2}} J_\eta(x^{\frac{1}{2}}) \|^2 \\ &= \liminf \| \sigma_{-1/2}(h_\lambda) J_\eta(x^{\frac{1}{2}}) \|^2 = \liminf \| \sigma_{-1/2}(h_\lambda) J_\eta(x^{\frac{1}{2}}) \|^2 \\ &= \| J_\eta(x^{\frac{1}{2}}) \|^2 = \| \eta(x^{\frac{1}{2}}) \|^2 = \varphi(x).\end{aligned}$$

Thus $\psi \leq \varphi$.

By [1] Lemma 2.3, there exists T in $\pi_p(M)'$ such that

$$0 \leq T \leq 1, \quad \psi(y*x) = (\eta(x) \mid T \eta(y))$$

for x, y in n_p . Then we have

$$\begin{aligned}\psi(h_\lambda x h_\lambda) &= (\eta(x^{\frac{1}{2}} h_\lambda) \mid T \eta(x^{\frac{1}{2}} h_\lambda)) \\ &= \| T^{\frac{1}{2}} \eta(x^{\frac{1}{2}} h_\lambda) \|^2.\end{aligned}$$

By the same argument above

$$\psi(h_\lambda x h_\lambda) = \left\| T^{\frac{1}{2}} J \sigma_{-1/2}(h) J \eta(x^2) \right\|^2.$$

$$\begin{aligned} \text{Then we have } \lim_{\lambda} \psi(h_\lambda x h_\lambda) &= \lim_{\lambda} \left\| T^{\frac{1}{2}} J \sigma_{-1/2}(h_\lambda) J \eta(x^2) \right\|^2 \\ &= \left\| T^{\frac{1}{2}} J \cdot J \eta(x^2) \right\|^2 \\ &= \left\| T \eta(x^2) \right\|^2 = \psi(x). \end{aligned}$$

Therefore $\psi(x) = \varphi(x)$ for all x in $(m_\varphi)_+$.

We refer to [5] Lemma 3.1 with respect to the faithfulness of ψ .

Proposition 3.2. ([5] proposition 5.9) If ψ is Σ -invariant normal semi-finite weight on M which is equal to φ on a σ -weakly dense Σ -invariant $*$ -subalgebra of m_φ then $\varphi = \psi$.

Proposition 3.3. Let φ and ψ be faithful normal semi-finite weights on von Neumann algebras M, N , σ_t and ρ_t one-parameter groups of automorphisms of φ and ψ , which are denoted by Σ and Σ^ψ respectively. There exists a unique $\Sigma \otimes \Sigma^\psi$ -invariant normal semi-finite weight θ on $M \otimes N$ such that

$$m_\theta \supset m_\varphi \otimes_a m_\psi, \quad \theta(x \otimes y) = \varphi(x) \cdot \psi(y)$$

for all x in $(m_\varphi)_+$, y in $(m_\psi)_+$. Moreover let g be a normal semi-finite weight on $M \otimes N$ such that $m_g \supset m_\varphi \otimes_a m_\psi$, $g(x \otimes y) = \varphi(x)\psi(y)$ for x in $(m_\varphi)_+$, y in $(m_\psi)_+$. Then we have $g \leq \theta$, $g|_{m_\theta} = \theta$ and g is faithful.

Proof. We may assume that $M = \mathcal{L}(\mathcal{U}_{\varphi_0})$, $N = \mathcal{L}(\mathcal{U}_{\psi_0})$ where $\mathcal{U}_{\varphi_0}$ and $\mathcal{U}_{\psi_0}$ are the maximal modular algebras associated with $\mathcal{U}_\varphi$ and $\mathcal{U}_\psi$ in [2] Theorem 2.13 respectively. By [4] Theorem 11.1. $\{\mathcal{U}_{\varphi_0} \otimes_a \mathcal{U}_{\psi_0}; \Delta_1(\alpha) \otimes_a \Delta_2(\alpha), \alpha \in \mathbb{C}\}$ is also a modular algebra, moreover we get

$$\mathcal{L}(\mathcal{U}_{\varphi_0} \otimes_a \mathcal{U}_{\psi_0}) = \mathcal{L}(\mathcal{U}_{\varphi_0}) \otimes \mathcal{L}(\mathcal{U}_{\psi_0}).$$

By [4] Lemma 2.1 there exists a unique positive self-adjoint non-singular operator Δ on $H_\varphi \otimes H_\psi$ such that Δ^α is the closure of $\Delta_1(\alpha) \otimes_a \Delta_2(\alpha)$ for all α in $\mathcal{C}$, therefore $\Delta^{it} = \Delta_1^{it} \otimes \Delta_2^{it}$ for all t in $\mathbb{R}$.

For each η_1 in $\mathcal{U}_{\varphi_0}$ and η_2 in $\mathcal{U}_{\psi_0}$ we get

$$\begin{aligned} \sigma_t \otimes \rho_t(\pi(\eta_1) \otimes \pi(\eta_2)) &= \sigma_t(\pi(\eta_1)) \otimes \rho_t(\pi(\eta_2)) \\ &= (\Delta_1^{it} \pi(\eta_1) \Delta_1^{-it}) \otimes (\Delta_2^{it} \pi(\eta_2) \Delta_2^{-it}) \\ &= (\Delta_1^{it} \otimes \Delta_2^{it})(\pi(\eta_1) \otimes \pi(\eta_2))(\Delta_1^{-it} \otimes \Delta_2^{-it}) \\ &= \Delta^{it} \pi(\eta_1 \otimes \eta_2) \Delta^{-it}. \end{aligned}$$

Then $\sigma_t \otimes \rho_t$ coincides with the modular automorphism group of $\mathcal{U}_{\varphi_0} \otimes_a \mathcal{U}_{\psi_0}$ on a σ -weakly dense sub-algebra $\pi(\mathcal{U}_{\varphi_0}) \otimes_a \pi(\mathcal{U}_{\psi_0})$. Therefore $\sigma_t \otimes \rho_t$ is equal to it.

Let θ be the canonical weight of $\mathcal{U}_{\varphi_0} \otimes_a \mathcal{U}_{\psi_0}$ defined in [2]

Theorem 2.11. By [2] Proposition 4.4 θ is a faithful normal semi-finite K.M.S. weight with respect to $\sigma_t \otimes \rho_t$, $\beta = 1$. Since $\mathcal{U}_{\varphi_0}$ (resp $\mathcal{U}_{\psi_0}$) is equivalent to $\mathcal{U}_\varphi$ (resp $\mathcal{U}_\psi$) we have $\xi \otimes \eta$ in $(\mathcal{U}_{\varphi_0} \otimes_a \mathcal{U}_{\psi_0})''$ for each ξ in $\mathcal{U}_\varphi$ and η in $\mathcal{U}_\psi$ and $\pi(\xi \otimes \eta) = \pi(\xi) \otimes \pi(\eta)$. We get

$$\begin{aligned} \theta((\pi(\xi) \otimes \pi(\eta))^*(\pi(\xi) \otimes \pi(\eta))) &= \theta(\pi(\xi \otimes \eta)^* \pi(\xi \otimes \eta)) \\ &= (\xi \otimes \eta \mid \xi \otimes \eta) \\ &= \|\xi\|^2 \cdot \|\eta\|^2 \\ &= \varphi(\pi(\xi)^* \pi(\xi)) \cdot \psi(\pi(\eta)^* \pi(\eta)). \end{aligned}$$

By [2] Lemma 2.4 for each x in $(m_\varphi)_+$ and y in $(m_\psi)_+$ there exist ξ in $\mathcal{U}_\varphi$ and η in $\mathcal{U}_\psi$ such that $\frac{1}{x^2} = \pi(\xi)$, $\frac{1}{y^2} = \pi(\eta)$, then we have $\theta(x \otimes y) = \varphi(x) \cdot \psi(y)$.

Let g be an another $\Sigma \otimes \Sigma^\psi$ -invariant normal semi-finite weight on $M \otimes N$ such that ;

$$m_g \supset m_\varphi \otimes_a m_\psi, \quad g(x \otimes y) = \varphi(x) \cdot \psi(y)$$

for all $x \in (m_\varphi)_+, y \in (m_\psi)_+$. By Proposition 3.2 we have $\theta = g$. The last part of Proposition 3.3 is clear by Lemma 3.1.

Theorem 3.4. Let φ_1 and ψ_1 be normal semi-finite weights on M and N , p and q the support projections of φ_1 and ψ_1 respectively. There exists a unique normal semi-finite weight θ_1 on $M \otimes N$ such that ;

$$(1) \quad m_{\theta_1} \supset m_{\varphi_1} \otimes_a m_{\psi_1}$$

$$(11) \quad \theta_1(x \otimes y) = \varphi_1(x) \cdot \psi_1(y)$$

for each $x \in (m_{\varphi_1})_+$ and $y \in (m_{\psi_1})_+$, and that θ_1 is $\Sigma^{\varphi_1} \otimes \Sigma^{\psi_1}$ -invariant on the von Neumann algebra $p \otimes q (M \otimes N) p \otimes q$. Furthermore θ_1 is the maximal normal semi-finite weight with the properties (1), (11) and its support projection is the tensor product $p \otimes q$.

Proof. It follows from Proposition 3.3.

Definition 3.5. The maximal weight above is called the tensor product of φ_1 and ψ_1 , which is denoted by $\varphi_1 \otimes \psi_1$.

Corollary 3.6. ([3] Proposition 6.2) Let M and N be two von Neumann algebras, ν and μ two normal strictly semi-finite weights on M^+ and N^+ , $(f_i)_{i \in I}$ [resp. $(g_j)_{j \in J}$] a family of positive normal linear functionals such that $\sum_{i \in I} f_i = \nu$ on M^+ and their supports are mutually orthogonal [resp. $\sum_{j \in J} g_j = \mu$, N^+].

(1) $\tau = \sum_{i,j} f_i \otimes g_j$ is a strictly semi-finite normal weight on $(M \otimes N)^+$.

This weight does not depend on the choice of $(f_i)_{i \in I}$, $(g_j)_{j \in J}$, and

its support is the tensor product of the supports of ν and μ . The algebra m_τ contains $m_\nu \otimes_a m_\mu$ and we have $\tau|_{m_\nu \otimes_a m_\mu} = \nu \otimes_a \mu$. Let θ be another normal semi-finite weight on $(M \otimes N)^+$ with the above properties. Then we get ;

$$m_\theta \supset m_\tau \quad \text{and} \quad \tau = \theta|_{m_\tau}$$

(ii) We suppose that ν [resp. μ] is K.M.S. with respect to a one-parameter automorphism group $\{\omega_t\}$ [resp. $\{\chi_t\}$], $\beta = 1$. Then there exists a unique normal weight τ on $(M \otimes N)^+$ such that $m_\tau \supset m_\nu \otimes_a m_\mu$, $\tau|_{m_\nu \otimes_a m_\mu} = \nu \otimes_a \mu$ and τ is K.M.S. with respect to $\{\omega_t \otimes \chi_t\}$ on $M \otimes N$, $\beta = 1$. This weight is equal to the weight defined above,

Proof. (i) By the choice of f_1 , $f_1(\cdot)$ is equal to $\nu(p_1 \cdot p_1)$ where p_1 is the support projection of f_1 , therefore f_1 is Σ^ν -invariant on pMp where p is the support projection of ν . Similarly g_1 is Σ^μ -invariant on qNq where q is that of μ . Since $\tau = \Sigma_{1,j}$, $f_1 \otimes g_1$ is $\Sigma^\nu \otimes \Sigma^\mu$ -invariant on $p \otimes q(M \otimes N)p \otimes q$, τ is the maximal weight in Theorem 3.4.

(ii) By the uniqueness of K.M.S. in [2] Proposition 4.8 we have ;

$$p\omega_t(\cdot)p = \sigma_t^\nu$$

$$q\chi_t(\cdot)q = \sigma_t^\mu$$

Therefore τ is $\Sigma^\nu \otimes \Sigma^\mu$ -invariant on $p \otimes q(M \otimes N)p \otimes q$. It follows from Theorem 3.4.

Corollary 3.7. ([2] Corollary 6.5) Let ν and μ be two normal semi-finite traces on von Neumann algebras M and N . $\tau = \nu \otimes \mu$ is a unique normal semi-finite trace of $M \otimes N$ such that $m_\nu \otimes_a m_\mu \subset m_\tau$ and $\tau|_{m_\nu \otimes_a m_\mu} = \nu \otimes_a \mu$.

Remark 3.8. (i) If ν and μ are strictly semi-finite normal weights on M and N , $\nu \otimes \mu$ is a normal strictly semi-finite weight on $M \otimes N$.

(ii) If ν and μ are normal semi-finite traces on M and N , $\nu \otimes \mu$ is a normal semi-finite trace on $M \otimes N$.

Corollary 3.9. (The extension of [2] Corollary 6.4) Let φ and ψ be two faithful normal semi-finite weights on M and N , $\mathcal{U}_\varphi$, $\mathcal{U}_\psi$ and $\mathcal{U}_{\varphi \otimes \psi}$ be the generalized Hilbert algebras defined by φ , ψ and $\varphi \otimes \psi$ respectively. Then $\mathcal{U}_{\varphi \otimes \psi}$ is isomorphic to the ~~achieved~~ generalized Hilbert algebra of $\mathcal{U}_\varphi \otimes \mathcal{U}_\psi$. Furthermore the modular operator Δ of $\mathcal{U}_{\varphi \otimes \psi}$ is the tensor product of modular operators of $\mathcal{U}_\varphi$ and $\mathcal{U}_\psi$.

Proof. It has already been proved in Proposition 3.3.

4. The Radon-Nikodym Theorem in the Tensor Product.

Theorem 4.1 Let φ and ψ be faithful normal semi-finite weights on M and N , $\varphi_1 = \varphi(h \cdot)$ and $\psi_1 = \psi(k \cdot)$ where positive self-adjoint operators h and k are affiliated with M^{Σ^φ} and N^{Σ^ψ} respectively.

Then we get

$$\varphi_1 \otimes \psi_1(\cdot) = \varphi \otimes \psi(h \otimes k \cdot)$$

where $h \otimes k$ has been defined in §2.

That is $\varphi(h \cdot) \otimes \psi(k \cdot) = \varphi \otimes \psi(h \otimes k \cdot)$.

Proof. For each $x \in (m_{\varphi_1})_+$ and $y \in (m_{\psi_1})_+$

$$\begin{aligned}\varphi_1(x) &= \lim_{\epsilon} \varphi(h_\epsilon \cdot x) \\ \psi_1(y) &= \lim_{\delta} \psi(k_\delta \cdot y).\end{aligned}$$

By [5] Proposition 4.2 we have

$$\begin{aligned}\frac{1}{h_\epsilon^2} x \frac{1}{h_\epsilon^2} &\text{ in } m_\varphi \\ \frac{1}{k_\delta^2} y \frac{1}{k_\delta^2} &\text{ in } m_\psi\end{aligned}$$

for all $\varepsilon > 0$ $\delta > 0$,

$$\begin{aligned}\varphi_1 \otimes \psi_1(x \otimes y) &= \varphi_1(x)\psi_1(y) \\ &= \lim_{(\delta, \varepsilon)} \varphi(h_\varepsilon x)\psi(k_\delta y) \\ &= \lim_{(\delta, \varepsilon)} \varphi \otimes \psi(h_\varepsilon \otimes k_\delta \cdot x \otimes y).\end{aligned}$$

By Lemma 1.2 and [5] Proposition 4.2 we get

$$\varphi_1 \otimes \psi_1(x \otimes y) = \varphi \otimes \psi(h \otimes k \cdot x \otimes y)$$

for $x \in (m_{\varphi_1})_+$ $y \in (m_{\psi_1})_+$.

[5] Theorem 4.6 says that

$$\begin{aligned}\sigma_t^\varphi &= h^{it} \sigma_t^\varphi (\cdot) h^{-it} \\ \sigma_t^\psi &= k^{it} \sigma_t^\psi (\cdot) k^{-it}\end{aligned}$$

By the definition of $\varphi_1 \otimes \psi_1$, $\sigma_t^{\varphi_1 \otimes \psi_1} = (h^{it} \otimes k^{it}) \sigma_t^\varphi \otimes \sigma_t^\psi (\cdot) (h^{-it} \otimes k^{-it})$.

Since $h^{it} \otimes k^{it} \in M^{\Sigma^\varphi} \otimes N^{\Sigma^\psi}$ and $h \otimes k$ commutes with $h^{it} \otimes k^{it}$ $\varphi \otimes \psi(h \otimes k \cdot)$ is $\sigma_t^{\varphi_1 \otimes \psi_1}$ -invariant on $[h] \otimes [k](M \otimes N)[h] \otimes [k]$ where $[h]$ and $[k]$ are the range projections of h and k respectively. By Theorem 3.4 we get $\varphi \otimes \psi(h \otimes k \cdot) = \varphi_1 \otimes \psi_1$.

Corollary 4.2. In Theorem 4.1 we suppose that φ_1 and ψ_1 are K.M.S. weights with respect to σ_t and ρ_t respectively.

Then $\varphi_1 \otimes \psi_1$ is K.M.S. weight with respect to $\sigma_t \otimes \rho_t$.

Proof. It follows from Theorem 4.1 and [5] Corollary 4.1.

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