Strong Full Exceptional Collections on Certain Toric Varieties via Mutations

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Exceptional Collections and Mutations

Def.
$\mathcal{D}$: triangulated category / $\mathbb{k}$.

1. $\mathcal{E} \in \mathcal{D}$: **exceptional object (EO)**
$$:\Leftrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}, \mathcal{E}[i]) = \begin{cases} \mathbb{k} \text{ if } i = 0, \\ 0 \text{ if } i \neq 0. \end{cases}$$
2. A sequence of **EO**s $\mathcal{E}_1, \dots, \mathcal{E}_r$: **exceptional collection (EC)**
$:\Leftrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_l, \mathcal{E}_k[i]) = 0$ for all $1 \leq k < l \leq r$ and all $i \in \mathbb{Z}$.
3. **EC** $\mathcal{E}_1, \dots, \mathcal{E}_r$: **full (F)** $:\Leftrightarrow \mathcal{D} = \langle \mathcal{E}_1, \dots, \mathcal{E}_r \rangle$.
4. **EC** $\mathcal{E}_1, \dots, \mathcal{E}_r$: **strong (S)** $:\Leftrightarrow \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}_k, \mathcal{E}_l[i]) = 0$ for all $1 \leq k < l \leq r$ and all $i \neq 0$.

Def.
$\mathcal{E} \in \mathcal{D}$: **EO**.
For $\mathcal{F}$ in ${}^{\perp}\mathcal{E}$, we define the **left mutation of $\mathcal{F}$ through $\mathcal{E}$** as $\mathbb{L}_{\mathcal{E}}(\mathcal{F})$ in $\mathcal{E}^{\perp}$ that lies in an exact triangle
$$\mathrm{RHom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathbb{L}_{\mathcal{E}}(\mathcal{F}).$$
Similarly, for $\mathcal{G}$ in $\mathcal{E}^{\perp}$, we define the **right mutation of $\mathcal{G}$ through $\mathcal{E}$** as $\mathbb{R}_{\mathcal{E}}(\mathcal{G})$ in ${}^{\perp}\mathcal{E}$ which lies in an exact triangle
$$\mathbb{R}_{\mathcal{E}}(\mathcal{G}) \longrightarrow \mathcal{G} \longrightarrow \mathrm{RHom}(\mathcal{G}, \mathcal{E})^* \otimes \mathcal{E}.$$

Lem.
$\mathcal{E}_1, \mathcal{E}_2$: **EC** of length 2.
Then, $\mathbb{L}_{\mathcal{E}_1}(\mathcal{E}_2)$ and $\mathbb{R}_{\mathcal{E}_2}(\mathcal{E}_1)$ are again **EO**s.

Lem.
$\mathcal{E}_1, \dots, \mathcal{E}_r$: **FEC**. Then, The collection
$$\mathcal{E}_1, \dots, \mathcal{E}_{i-1}, \mathbb{L}_{\mathcal{E}_i}(\mathcal{E}_{i+1}), \mathcal{E}_i, \mathcal{E}_{i+2}, \dots, \mathcal{E}_r$$
is again **FEC** for each $1 \leq i \leq r-1$. Similarly, the collection
$$\mathcal{E}_1, \dots, \mathcal{E}_{i-2}, \mathcal{E}_i, \mathbb{R}_{\mathcal{E}_i}(\mathcal{E}_{i-1}), \mathcal{E}_{i+1}, \dots, \mathcal{E}_r$$
is again **FEC** for each $2 \leq i \leq r$.

Remark
In general, it is very difficult to calculate $\mathbb{L}_{\mathcal{E}_i}(\mathcal{E}_{i+1})$ and $\mathbb{R}_{\mathcal{E}_i}(\mathcal{E}_{i-1})$ explicitly.

Main Theorem and Sketch of Proof

The following question is due to L. Costa, R.M. Miró-Roig and A. King :

Question.
For any smooth toric (Fano) variety X, does its derived category $\mathrm{D^b}(X)$ have a Strong Full Exceptional Collection (SFEC) of line bundles?

Main Theorem ([7])
X : sm proj toric variey with $\rho(X) = 2$,
$Y \subset X$: torus invariant closed subvariety with $\mathrm{codim}_X Y \leq 3$,
$\widetilde{X} := \mathrm{Bl}_Y X \quad (\rho(\widetilde{X}) = 3)$,
$\Longrightarrow \mathrm{D^b}(\widetilde{X})$ has a
Strong Full Exceptional Collection (SFEC) of line bundles.

For the proof, we need following two formula due to D. Orlov.

Thm. (D. Orlov [10])
X : sm proj var, $\mathcal{E}$: vect bdl of rank r+1 on X, $p : \widetilde{X} = \mathbb{P}(\mathcal{E}) \to X$.
$\Longrightarrow$ the functor $p^* : \mathrm{D^b}(X) \to \mathrm{D^b}(\widetilde{X})$ is fully faithful, and $\mathrm{D^b}(\widetilde{X})$ has a SOD
$$\mathrm{D^b}(\widetilde{X}) = \langle p^*\mathrm{D^b}(X), \dots, p^*\mathrm{D^b}(X) \otimes \mathcal{O}_p(r) \rangle$$
where $\mathcal{O}_p(1)$ is the tautological line bundle of $\mathbb{P}_X(\mathcal{E})$.

$\rho(X) = 2 \Rightarrow X$: proj sp bdl / $\mathbb{P}^s \Rightarrow \mathrm{D^b}(X)$ has a **FEC**

Thm. (D. Orlov [10])
X : sm proj var,
$Y \subset X$: sm closed subvar of $\mathrm{codim}_Y X = c$,
$\widetilde{X} := \mathrm{Bl}_Y X$,
E : exceptional divisor,
$\Longrightarrow \exists\, \mathrm{D^b}(X) \to \mathrm{D^b}(\widetilde{X})$ and $\exists\, \mathrm{D^b}(Y) \to \mathrm{D^b}(\widetilde{X})$ fully faithful functors, and $\mathrm{D^b}(\widetilde{X})$ has a SOD
$$\mathrm{D^b}(\widetilde{X}) = \langle \mathrm{D^b}(Y) \otimes \mathcal{O}((c-1)E), \dots, \mathrm{D^b}(Y) \otimes \mathcal{O}(E), \mathrm{D^b}(X) \rangle.$$

$\Rightarrow \mathrm{D^b}(\widetilde{X})$ has a **FEC** $\xrightarrow{\text{Mutation}}$ **SFEC** of line bdls

We find a new procedure of mutation operations which we can calcurate, and finally we construct a **SFEC** of line bdls.

Known Result for Existance of FEC

Thm. (Y. Kawamata [8])
$\mathcal{X}$: sm proj toric DM stack $\Longrightarrow \mathrm{D^b}(\mathcal{X})$ has a **FEC**.

E.g. (A. Beilinson [1])
$\mathrm{D^b}(\mathbb{P}^n)$ has a **SFEC** of line bdls, called Beilinson collection
$$\mathrm{D^b}(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \dots, \mathcal{O}(n) \rangle.$$

Thm. (L. Costa, R.M. Miró-Roig [2])
X : sm proj toric var with $\rho(X) = 1$ or 2
$\Longrightarrow \mathrm{D^b}(X)$ has a **SFEC** of line bdls.

Thm. (L. Costa, R.M. Miró-Roig [3])
X : toric Fano variety of $\dim X = n$ and $\rho(X) = 2n-1$ or $2n$
$\Longrightarrow \mathrm{D^b}(X)$ has a **SFEC** of line bdls.

But in Picard number three, A. Efimov proved the following.

Thm. (A. Efimov [6])
There are **infinitely** many toric Fano varieties with $\rho = 3$ **without FEC** of line bdls.

So the next question is as follow:

Question.
When a sm proj toric var X with $\rho(X) = 3$ has a **SFEC** of line bdls?

There are some previous works for this question ([4], [5], [9]). They use the method of Bondal's Frobenius splitting. Our method of the proof is **very different from this**, and our result **fully includes** these previous results.

Reference

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