

Mass formula for Jacobi weight enumerators of type II binary codes and some relationships of it with Jacobi forms

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1 Definitions from binary linear codes

1.1 Binary codes

Let $\mathbb{F}_2 = GF(2)$ be the field of 2 elements. Let $V = \mathbb{F}_2^n$ be the vector space of dimension n over $\mathbb{F}_2$. A linear $[n, k]$ code C is a vector subspace of V of dimension k . An element $\mathbf{x}$ in C is called a codeword of C . The inner product on V , which is denoted by $\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y}$ in V , is defined as usual. Two codes C_1 and C_2 are said to be equivalent if and only if after a suitable change of coordinate positions of C_1 all the codewords in both codes coincide.

Let C be a binary code of length n . An automorphism σ of the code C is an element of the permutation group of n letters S_n which leaves C invariant. All automorphisms of the code C form a group and it is denoted by $Aut(C)$.

The dual code $C^\perp$ of C is defined by

$$C^\perp = \{\mathbf{u} \in V \mid \mathbf{u} \cdot \mathbf{v} = 0 \quad \forall \mathbf{v} \in C\}.$$

The code C is called self-orthogonal if it satisfies $C \subseteq C^\perp$, and the code C is called self-dual if it satisfies $C = C^\perp$. Self-dual codes exist only if $n \equiv 0 \pmod{2}$. For even n we let S_n denote the set of all self-dual binary codes of length n . Let

$$\mathbf{x} = (x_1, x_2, \dots, x_n)$$

be a vector in V , then the Hamming weight $wt(\mathbf{x})$ of the vector $\mathbf{x}$ is defined to be the number of i 's such that $x_i \neq 0$. The Hamming distance d on V is also defined by $d(\mathbf{x}, \mathbf{y}) = wt(\mathbf{x} - \mathbf{y})$. Let C be a code, then d of the code C is defined by

$$\begin{aligned} d &= \min_{\mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}} d(\mathbf{x}, \mathbf{y}) \\ &= \min_{\mathbf{x} \in C, \mathbf{x} \neq 0} wt(\mathbf{x}). \end{aligned}$$

Let C be a self-dual binary code, then the weight $wt(\mathbf{x})$ of each codeword $\mathbf{x}$ in C is even. Further, if the weight of each codeword $\mathbf{x}$ in C is divisible by 4, then the code is called doubly even. It is known that a doubly even self-dual binary codes C exist only when the length n of C is a multiple of 8. In short a doubly even self-dual binary code is type II binary code.

Let C be a self-dual doubly even code of length n , which are embedded in $\mathbb{F}_2^n$. Let $\mathbf{u} = (u_1, u_2, \dots, u_n), \mathbf{v} = (v_1, v_2, \dots, v_n)$ be any pair of vectors in $\mathbb{F}_2^n$, then the number of common 1's of the corresponding coordinates for $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} * \mathbf{v}$. This is called the intersection number of $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u} * \mathbf{u}$ is nothing else $wt(\mathbf{u})$.

Let C be a type II binary $[n, \frac{n}{2}]$ code. The homogeneous weight enumerator $W_C(x, y)$ of the code C is defined by

$$W_C(x, y) = \sum_{\mathbf{v} \in C} x^{n-wt(\mathbf{v})} y^{wt(\mathbf{v})}$$

Following identity is known as the MacWilliams identity:

$$\begin{aligned} W_C(x, y) &= \frac{1}{2^{\frac{n}{2}}} W_C(x+y, x-y) \\ &= W_C\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right), \end{aligned} \quad (1)$$

Since C is doubly even, each codeword u of C has weight divisible by 4, and we know that

$$W_C(x, iy) = W_C(x, y). \quad (2)$$

Let G_1 be the group generated by

$$\sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

The above two equations (1) and (2) show that the homogeneous weight enumerator of a type II binary code is invariant under linear action of the elements of the group G_1 . Let $\mathbb{C}[x, y]$ be the polynomial ring over the field of complex numbers $\mathbb{C}$. We let $\mathbb{C}[x, y]^{G_1}$ to denote the subring of $\mathbb{C}[x, y]$ consisting of all elements in $\mathbb{C}[x, y]$ invariant under linear action of G_1 . The following theorem is due to A. Gleason [9]

Theorem 1.1 *It holds that*

$$\mathbb{C}[x, y]^{G_1} = \mathbb{C}[W_{e_8}(x, y), W_{gol_{24}}(x, y)],$$

where $W_{e_8}(x, y)$ is the weight enumerator of the extended Hamming code of length 8, and $W_{gol_{24}}(x, y)$ is the weight enumerator of the binary Golay code of length 24.

Let H_1 be a subgroup of G_1 generated by $\sigma_1\sigma_2\sigma_1$ and σ_1 . This subgroup is of index 2 in G_1 . Let $\mathbb{C}[x, y]^{H_1}$ be the ring of invariants for H_1 . Then it is known that (see for instance [19])

Theorem 1.2 *It holds that*

$$\mathbb{C}[x, y]^{H_1} = \mathbb{C}[W_{e_8}(x, y), E_{12}(x, y)],$$

where $E_{12}(x, y) = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$.

1.2 Jacobi weight enumerator

Definition: Jacobi polynomials for binary codes

Jacobi polynomial $Jac(C, \mathbf{v} \mid X, Z)$ for C with respect to $\mathbf{v} \in \mathbb{F}_2^n$ is defined by

$$Jac(C, \mathbf{v} \mid X, Z) = \sum_{\mathbf{u} \in C} X^{\mathbf{u} \cdot \mathbf{u}} Z^{\mathbf{u} \cdot \mathbf{v}}.$$

The homogeneous form of $Jac(C, \mathbf{v} \mid X, Z)$ is given by

$$Jac(C, \mathbf{v}; x, y, u, v) = \sum_{\mathbf{t} \in C} x^{n - wt(\mathbf{v}) - wt(\mathbf{t}) + \mathbf{t} \cdot \mathbf{v}} y^{wt(\mathbf{t}) - \mathbf{t} \cdot \mathbf{v}} u^{wt(\mathbf{v}) - \mathbf{t} \cdot \mathbf{v}} v^{\mathbf{t} \cdot \mathbf{v}}.$$

Theorem 1.3 *Let the notations be as above, then we have*

$$Jac(C, \mathbf{v}; x', y', u', v') = Jac(C, \mathbf{v}; x, y, u, v), \quad (3)$$

where

$$\begin{pmatrix} x' \\ y' \\ u' \\ v' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}$$

It may be remarked here that it holds

$$Jac(\mathbb{C}, \mathbf{v}; x, iy, u, iv) = Jac(\mathbb{C}, \mathbf{v}; x, y, u, v) \quad (4)$$

Let $G_1 \oplus G_1$ be the group generated by $diag(\sigma_1, \sigma_1)$ and $diag(\sigma_2, \sigma_2)$, and $\mathbb{C}[x, y, u, v]$ be the polynomial ring in 4 independent variables over $\mathbb{C}$. We let $\mathbb{C}[x, y, u, v]^{G_1 \oplus G_1}$ to denote the subring of $\mathbb{C}[x, y, u, v]$ invariant under the linear action of each element of $G_1 \oplus G_1$. The above equations (3) and (4) implies that $Jac(\mathbb{C}, \mathbf{v}; x, y, u, v)$ belongs to $\mathbb{C}[x, y, u, v]^{G_1 \oplus G_1}$. We have a Gleason type result for $\mathbb{C}[x, y, u, v]^{G_1 \oplus G_1}$ ([4]).

Let $H_1 \oplus H_1$ be the group generated by $diag(\sigma_1, \sigma_1)$ and $diag(\sigma_1 \sigma_2 \sigma_1, \sigma_1 \sigma_2 \sigma_1)$, and $R = \mathbb{C}[x, y, u, v]^{H_1 \oplus H_1}$ be the ring of invariants for the group $H_1 \oplus H_1$. We also have a Gleason type result for R . Here we briefly describe the result. When a polynomial $f(x, y, u, v)$ of total degree n belongs to R we call the partial degree of f with respect to the variables u and v the index of f . The Molien series for $H_1 \oplus H_1$ is given by

$$\begin{aligned} \Phi_{H_1 \oplus H_1}(t) &= \sum_{n \geq 0} \dim_{\mathbb{C}}(FJac_n) t^n \\ &= \frac{1 + 8t^8 + 18t^{12} + 21t^{16} + 19t^{20} + 21t^{24} + 7t^{28} + t^{32}}{(1 - t^8)^2(1 - t^{12})^2} \\ &= 1 + 10t^8 + 20t^{12} + 40t^{16} + 75t^{20} + 130t^{24} + 179t^{28} + 283t^{32} + \\ &\quad 383t^{36} + 513t^{40} + 678t^{44} + 883t^{48} + 1078t^{52} + 1372t^{56} + \\ &\quad + 1658t^{60} + 1994t^{64} + 2385t^{68} + 2836t^{72} + \dots \end{aligned}$$

We decompose this ring R into a direct sum :

$$R = \bigoplus_{n \geq 0} R_n,$$

where R_n is the n -th homogeneous part of R . Further we decompose R_n as

$$R_n = \bigoplus_{0 \leq m \leq n} R_{n,m},$$

where $R_{n,m}$ is the set of polynomials $f(x, y, u, v) \in R_n$ with partial degree with respect to u and v equal to m . This set $R_{n,m}$ forms a vector subspace of R .

2 Jacobi forms

2.1 Definition of Jacobi forms

Let $\mathbb{H}$ be the complex upper half plane and τ be a variable on $\mathbb{H}$. Let $\mathbb{C}$ be the complex plane and z be a variable on $\mathbb{C}$. A complex valued holomorphic function $\phi(\tau, z)$ defined on $\mathbb{H} \times \mathbb{C}$ is called a Jacobi form of weight k and index h with respect to the pair $(SL_2(\mathbb{Z}), \mathbb{Z})$ if it satisfies the conditions (5), (6) and (7) below:

$$\phi(\tau, z) = (c\tau + d)^{-k} e^{2\pi i h \left(\frac{-cz^2}{c\tau + d} \right)} \phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) \text{ holds for } \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \quad (5)$$

$$\phi(\tau, z) = e^{2\pi i h(\lambda^2 \tau + 2\lambda z)} \phi(\tau, z + \lambda\tau + \mu) \text{ for } \lambda, \mu \in \mathbb{Z} \quad (6)$$

$\phi(\tau, z)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n \geq r^2/4h} c(n, r) q^n \zeta^r \quad (7)$$

2.2 Eisenstein Jacobi forms

One major construction method of Jacobi forms is Eisenstein Jacobi forms (c.f. [8], pages 17-18).

$$\begin{aligned}
 E_{k,m}(\tau, z) &= \\
 &= \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \sum_{\lambda \in \mathbb{Z}} (c\tau + d)^{-k} e^m \left(\lambda^2 \frac{a\tau + b}{c\tau + d} + 2\lambda \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d} \right) \\
 &= \sum_{\substack{n,r \in \mathbb{Z} \\ 4nm \geq r^2}} e_{k,m}(n, r) q^n \zeta^r
 \end{aligned}$$

where a, b are chosen so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

3 Massformula for Jacobi weight enumerators

3.1 Mass formula for ordinary weight enumerators

For $1 \leq h < \frac{n}{2}$ let C_0 be a binary self-orthogonal code of length n and dimension h containing all one vector 1 in $\mathbb{F}_2^n$. We denote by

$$\nu(n, h) = \#\{C \in \mathcal{D}_n \mid C \supset C_0\}.$$

This is independent of the choice of C_0 .

We recall that $\mathcal{S}_n$ is the set of all binary self-dual codes of length n for each even integer n . We denote by

$$\mu(n, h) = \#\{C \in \mathcal{S}_n \mid C \supset C_0\}.$$

We quote a well-known result

Proposition 3.1 ([20]) *It holds that*

$$\nu(n, h) = \prod_{j=0}^{\frac{n}{2}-h-1} (2^j + 1).$$

Proposition 3.2 ([20]) *For h with $1 \leq h < \frac{n}{2}$ it holds that*

$$\mu(n, h) = \prod_{j=1}^{\frac{n}{2}-h} (2^j + 1).$$

J.G. Thompson [20] proved that

$$\sum_{C \in \mathcal{D}_n} W_1(x, y; C) = \nu(n, 1)(x^n + y^n) + \nu(n, 2) \sum_{\substack{0 \leq j < n \\ 4 \mid j}} \binom{n}{j} x^{n-j} y^j.$$

If we define

$$\begin{aligned}
 W_1^{(n)}(x, y) &= \sum_{4 \mid j} \binom{n}{j} x^{n-j} y^j \\
 &= \frac{1}{4} ((x+y)^n + (x-y)^n + (x+iy)^n + (x-iy)^n),
 \end{aligned}$$

then

$$\sum_{C \in \mathcal{D}_n} W_1(x, y; C) = \nu(n, 2)(2^{n/2-2}(x^n + y^n) + W_1^{(n)}(x, y)).$$

Recall that the root system D_4 consists of the 24 roots listed below:

$$\begin{aligned} \pm\sqrt{2}e_j \quad (j = 1, 2, 3, 4), \\ \frac{1}{\sqrt{2}}(\pm 1, \pm 1, \pm 1, \pm 1). \end{aligned}$$

We imbed these vectors into $\mathbb{C}^2$ as follows.

$$\begin{aligned} i^k\sqrt{2}e_j \quad (j = 1, 2, k = 0, 1, 2, 3), \\ \zeta^j e_1 + \zeta^k e_2 \quad (j, k = 1, 3, 5, 7), \end{aligned}$$

where $\zeta = e^{\pi i/4}$. Now, let D_4 denote the set of 24 vectors above. If $n \equiv 0 \pmod{4}$, then

$$\begin{aligned} \sum_{\alpha \in D_4} (\alpha_1 x + \alpha_2 y)^n &= 2^{n/2+2}(x^n + y^n) + \sum_{j,k=1,3,5,7} (\zeta^j x + \zeta^k y)^n \\ &= 2^{n/2+2}(x^n + y^n) + (-1)^{n/4} \sum_{j,k=0,2,4,6} (\zeta^j x + \zeta^k y)^n \\ &= 16(2^{n/2-2}(x^n + y^n) + (-1)^{n/4} W_n^{(1)}(x, y)). \end{aligned} \quad (8)$$

3.2 A Theorem

Using the notation introduced in the previous section, the mass formula for the Jacobi weight enumerator polynomial can easily be established. The Jacobi weight enumerator polynomial for a code C with respect to a reference vector u is defined by

$$Jac(C, u; x_{00}, x_{01}, x_{10}, x_{11}) = \sum_{v \in C} X(u, v).$$

Denote by $\overline{Jac}_{n,k}$ the sum of the Jacobi weight enumerator polynomial with respect to a fixed reference vector of weight k for all $C \in \mathcal{D}_n$. Note that $\overline{Jac}_{n,k}$ is independent of the choice of u . We prove

Theorem 3.3 (Munemasa-Ozeki [13])

$$\overline{Jac}_{n,k} = \frac{1}{16} \nu(n, 2) \sum_{\alpha=(\alpha_1, \alpha_2) \in D_4} (\alpha_1 x_{00} + \alpha_2 x_{01})^{n-k} (\alpha_1 x_{10} + \alpha_2 x_{11})^k. \quad (9)$$

4 An application of the mass formula to the construction of Jacobi forms

4.1 Some instances

If we apply the so called Bannai-Ozeki map (c.f. [2]) to the right hand side of (9), we obtain many important Jacobi forms of weight $n/2$ and index k . As the mass formula the both hands are meaningful only when n is divisible by 8. However the polynomials in the right hand side are useful even if $n \equiv 4 \pmod{8}$ in constructing Jacobi forms. Here we give few instances of the construction.

To do this we recall Jacobi's theta functions:

$$\begin{aligned} \theta_0(\tau, z) &= \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi n^2 \tau + 2n\pi iz}, \\ \theta_2(\tau, z) &= \sum_{n \in \mathbb{Z}} e^{\pi(n+1/2)^2 \tau + (2n+1)\pi iz}, \\ \theta_3(\tau, z) &= \sum_{n \in \mathbb{Z}} e^{\pi n^2 \tau + 2n\pi iz}. \end{aligned}$$

We put $\varphi_i(\tau, z) = \theta(2\tau, 2z)$, and $\varphi_i(\tau) = \varphi_i(\tau, 0)$ ($i = 2, 3$).

When $n = 8$ and $k = 1$ the right hand side of (9) becomes 30 times of

$$7y^4x^3u + 7x^4y^3v + y^7v + x^7u.$$

Substituting $x = \varphi_2(\tau)$, $y = \varphi_3(\tau)$, $u = \varphi_2(\tau, z)$, $v = \varphi_3(\tau, z)$ in this polynomial we get a Jacobi form of weight 4 and index 1:

$$\begin{aligned}\psi_{4,1} = & 1 + (\zeta^2 + 56\zeta + 56\zeta^{-1} + \zeta^{-2} + 126)q \\ & + (126\zeta^2 + 576\zeta + 576\zeta^{-1} + 126\zeta^{-2} + 756)q^2 \\ & + (56\zeta^3 + 756\zeta^2 + 1512\zeta + 1512\zeta^{-1} + 756\zeta^{-2} + 56\zeta^{-3} + 2072)q^3 \\ & + (\zeta^4 + 576\zeta^3 + 2072\zeta^2 + 4032\zeta + 4032\zeta^{-1} + 2072\zeta^{-2} + 576\zeta^{-3} + \zeta^{-4} + 4158)q^4 \\ & (126\zeta^4 + 1512\zeta^3 + 4158\zeta^2 + 5544\zeta + 5544\zeta^{-1} + 4158\zeta^{-2} + 1512\zeta^{-3} + 126\zeta^{-4} + 7560)q^5 + \dots\end{aligned}$$

When $n = 8$ and $k = 2$ the right hand side of (9) becomes 30 times of

$$3x^4y^2v^2 + 8x^3y^3uv + 3x^2y^4u^2 + x^6u^2 + y^6v^2.$$

The last polynomial leads to a Jacobi form of weight 4 and index 2:

$$\begin{aligned}\psi_{4,2} = & 1 + [14(\zeta^2 + \zeta^{-2}) + 64(\zeta + \zeta^{-1}) + 84]q + \\ & + [\zeta^4 + \zeta^{-4} + 64(\zeta^3 + \zeta^{-3}) + 280(\zeta^2 + \zeta^{-2}) + 448(\zeta + \zeta^{-1}) + 574]q^2 \\ & + (84\zeta^4 + 448\zeta^3 + 840\zeta^2 + 1344\zeta + 1344\zeta^{-1} + 840\zeta^{-2} + 448\zeta^{-3} + 84\zeta^{-4} + 1288)q^3 \\ & + [64(\zeta^5 + \zeta^{-5}) + 574(\zeta^4 + \zeta^{-4}) + 1344(\zeta^3 + \zeta^{-3}) + 2368(\zeta^2 + \zeta^{-2}) + 2688(\zeta + \zeta^{-1}) + 3444]q^4 \\ & + [14(\zeta^6 + \zeta^{-6}) + 448(\zeta^5 + \zeta^{-5}) + 1288(\zeta^4 + \zeta^{-4}) + 2688(\zeta^3 + \zeta^{-3}) \\ & + 3542(\zeta^2 + \zeta^{-2}) + 4928(\zeta + \zeta^{-1}) + 4424]q^5 + \dots\end{aligned}$$

When $n = 12$ and $k = 1$ the right hand side of (9) is a polynomial that is 4050 times of

$$-22x^4y^7v - 11x^8y^3v - 11y^8x^3u - 22y^4x^7u + x^{11}u + y^{11}v.$$

This leads to a Jacobi form of weight 6 and index 1

$$\begin{aligned}\psi_{6,1} = & 1 + (\zeta^2 - 88\zeta - 88\zeta^{-1} + \zeta^{-2} - 330)q \\ & + (-330\zeta^2 - 4224\zeta - 4224\zeta^{-1} - 330\zeta^{-2} - 7524)q^2 \\ & + (-88\zeta^3 - 7524\zeta^2 - 30600\zeta - 30600\zeta^{-1} - 7524\zeta^{-2} - 88\zeta^{-3} - 46552)q^3 \\ & + (\zeta^4 - 4224\zeta^3 - 46552\zeta^2 - 130944\zeta \\ & - 130944\zeta^{-1} - 46552\zeta^{-2} - 4224\zeta^{-3} + \zeta^{-4} - 169290)q^4 \\ & (-330\zeta^4 - 30600\zeta^3 - 169290\zeta^2 - 355080\zeta \\ & - 355080\zeta^{-1} - 169290\zeta^{-2} - 30600\zeta^{-3} - 330\zeta^{-4} - 464904)q^5 + \dots\end{aligned}$$

When $n = 12$ and $k = 2$ the right hand side of (9) is 4050 times of the polynomial

$$-14y^4x^6u^2 - 14y^6x^4v^2 - 3y^2x^8v^2 - 3y^8x^2u^2 + x^{10}u^2 + y^{10}v^2 - 16y^3x^7uv - 16y^7x^3uv.$$

This leads to a Jacobi form of weight 6 and index 2:

$$\begin{aligned}
\psi_{6,2} = & 1 + (-10\zeta^2 - 128\zeta - 128\zeta^{-1} - 10\zeta^{-2} - 228)q \\
& + (\zeta^4 - 128\zeta^3 - 1496\zeta^2 - 3968\zeta \\
& - 3968\zeta^{-1} - 1496\zeta^{-2} - 128\zeta^{-3} + \zeta^{-4} - 5450)q^2 \\
& + (-228\zeta^4 - 3968\zeta^3 - 14088\zeta^2 - 27264\zeta \\
& - 27264\zeta^{-1} - 14088\zeta^{-2} - 3968\zeta^{-3} - 228\zeta^{-4} - 31880)q^3 \\
& + (-128\zeta^5 - 5450\zeta^4 - 27264\zeta^3 - 67712\zeta^2 - 103680\zeta \\
& - 103680\zeta^{-1} - 67712\zeta^{-2} - 27264\zeta^{-3} - 5450\zeta^{-4} - 128\zeta^{-5} - 124260)q^4 \\
& + (-10\zeta^6 - 3968\zeta^5 - 31880\zeta^4 - 103680\zeta^3 - 197650\zeta^2 - 292480\zeta \\
& - 292480\zeta^{-1} - 197650\zeta^{-2} - 103680\zeta^{-3} - 31880\zeta^{-4} - 3968\zeta^{-5} - 10\zeta^{-6} - 316168)q^5 + \dots
\end{aligned}$$

In this way we obtain an infinite family of Jacobi forms of various weights and various indices.

4.2 A comparison of two constructions

In [8] only the values $e_{k,m}(n,r)$ ($k \leq 8, m = 1$) of the Fourier coefficients of $E_{k,m}(\tau, z)$ are given explicitly.

Here we explain a method to compute $e_{k,m}(n,r)$ for any even k and $m \geq 1$. For this we start from the formula given in [8] page 22:

$$e_{k,m}(n,r) = \frac{\sigma_{k-1}(m)^{-1}}{\zeta(3-2k)} \sum_{d|(n,r,m)} d^{k-1} H(k-1, \frac{4nm-r^2}{d^2}),$$

and

$$e_{k,1}(n,r) = \frac{H(k-1, 4n-r^2)}{\zeta(3-2k)},$$

where $\zeta(3-2k)$ is the special value of Riemann's zeta function. The quantity $H(k-1, N)$ is described at page 30 in [8]:

$$H(k-1, N) = \begin{cases} L_{-N}(2-k) & \text{if } N > 0 \text{ and } N \equiv 0 \text{ or } 3 \pmod{4}, \\ \zeta(3-2k) & \text{if } N = 0, \\ 0 & \text{if } N > 0 \text{ and } N \equiv 1 \text{ or } 2 \pmod{4}. \end{cases}$$

When $-N \equiv 0$ or $1 \pmod{4}$ we put $-N = (-N_0)u^2$ $u \in \mathbb{N}$ so that $-N_0$ is the discriminant of the quadratic number field $\mathbb{Q}(\sqrt{-N})$. The number $L_{-N}(2-k)$ comes from the L-function $L_{-N_0}(s)$ by way of

$$L_{-N}(s) = L_{-N_0}(s) \sum_{d|u} \mu(d) \left(\frac{-N_0}{d} \right) d^{-s} \sigma_{1-2s}\left(\frac{u}{d}\right),$$

and

$$L_{-N_0}(s) = L(s, \left(\frac{-N_0}{*} \right)) = \sum_{n=1}^{\infty} \frac{\left(\frac{-N_0}{d} \right)}{n^s}.$$

To make the value $e_{k,1}(n,r)$ explicit it is necessary to know the values $\zeta(3-2k)$ and $L_{-N_0}(1-m)$. As to the values $\zeta(3-2k)$ there are many literature available and they tell us that

$$\begin{aligned}
\zeta(2k) &= \frac{(-1)^{k-1} (2\pi)^{2k}}{2} B_{2k} \quad (k \geq 1) \\
\zeta(1-n) &= (-1)^{n-1} \frac{B_n}{n} \quad (n = 1, 2, \dots),
\end{aligned}$$

where B_n is the n -th Bernoulli number. The beginning few numbers are

$$B_1 = \frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, \dots, B_{\text{odd} \geq 3} = 0.$$

By these one gets

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \zeta(8) = \frac{\pi^8}{9450}, \zeta(10) = \frac{\pi^{10}}{93555}, \dots$$

$$\zeta(-1) = -\frac{1}{12}, \zeta(-2) = 0, \zeta(-3) = \frac{1}{120}, \zeta(-5) = -\frac{1}{252}, \zeta(-7) = \frac{1}{240}, \zeta(-9) = -\frac{1}{132}, \dots$$

It is much complicated to get the values $L_{-N_0}(1-m)$. On reading the book [1] we find a suitable formula to do this task. Note that a similar formula has been given in [11] Chapter XIII in a not straight way.

Theorem 4.1 (Arakawa-Ibukiyama-Kaneko) Let χ be a primitive character mod f and m be a positive integer, then

$$L(1-m, \chi) = -\frac{B_{m,\chi}}{m},$$

where $B_{m,\chi}$ is the generalized Bernoulli number associated with χ :

$$B_{m,\chi} = f^{m-1} \sum_{a=1}^f \chi(a) B_m\left(\frac{a}{f}\right),$$

and $B_m(x)$ is the Bernoulli polynomial of degree m .

The Bernoulli polynomials are given by

$$B_m(x) = \sum_{j=0}^m (-1)^j \binom{m}{j} B_j x^{m-j}.$$

With the above Theorem we compute $e_{k,m}(n,r)$, and we give small tables of them, that are not contained in [8].

$4n-r^2$	0	3	4	7	8	11	12	15
$e_{10,1}(n,r)$	1	$-\frac{860776}{43867}$	$-\frac{9947070}{43867}$	$-\frac{1159757568}{43867}$	$-\frac{3601586268}{43867}$	$-\frac{53854227000}{43867}$	$-\frac{113044851304}{43867}$	$-\frac{754799931648}{43867}$
$e_{12,1}(n,r)$	1	$\frac{339848}{77683}$	$\frac{6971898}{77683}$	$\frac{2485779648}{77683}$	$\frac{10096500348}{77683}$	$\frac{285849348696}{77683}$	$\frac{713061257096}{77683}$	$\frac{7428376170816}{77683}$

$4n-r^2$	16	19	20
$e_{10,1}(n,r)$	$-\frac{1303792306110}{43867}$	$-\frac{5607166776120}{43867}$	$-\frac{8689286943288}{43867}$
$e_{12,1}(n,r)$	$\frac{14621136806394}{77683}$	$\frac{88801830903192}{77683}$	$\frac{152244273101400}{77683}$

$8n-r^2$	0	4	7	8	12	15	16	20	23	24	28	31	32
$e_{4,2}(n,r)$	1	14	64	84	280	448	574	840	1344	1288	2368	2688	3444
$e_{6,2}(n,r)$	1	-10	-128	-228	-1496	-3968	-5450	-14088	-27264	-31880	-67712	-103680	-124260
$e_{8,2}(n,r)$	1	$\frac{122}{43}$	$\frac{4872}{43}$	$\frac{11052}{43}$	$\frac{3640}{43}$	$\frac{862464}{43}$	$\frac{1015162}{43}$	$\frac{4266360}{43}$	$\frac{10665792}{43}$	$\frac{13948984}{43}$	$\frac{38576704}{43}$	$\frac{74169984}{43}$	$\frac{91963692}{43}$

We remark that the functions $\psi_{4,1}, \psi_{6,1}, \psi_{8,1}$ respectively coincide with Eisenstein-Jacobi forms $E_{4,1}, E_{6,1}, E_{8,1}$ respectively of index 1 described in [8] pages 17-23. Explicit Fourier expansions of Eisenstein-Jacobi forms of index ≥ 2 are not given in [8]. We have verified that $\psi_{4,2}, \psi_{6,2}$ also coincide with Jacobi-Eisenstein series of index 2. This is done by using the relation (7) in [8], page 22. Besides these exceptional cases Eisenstein-Jacobi form $E_{k,m}$ differ from $\psi_{k,m}$. One may be interested with a problem to explore the further relations between these two constructions of Jacobi forms.

5 Eisenstein type polynomials in more variables

$$E_{k_1, k_2}(x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}) = \frac{1}{16} \nu(n, 2) \sum_{\alpha=(\alpha_1, \alpha_2) \in D_4} (\alpha_1 x_{00} + \alpha_2 x_{01})^{n-k_1-k_2} (\alpha_1 x_{10} + \alpha_2 x_{11})^{k_1} (\alpha_1 y_{10} + \alpha_2 y_{11})^{k_2}, \quad (10)$$

$$E_{k_1, k_2, k_3}(x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}, z_{10}, z_{11}) = \frac{1}{16} \nu(n, 2) \sum_{\alpha=(\alpha_1, \alpha_2) \in D_4} (\alpha_1 x_{00} + \alpha_2 x_{01})^{n-k_1-k_2-k_3} (\alpha_1 x_{10} + \alpha_2 x_{11})^{k_1} (\alpha_1 y_{10} + \alpha_2 y_{11})^{k_2} (\alpha_1 z_{10} + \alpha_2 z_{11})^{k_3} \quad (11)$$

⋮

where all exponents are non negative integers.

The right-hand side of (10) belongs to $\mathbb{C}[x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}]^{H_1 \oplus H_1 \oplus H_1}$, and the right-hand side of (11) belongs to $\mathbb{C}[x_{00}, x_{01}, x_{10}, x_{11}, y_{10}, y_{11}, z_{10}, z_{11}]^{H_1 \oplus H_1 \oplus H_1 \oplus H_1}$. As discussed in [2] these polynomials contribute to the construction of Jacobi forms.

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