

CONJECTURES OF DIOPHANTINE EQUATIONS ON PIATETSKI-SHAPIRO SEQUENCES

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ABSTRACT. Let $\alpha > 1$ be a non-integral real number. Let $\text{PS}(\alpha)$ be the set of positive integers of the form $\lfloor n^\alpha \rfloor$ for some $n \in \mathbb{N}$. In this article, we discuss the equation $x + y = z$, where $(x, y, z) \in \text{PS}(\alpha)^3$. The author conjectures that for almost all or all $2 < \alpha < 3$ the equation $x + y = z$ has infinitely many solutions $(x, y, z) \in \text{PS}(\alpha)^3$. In this article, we aim to present heuristic and numerical evidence of the conjecture.

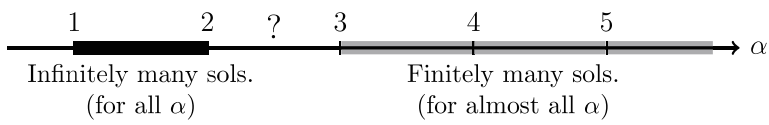
1. INTRODUCTION

Let $\mathbb{N}$ be the set of all positive integers. We say that an integral sequence $(a_n)_{n \in \mathbb{N}}$ is a Piatetski-Shapiro sequence if there exists a non-integral $\alpha > 1$ such that $a_n = \lfloor n^\alpha \rfloor$ for all $n \in \mathbb{N}$. Further, we define $\text{PS}(\alpha) = \{\lfloor n^\alpha \rfloor : n \in \mathbb{N}\}$. In this article, we discuss the following equation:

$$(1.1) \quad x + y = z,$$

where $(x, y, z) \in \text{PS}(\alpha)^3$. The equation (1.1) is an extension of Fermat's equation $(x^n + y^n = z^n)$ from the integral powers to real powers of integers. By Fermat's last theorem, the equation (1.1) does not have any solutions $(x, y, z) \in \text{PS}(\alpha)^3$ if $\alpha \geq 3$ and $\alpha \in \mathbb{N}$.

By using the result given by Frantzikinakis and Wierdl [FW09, Proposition 5.1], for every $\alpha \in (1, 2)$, there are infinitely many solutions $(x, y, z) \in \text{PS}(\alpha)^3$ to (1.1). Further, the author [Sai] showed that for almost all $\alpha > 3$ in the sense of the 1-dimensional Lebesgue measure, the equation (1.1) has at most finitely many solutions $(x, y, z) \in \text{PS}(\alpha)^3$. We describe these results in the following figure.



For all real numbers s, t with $1 < s < t$, we define

$$(1.2) \quad \mathcal{A}(s, t) = \{\alpha \in [s, t] : (1.1) \text{ has infinitely many solutions } (x, y, z) \in \text{PS}(\alpha)^3\}.$$

Matsusaka and the author [MS21, Theorem 1.1] showed that for all real numbers s, t with $2 < s < t$, the Hausdorff dimension of $\mathcal{A}(s, t)$ is not less than $1/s^3$. In particular, $\mathcal{A}(s, t)$ is dense in $[s, t]$. However, we do not know whether $\mathcal{A}(2, 3)$ has positive Lebesgue measure or not. Here, the author conjectures the following.

Conjecture 1.1. *For almost all or all $\alpha \in (2, 3)$, the equation (1.1) would have infinitely many solutions $(x, y, z) \in \text{PS}(\alpha)^3$.*

The goal of this article is to give heuristic and numerical observations on Conjecture 1.1. This problem was first proposed by Glasscock [Gla17]. He investigated the equation

$$(1.3) \quad y = ax + b,$$

where a and b are fixed real numbers satisfying that (1.3) has infinitely many solutions $(x, y) \in \mathbb{N}^2$. In [Gla17, Gla20], he showed that for almost all $\alpha > 1$

- (1) if $\alpha < 2$, then (1.3) has infinitely many solutions $(x, y) \in \text{PS}(\alpha)^2$;
- (2) if $\alpha > 2$, then (1.3) has at most finitely many solutions $(x, y) \in \text{PS}(\alpha)^2$.

The author also contributed to the solvability of (1.3) on $\text{PS}(\alpha)$. In [Sai22], if we additionally assume $0 \leq b < a$, then the author showed that for all $1 < \alpha < 2$, we find infinitely many solutions $(x, y) \in \text{PS}(\alpha)^2$ to (1.3). Thus, he improved (1) of Glasscock's result from "for almost all" to "for all" in the case $1 < \alpha < 2$. Moreover, in the same article, the author showed that for all $2 < s < t$, the Hausdorff dimension of the set

$$\{\alpha \in [s, t] : (1.3) \text{ has infinitely many } (x, y) \in \text{PS}(\alpha)^2\}$$

is coincident with $2/s$. Without the assumption $0 \leq b < a$, it is unknown whether his improvements are true or not.

Glasscock successfully discovered that $\alpha = 2$ is a threshold for the infiniteness or finiteness of solutions to (1.3) on $\text{PS}(\alpha)$. Further, he proposed the following interesting question.

Question 1.2 ([Gla17, Question 6]). *Does there exist $\mathcal{G} > 1$ with the property that for almost every or all $\alpha > 1$, the equation $x + y = z$ has infinitely many solutions $(x, y, z) \in \text{PS}(\alpha)^3$ or not, according as $\alpha < \mathcal{G}$ or $\alpha > \mathcal{G}$?*

The result of Matsusaka and the author [MS21, Theorem 1.1] leads to a negative answer to the "all" version of Question 1.2. The "almost all" version is still open, which is directly connected with Conjecture 1.1

Notation 1.3. For $x \in \mathbb{R}$, let $\{x\}$ denote the fractional part of x . For all intervals $I \subset \mathbb{R}$, let $I_{\mathbb{Z}} = I \cap \mathbb{Z}$. We write $f(x) = g(x) + o(h(x))$ if $(f(x) - g(x))/h(x) \rightarrow 0$ as $x \rightarrow \infty$, where $h(x)$ is a positive-valued function.

2. HEURISTIC OBSERVATIONS

For every $\alpha > 1$ and $N \in \mathbb{N}$, we define

$$s(\alpha, N) = \{([p^\alpha], [q^\alpha]) : p, q \in \mathbb{N}, p \leq q \leq N, [p^\alpha] + [q^\alpha] = [r^\alpha] \text{ for some } r \in \mathbb{N}\}.$$

In this section, by heuristic calculation, we propose the following conjecture. If Conjecture 2.1 was true for every $2 < \alpha < 3$, then Conjecture 1.1 would be also true.

Conjecture 2.1. *For almost all or all $\alpha \in (2, 3)$, we would have*

$$\#s(\alpha, N) = \left(\frac{C(\alpha)}{\alpha(3-\alpha)} + o(1) \right) N^{3-\alpha} \quad (\text{as } N \rightarrow \infty),$$

where $C(\alpha) := \int_0^1 (1+x^\alpha)^{1/\alpha-1} dx$.

Yoshida [Yos, Corollary 1.2] gave a quantitative result on the number of solutions (1.1). For all $1 < \alpha < (\sqrt{21} + 4)/5 = 1.7165 \dots$, he showed that

$$(2.1) \quad \#\{(\ell, m, n) \in \mathbb{N}^3 : \ell < x, [\ell^\alpha] + [m^\alpha] = [n^\alpha]\} = (\beta \alpha^{-\beta} \zeta(\beta) + o(1)) x^{\alpha(\beta-1)+1}$$

as $x \rightarrow \infty$, where $\beta = 1/(\alpha - 1)$ and $\zeta(s)$ denotes the Riemann zeta function. Note that the left-hand side of (2.1) is slightly different from $\#s(\alpha, N)$.

Heuristic evidence of Conjecture 2.1. By the definition of $s(\alpha, N)$, it follows that

$$\#s(\alpha, N) = \sum_{1 \leq q \leq N} \sum_{1 \leq p \leq q} \#[A, B]_{\mathbb{Z}},$$

where $A = (\lfloor p^\alpha \rfloor + \lfloor q^\alpha \rfloor)^{1/\alpha}$ and $B = (\lfloor p^\alpha \rfloor + \lfloor q^\alpha \rfloor + 1)^{1/\alpha}$. We now assume that

$$(2.2) \quad \#[A, B]_{\mathbb{Z}} = B - A.$$

By $0 < B - A < 1$, the assumption (2.2) never holds. We ignore this problem to calculate heuristically, which is not a small gap.

By ignoring the integral parts and applying Taylor's expansion, we see that

$$B - A \sim \frac{1}{\alpha} (p^\alpha + q^\alpha)^{1/\alpha-1} = \frac{q^{2-\alpha}}{\alpha} \left(1 + \left(\frac{p}{q} \right)^\alpha \right)^{1/\alpha-1} \frac{1}{q}.$$

Remark that we do not give the precise definition of " $X \sim Y$ ". The symbol means that X and Y share the almost same magnitude. By Riemann sum, we obtain

$$\lim_{q \rightarrow \infty} \sum_{1 \leq p \leq q} \left(1 + \left(\frac{p}{q} \right)^\alpha \right)^{1/\alpha-1} \frac{1}{q} = \int_0^1 (1 + x^\alpha)^{1/\alpha-1} dx = C(\alpha).$$

Therefore, we can expect that

$$\#s(\alpha, N) \sim \frac{1}{\alpha} \sum_{1 \leq q \leq N} \sum_{1 \leq p \leq q} (p^\alpha + q^\alpha)^{1/\alpha-1} \sim \frac{C(\alpha)}{\alpha} \sum_{1 \leq q \leq N} q^{2-\alpha} \sim \frac{C(\alpha)}{\alpha(3-\alpha)} N^{3-\alpha}.$$

□

By the above calculation, we should consider the set of $(p, q) \in \mathbb{N}^2$ such that $[A, B]_{\mathbb{Z}}$ is non-empty. However, the interval $[A, B]$ would be too short to be investigated.

3. NUMERICAL OBSERVATIONS

In this section, we give numerical evidence on Conjectures 1.1 and 2.1. The author calculates $\#s(\alpha, N)$ for $N = 50000$ and $\alpha = 2.00 + 0.05i$ ($i = 1, 2, \dots, 19$) by using Mathematica. The used program will be described at the end of this article. Mathematica outputs the following.

α	2.05	2.10	2.15	2.20	2.25	2.30	2.35	2.40	2.45	2.50
$\#s(\alpha, 5 \cdot 10^5)$	13035	7754	4763	2739	1653	1060	647	356	247	153

α	2.55	2.60	2.65	2.70	2.75	2.80	2.85	2.90	2.95	-
$\#s(\alpha, 5 \cdot 10^5)$	93	69	41	22	17	7	4	0	1	-

From the table, we do not guess the existence of a threshold like Glasscock's result on (1.3). Let us note that $N = 50000$ may be still small. Indeed by numerical calculation, for every $\alpha \in \{2.8, 2.85, 2.9, 2.95\}$, we obtain

$$(3.1) \quad \frac{C(\alpha)}{\alpha(3-\alpha)} 50000^{3-\alpha} \leq \frac{C(2.8)}{2.8(3-2.8)} 50000^{3-2.8} = 13.70 \dots$$

Thus, it is too small to guess whether Conjecture 1.1 is true or not. In the case $\alpha = 2.95$, if we chose $N \geq 1.671 \times 10^{59}$, then the most left-hand side of (3.1) would be greater than 1000.

On the other hand, we expect that Conjecture 2.1 is true for small $\alpha > 2$. Indeed, for every $\alpha > 1$ and $N \in \mathbb{N}$, we define

$$\delta(\alpha, N) = \#s(\alpha, N) \cdot \frac{\alpha(3 - \alpha)}{C(\alpha)} N^{\alpha-3} - 1.$$

If $\delta(\alpha, N) \rightarrow 0$ as $N \rightarrow \infty$, then Conjecture 2.1 would be true for α . The following figure describes $\{(N, \delta(\alpha, N)) \mid N = 1, 2, \dots, 50000 \text{ and } \delta(\alpha, N) \in [-0.10, 0.02]\}$ for every $\alpha \in \{2.05, 2.10, 2.15, 2.20, 2.25, 2.30\}$.

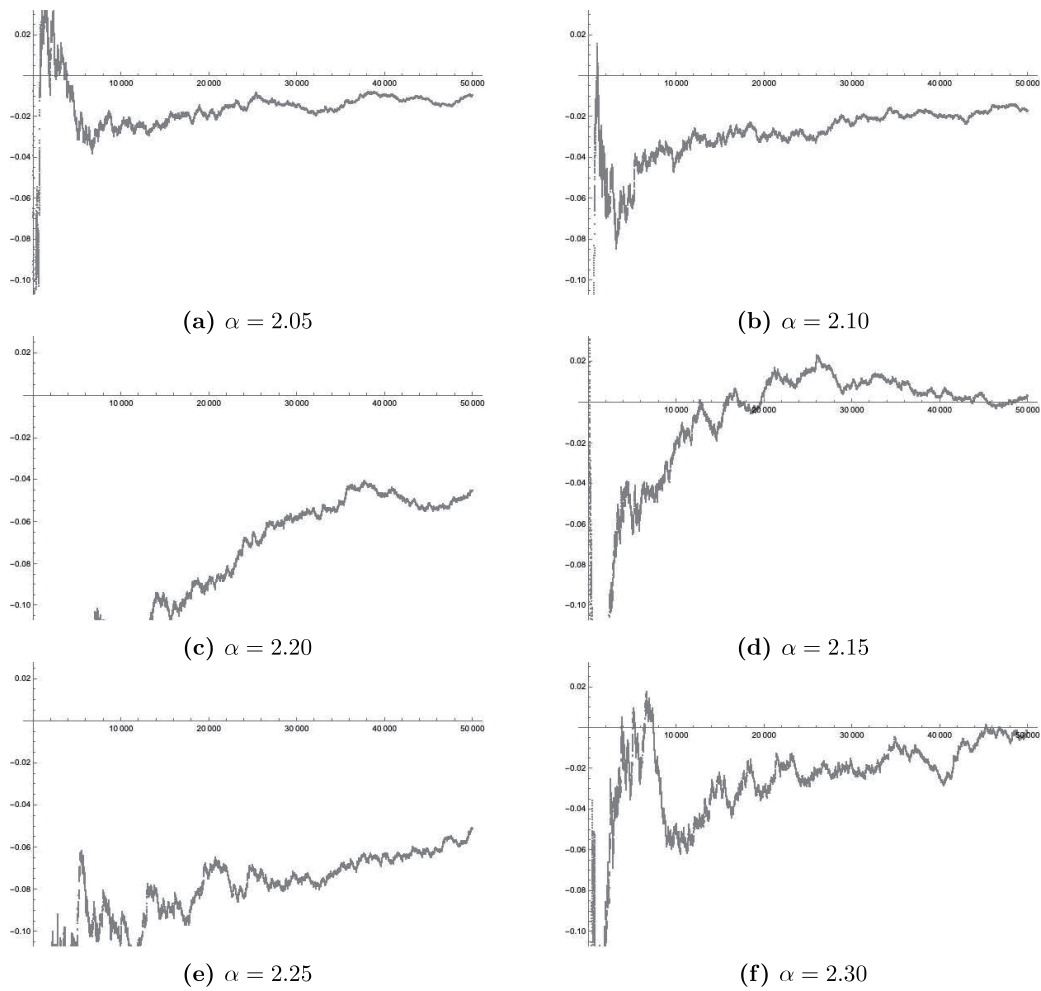


Figure 1. Graphs of $\delta(\alpha, 50000)$

We also give the following table of $\delta(\alpha, 5 \cdot 10^5)10^2$ to exhibit the calculation more clearly. Remark that it is not $\delta(\alpha, 5 \cdot 10^5)$, but $\delta(\alpha, 5 \cdot 10^5) \times 10^2$.

α	2.05	2.10	2.15	2.20	2.25	2.30	2.35	2.40	2.45	2.50
$\delta(\alpha, 5 \cdot 10^5)10^2$	-0.970	-1.73	0.303	-4.55	-5.10	-0.258	-0.781	-0.116	-1.43	-2.73
α	2.55	2.60	2.65	2.70	2.75	2.80	2.85	2.90	2.95	-
$\delta(\alpha, 5 \cdot 10^5)10^2$	-6.79	7.62	-2.07	-21.2	-11.2	-48.9	-61.7	-100	-90.3	-

Thus, $\delta(\alpha, N)$ seems small for several α 's. This supports Conjecture 2.1. Moreover, we observe that almost all values in the table are negative. Thus, we expect that almost every (or every) $\alpha \in (2, 3)$ would satisfy that

$$\#s(\alpha, N) \leq \frac{C(\alpha)}{\alpha(3-\alpha)} N^{3-\alpha}$$

for sufficiently large N . At last, we give a program to calculate $\#s(\alpha, N)$ in Mathematica.

```

ps[alpha_,n_]:= Floor[n^alpha];
.....
For[alpha = 2.05, alpha <= 2.95, alpha = alpha + 0.05,
  sols = {}; gamma = 1/alpha;
For[p = 1, p <= 50000, p = p + 1, For[q = p, q <= 50000, q = q + 1,
  sum = ps[alpha,p] + ps[alpha,q]; r = Ceiling[sum^gamma];
  If[sum == ps[alpha,r],
    sols = Join[sols, {{p,q}}];
];];];
Print[{alpha, Length[sols]}]; Print[sols];

```

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