

$\mathbf{R}$ 上の周期係数楕円型作用素のグリーン関数

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In the one dimensional case we shall show that the Green functions of elliptic operators with periodic coefficients are written as a product of an exponential function and a periodic function, and that the limiting absorption principle holds for all λ in the interior of the spectrum. We shall also calculate the resolvent kernel for all $\lambda \in \mathbf{R}$ in the resolvent set. The results are joint work with M. Murata, Tokyo Institute of Technology.

Let

$$L = -\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) + c(x),$$

where $a(x)$ and $c(x)$ are real-valued periodic functions with period 1. Assume that $a \in L^\infty(\mathbf{R})$ and $0 < \mu \leq a(x) \leq \mu^{-1}$ for some constant μ , and that $c \in L^1_{loc}(\mathbf{R})$. Corresponding to this operator, we consider the equation

$$\frac{d}{dx} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} 0 & a(x)^{-1} \\ c(x) - z & 0 \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \quad (1)$$

for $z \in \mathbf{C}$. By the standard iteration method of ordinary differential equations, we can find unique solutions to (1), $(c_1(x, z), c_2(x, z))$ and $(s_1(x, z), s_2(x, z))$ with the initial conditions

$$\begin{pmatrix} c_1(0, z) \\ c_2(0, z) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} s_1(0, z) \\ s_2(0, z) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively, in the space of $\mathbf{C}^2$ -valued absolutely continuous functions $AC(\mathbf{R})^2$. We can also see that $c_j(x, z)$ and $s_j(x, z)$ are $C([-R, R])$ -valued entire functions of z for any R .

For each $\zeta \in \mathbf{C}$, the eigenvalue problem

$$\begin{cases} y \in H^1_{loc}(\mathbf{R}) \\ Ly = zy \\ y(x+1) = e^{i\zeta}y(x) \quad (\zeta\text{-periodicity}) \end{cases} \quad (2)$$

is equivalent to

$$\begin{cases} (y_1, y_2) \in AC(\mathbf{R})^2 \\ (y_1, y_2) \text{ satisfies (1) and } y_1 \text{ satisfies the } \zeta\text{-periodicity} \end{cases}$$

under the relation $y_1 = y$, $y_2 = ay'$. Writing a solution to (2) as $y(x) = \alpha_1 c_1(x, z) + \alpha_2 s_1(x, z)$, $|\alpha_1|^2 + |\alpha_2|^2 \neq 0$, by the ζ -periodicity we have $(M(z) - e^{i\zeta}I)\alpha = 0$, where

$$M(z) := \begin{pmatrix} c_1(1, z) & s_1(1, z) \\ c_2(1, z) & s_2(1, z) \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

We see that $\det(M(z) - e^{i\zeta}I) = 0$ if and only if

$$D(z) = e^{i\zeta} + e^{-i\zeta}, \quad (3)$$

where $D(z) := c_1(1, z) + s_2(1, z)$ is the discriminant, which is an entire function. Hence the existence of non-trivial solution of (2) is equivalent to (3).

A function y is an eigenfunction of (2) if and only if $u(x) = e^{-ix\zeta}y(x)$ is an eigenfunction of $L(\zeta)$ with the same eigenvalue. Here $L(\zeta) = e^{-ix\zeta}Le^{ix\zeta}$ is an operator on $L^2(\mathbf{T})$ with compact resolvent with the domain $D(L(\zeta)) = \{u \in H^1(\mathbf{T}); L(\zeta)u \in L^2(\mathbf{T})\}$. Regarding L as the selfadjoint operator on $L^2(\mathbf{R})$ with the domain $D(L) = \{u \in H^1(\mathbf{R}); Lu \in L^2(\mathbf{R})\}$, we have the direct integral decomposition $\mathcal{U}L\mathcal{U}^{-1} = \int_{[-\pi, \pi)}^\oplus L(\xi)d\xi$, where $\mathcal{U}$ is a unitary operator (cf. [RS]).

We denote the eigenvalues of $L(\xi)$ by $\lambda_1(\xi) \leq \lambda_2(\xi) \leq \dots$ for $\xi \in \mathbf{R}$ counted with multiplicities. Each $\lambda_n(\xi)$ is known to be continuous on $\mathbf{R}$. We summarize several facts, which can be proved in ways similar to those in [E], [Ku], [Ma], and [RS]. Each $\lambda_n(\xi)$ is real analytic on $(0, \pi)$, and for $\xi \in (0, \pi)$, $\lambda_n(\xi)$ is a nondegenerate eigenvalue of $L(\xi)$. There exists a sequence of real numbers

$$-\infty < \mu_1 < \nu_1 \leq \nu_2 < \mu_2 \leq \mu_3 < \nu_3 \leq \dots$$

such that it tends to infinity and has the following properties:

(i) The spectrum $\sigma(L)$ of L is $\cup_{n=1}^\infty ([\mu_{2n-1}, \nu_{2n-1}] \cup [\nu_{2n}, \mu_{2n}])$; and $|D(\lambda)| \leq 2$, $\lambda \in \mathbf{R}$, if and only if $\lambda \in \sigma(L)$.

(ii) $D(\lambda) = 2$ only at $\lambda = \mu_j$, and $D(\lambda) = -2$ only at $\lambda = \nu_j$.

(iii) $D'(\lambda) < 0$ on $(-\infty, \nu_1)$ and (μ_{2n-1}, ν_{2n-1}) , and $D'(\lambda) > 0$ on (ν_{2n}, μ_{2n}) .

(iv) $\lambda'_{2n-1}(\xi) > 0$ and $\lambda'_{2n}(\xi) < 0$ on $(0, \pi)$; in the interval $[0, \pi]$, $\lambda_{2n-1}(\xi)$ increases from μ_{2n-1} to ν_{2n-1} , and $\lambda_{2n}(\xi)$ decreases from μ_{2n} to ν_{2n} ; $\lambda_n(k\pi + \xi) = \lambda_n(k\pi - \xi)$ for any integer k and real ξ .

(v) If $\lambda_{2n-1}(\pi) = \lambda_{2n}(\pi)$, then $\lambda_{2n-1}(\pi - 0) \neq 0$; if $\lambda_{2n}(0) = \lambda_{2n+1}(0)$, then $\lambda_{2n+1}(0 + 0) \neq 0$.

(vi) If $\nu_{2n-1} \neq \nu_{2n}$, then $D'(\nu_{2n-1}) \neq 0$ and $D'(\nu_{2n}) \neq 0$, and ν_{2n-1} and ν_{2n} are nondegenerate eigenvalues of $L(\pi)$; if $\mu_{2n} \neq \mu_{2n+1}$, then $D'(\mu_{2n}) \neq 0$ and $D'(\mu_{2n+1}) \neq 0$ and μ_{2n} and μ_{2n+1} are nondegenerate eigenvalues of $L(0)$; if $\nu_{2n-1} = \nu_{2n}$ or $\mu_{2n} = \mu_{2n+1}$, then $D' = 0$ at these points, and these are doubly degenerate eigenvalues of $L(\pi)$ or $L(0)$, respectively; if $D(\lambda) \geq 2$ (≤ -2) and $D'(\lambda) = 0$, then $D''(\lambda) < 0$ (> 0).

We denote by $G_z(x, y)$ the integral kernel of the resolvent $R(z) := (L - z)^{-1}$ for z in the resolvent set. We use the notations $(u, v) = \int_0^1 u(x)\overline{v(x)}dx$ and $\|u\|^2 = (u, u)$.

First, let λ be in the interior of $\sigma(L)$. Then the only one of the following four cases holds:

(I) $\lambda = \lambda_{2n-1}(\xi) \in (\mu_{2n-1}, \nu_{2n-1})$ for some $\xi \in (0, \pi)$,

(II) $\lambda = \lambda_{2n}(\xi) \in (\nu_{2n}, \mu_{2n})$ for some $\xi \in (-\pi, 0)$,

(III) $\lambda = \lambda_{2n-1}(\pi) = \lambda_{2n}(\pi) = \nu_{2n-1} = \nu_{2n}$,

(IV) $\lambda = \lambda_{2n}(0) = \lambda_{2n+1}(0) = \mu_{2n} = \mu_{2n+1}$.

Theorem 1. Assume that λ is in the interior of $\sigma(L)$. There exists the limit

$\lim_{\varepsilon \downarrow 0} \left(\frac{d}{d\lambda}\right)^m R(\lambda \pm i\varepsilon)f(x)$ in $L^2_{loc}(\mathbf{R})$ for $m \geq 0$ and $f \in L^2(\mathbf{R})$ with compact support, and the convergence is locally uniform in the interior of $\sigma(L)$. The integral kernels $G_{\lambda+i0}(x, y)$ and $G_{\lambda+i0}^{(m)}(x, y)$ of $\lim_{\varepsilon \downarrow 0} R(\lambda + i\varepsilon)$ and $\lim_{\varepsilon \downarrow 0} \left(\frac{d}{d\lambda}\right)^m R(\lambda + i\varepsilon)$, $m \geq 1$, admit the following expressions:

Case (I).

$$G_{\lambda+i0}(x, y) = G_{\lambda+i0}(y, x) = \frac{ie^{i(x-y)\xi}}{\lambda'_{2n-1}(\xi)} \frac{u_\xi(x)\overline{u_\xi(y)}}{\|u_\xi\|^2}, \quad y \leq x,$$

$$\begin{aligned} G_{\lambda+i0}^{(m)}(x, y) &= G_{\lambda+i0}^{(m)}(y, x) \\ &= \left(\frac{i}{\lambda'_{2n-1}(\xi)}\right)^{m+1} (x-y)^m e^{i(x-y)\xi} \frac{u_\xi(x)\overline{u_\xi(y)}}{\|u_\xi\|^2} (1 + O(|x-y|^{-1})), \quad y \leq x. \end{aligned}$$

Here u_ξ is an eigenfunction corresponding to the eigenvalue $\lambda_{2n-1}(\xi)$.

Case (II). $G_{\lambda+i0}(x, y)$ and $G_{\lambda+i0}^{(m)}(x, y)$ admit the same expressions as in (I) with $\lambda'_{2n-1}(\xi)$ replaced by $\lambda'_{2n}(\xi)$, and with u_ξ being an eigenfunction corresponding to the eigenvalue $\lambda_{2n}(\xi)$.

Case (III). With u_ξ being a $C(\mathbf{T})$ -valued holomorphic function in a neighborhood of π such that $\|u_\xi\| \neq 0$, $(L(\xi) - \lambda_{2n-1}(\xi))u_\xi = 0$ for $\xi \leq \pi$, and $(L(\xi) - \lambda_{2n}(\xi))u_\xi = 0$ for $\pi < \xi$,

$$G_{\lambda+i0}(x, y) = G_{\lambda+i0}(y, x) = \frac{ie^{i(x-y)\pi}}{\lambda'_{2n-1}(\pi-0)} \frac{u_\pi(x)\overline{u_\pi(y)}}{\|u_\pi\|^2}, \quad y \leq x,$$

$$\begin{aligned} G_{\lambda+i0}^{(m)}(x, y) &= G_{\lambda+i0}^{(m)}(y, x) \\ &= \left(\frac{i}{\lambda'_{2n-1}(\pi-0)}\right)^{m+1} (x-y)^m e^{i(x-y)\pi} \frac{u_\pi(x)\overline{u_\pi(y)}}{\|u_\pi\|^2} (1 + O(|x-y|^{-1})), \quad y \leq x. \end{aligned}$$

Case (IV). With u_ξ being a $C(\mathbf{T})$ -valued holomorphic function in a neighborhood of 0 such that $\|u_\xi\| \neq 0$, $(L(\xi) - \lambda_{2n+1}(\xi))u_\xi = 0$ for $0 \leq \xi$, and $(L(\xi) - \lambda_{2n}(\xi))u_\xi = 0$ for $\xi < 0$,

$$G_{\lambda+i0}(x, y) = G_{\lambda+i0}(y, x) = \frac{i}{\lambda'_{2n+1}(0+0)} \frac{u_0(x)\overline{u_0(y)}}{\|u_0\|^2}, \quad y \leq x,$$

$$\begin{aligned} G_{\lambda+i0}^{(m)}(x, y) &= G_{\lambda+i0}^{(m)}(y, x) \\ &= \left(\frac{i}{\lambda'_{2n+1}(0+0)}\right)^{m+1} (x-y)^m \frac{u_0(x)\overline{u_0(y)}}{\|u_0\|^2} (1 + O(|x-y|^{-1})), \quad y \leq x. \end{aligned}$$

Proof. (I) Since $D'(\lambda) < 0$ on (μ_{2n-1}, ν_{2n-1}) , there exists a holomorphic inverse function D^{-1} of D on an open set containing $(-2, 2)$. Put $\lambda(\zeta) := D^{-1}(e^{i\zeta} + e^{-i\zeta})$ for ζ in an open set containing $(0, \pi)$. We have $\lambda(\xi) = \lambda_{2n-1}(\xi)$ for $\xi \in (0, \pi)$. Let

$$\alpha(\zeta) = (\alpha_1(\zeta), \alpha_2(\zeta)) := (-s_1(1, \lambda(\zeta)), c_1(1, \lambda(\zeta)) - e^{i\zeta}).$$

Since $\alpha(\xi) \neq 0$ for $\xi \in (0, \pi)$, $\alpha(\zeta)$ is an eigenvector of $M(\lambda(\zeta))$ corresponding to the eigenvalue $e^{i\zeta}$ for ζ in an open set containing $(0, \pi)$. Thus $y_\zeta(x) := \alpha_1(\zeta)c_1(x, \lambda(\zeta)) +$

$\alpha_2(\zeta)s_1(x, \lambda(\zeta))$ satisfies (2) with z replaced by $\lambda(\zeta)$. So $u_\zeta(x) := e^{-i\zeta x}y_\zeta(x)$ is a $C(\mathbf{T})$ -valued holomorphic eigenfunction of $L(\zeta)$ corresponding to the eigenvalue $\lambda(\zeta)$. Since $\lambda'_{2n-1}(\xi) > 0$ on $(0, \pi)$, the inverse function theorem implies that there exists a holomorphic function $\zeta(z)$ on an open set containing (μ_{2n-1}, ν_{2n-1}) such that $\lambda(\zeta(z)) = z$. For each $\lambda \in (\mu_{2n-1}, \nu_{2n-1})$, if $\varepsilon > 0$ is small enough, $y_{\zeta(\lambda+i\varepsilon)}(x)$ is a solution to the equation $Ly = (\lambda + i\varepsilon)y$. Taking the complex conjugate of this equation and replacing ε by $-\varepsilon$, we obtain that $\overline{y_{\zeta(\lambda-i\varepsilon)}(x)}$ is also a solution. Since $\zeta'(\lambda) > 0$, we obtain the linearly independent solutions to $Ly = (\lambda + i\varepsilon)y$:

$$\begin{aligned} y_{\zeta(\lambda+i\varepsilon)}(x) &= e^{i\zeta(\lambda+i\varepsilon)x} u_{\zeta(\lambda+i\varepsilon)}(x) = \exp[(i\zeta(\lambda) - \varepsilon\zeta'(\lambda) + O(\varepsilon^2))x] u_{\zeta(\lambda+i\varepsilon)}(x), \\ \overline{y_{\zeta(\lambda-i\varepsilon)}(x)} &= e^{-i\overline{\zeta(\lambda-i\varepsilon)}x} \overline{u_{\zeta(\lambda-i\varepsilon)}(x)} = \exp[(-i\zeta(\lambda) + \varepsilon\zeta'(\lambda) + O(\varepsilon^2))x] \overline{u_{\zeta(\lambda-i\varepsilon)}(x)}. \end{aligned}$$

Let $[y, \tilde{y}](x) := a(x)(y(x)\tilde{y}'(x) - y'(x)\tilde{y}(x))$ be the Wronskian of two solutions y and $\tilde{y}$. Then

$$G_{\lambda+i\varepsilon}(x, y) = \begin{cases} y_{\zeta(\lambda+i\varepsilon)}(x) \overline{y_{\zeta(\lambda-i\varepsilon)}(y)} / [y_{\zeta(\lambda+i\varepsilon)}, \overline{y_{\zeta(\lambda-i\varepsilon)}}](0), & y \leq x, \\ y_{\zeta(\lambda+i\varepsilon)}(y) \overline{y_{\zeta(\lambda-i\varepsilon)}(x)} / [y_{\zeta(\lambda+i\varepsilon)}, \overline{y_{\zeta(\lambda-i\varepsilon)}}](0), & x \leq y, \end{cases}$$

(cf. §5.3 in [E]). Since $[y_{\zeta(\lambda+i\varepsilon)}, \overline{y_{\zeta(\lambda-i\varepsilon)}}](x)$ is a constant independent of x and $\zeta(\lambda + i\varepsilon) = \overline{\zeta(\lambda - i\varepsilon)}$, it follows that

$$\begin{aligned} &[y_{\zeta(\lambda+i\varepsilon)}, \overline{y_{\zeta(\lambda-i\varepsilon)}}](0) \\ &= \int_0^1 ([u_{\zeta(\lambda+i\varepsilon)}, \overline{u_{\zeta(\lambda-i\varepsilon)}}](x) - 2i\zeta(\lambda + i\varepsilon)a(x)u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(x)})dx. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_0^1 [a(x)(\frac{d}{dx} + i\zeta(\lambda + i\varepsilon))u_{\zeta(\lambda+i\varepsilon)}(x)(\frac{d}{dx} - i\zeta(\lambda + i\varepsilon))\overline{u_{\zeta(\lambda-i\varepsilon)}(x)} \\ &\quad + c(x)u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(x)}]dx = (\lambda + i\varepsilon)(u_{\zeta(\lambda+i\varepsilon)}, u_{\zeta(\lambda-i\varepsilon)}). \end{aligned}$$

Differentiating both sides of this equation with respect to λ , we have

$$\begin{aligned} &i\zeta'(\lambda + i\varepsilon) \int_0^1 ([u_{\zeta(\lambda+i\varepsilon)}, \overline{u_{\zeta(\lambda-i\varepsilon)}}](x) - 2i\zeta(\lambda + i\varepsilon)a(x)u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(x)})dx \\ &= (u_{\zeta(\lambda+i\varepsilon)}, u_{\zeta(\lambda-i\varepsilon)}). \end{aligned}$$

Thus

$$i\zeta'(\lambda + i\varepsilon)[y_{\zeta(\lambda+i\varepsilon)}, \overline{y_{\zeta(\lambda-i\varepsilon)}}](0) = (u_{\zeta(\lambda+i\varepsilon)}, u_{\zeta(\lambda-i\varepsilon)}).$$

Therefore we have

$$G_{\lambda+i\varepsilon}(x, y) = G_{\lambda+i\varepsilon}(y, x) = i\zeta'(\lambda + i\varepsilon)e^{i\zeta(\lambda+i\varepsilon)(x-y)} \frac{u_{\zeta(\lambda+i\varepsilon)}(x)\overline{u_{\zeta(\lambda-i\varepsilon)}(y)}}{(u_{\zeta(\lambda+i\varepsilon)}, u_{\zeta(\lambda-i\varepsilon)}), \quad y \leq x.$$

Taking the limit $\varepsilon \downarrow 0$, we have the existence of the limit $\lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon)f(x)$ and

$$G_{\lambda+i0}(x, y) = \lim_{\varepsilon \downarrow 0} G_{\lambda+i\varepsilon}(x, y) = \frac{ie^{i(x-y)\xi} u_\xi(x) \overline{u_\xi(y)}}{\lambda'_{2n-1}(\xi) \|u_\xi\|^2}, \quad y \leq x,$$

where $\xi = \zeta(\lambda)$, i.e., $\lambda_{2n-1}(\xi) = \lambda$. Furthermore, we can see that for any integer $m \geq 1$, the limit $\lim_{\varepsilon \downarrow 0} \left(\frac{d}{d\lambda}\right)^m R(\lambda \pm i\varepsilon)f(x)$ exists and

$$\begin{aligned} G_{\lambda+i0}^{(m)}(x, y) &= \lim_{\varepsilon \downarrow 0} \left(\frac{d}{d\lambda}\right)^m G_{\lambda+i\varepsilon}(x, y) \\ &= \left(\frac{i}{\lambda'_{2n-1}(\xi)}\right)^{m+1} (x-y)^m e^{i(x-y)\xi} \frac{u_\xi(x) \overline{u_\xi(y)}}{\|u_\xi\|^2} (1 + O(|x-y|^{-1})), \quad y \leq x. \end{aligned}$$

We have thus proved the case (I). The case (II) is proved in the same way as (I).

(III) Assume that $\lambda_{2n-1}(\pi) = \lambda_{2n}(\pi) = \nu_{2n-1} = \nu_{2n}$. Since ν_{2n} is a doubly degenerate eigenvalue and $L(\xi)$ is selfadjoint for ξ real, Theorem XII.13 in [RS] implies that there exist holomorphic eigenvalues $E_1(\zeta)$ and $E_2(\zeta)$ of $L(\zeta)$ near $\zeta = \pi$ such that $E_1(\pi) = E_2(\pi) = \nu_{2n}$. If $\xi \in \mathbf{R}$, each of $\lambda_{2n-1}(\xi)$ and $\lambda_{2n}(\xi)$ must be equal to one of $E_j(\xi)$, $j = 1, 2$. Since $D(E_j(\xi)) = 2 \cos \xi$ near $\xi = \pi$, we have

$$D''(E_j(\xi))E'_j(\xi)^2 + D'(E_j(\xi))E''_j(\xi) = -2 \cos \xi.$$

So, since $D'(\nu_{2n}) = 0$ and $D''(\nu_{2n}) > 0$, we obtain that $E'_j(\pi) \neq 0$ (which implies the fact (v) stated before Theorem 1). Since

$$\begin{cases} \lambda'_{2n-1}(\xi) > 0, & \xi < \pi, \\ \lambda'_{2n}(\xi) > 0, & \pi < \xi, \end{cases} \quad \text{and} \quad \begin{cases} \lambda'_{2n-1}(\xi) < 0, & \pi < \xi, \\ \lambda'_{2n}(\xi) < 0, & \xi < \pi, \end{cases}$$

we conclude that there exist holomorphic functions $E_1(\zeta)$ and $E_2(\zeta)$ on an open set containing $(0, 2\pi)$ such that

$$E_1(\xi) = \begin{cases} \lambda_{2n-1}(\xi), & 0 \leq \xi \leq \pi, \\ \lambda_{2n}(\xi), & \pi \leq \xi \leq 2\pi, \end{cases} \quad E_2(\xi) = \begin{cases} \lambda_{2n}(\xi), & 0 \leq \xi \leq \pi, \\ \lambda_{2n-1}(\xi), & \pi \leq \xi \leq 2\pi. \end{cases}$$

Since $E'_1(\xi) > 0$ on $(0, 2\pi)$, the inverse function theorem implies that there exists a holomorphic function $\zeta(z)$ on an open set containing (μ_{2n-1}, μ_{2n}) such that $E_1(\zeta(z)) = z$.

Let $p(\xi)$ be the eigenprojection for the eigenvalue $e^{i\xi}$ of $M(E_1(\xi))$ for $\xi \in (0, \pi) \cup (\pi, 2\pi)$:

$$\begin{aligned} p(\xi) &:= (-2\pi i)^{-1} \oint_{|z-e^{i\xi}|=\delta} (M(E_1(\xi)) - z)^{-1} dz \\ &= \frac{-1}{e^{i\xi} - e^{-i\xi}} \begin{pmatrix} s_2(1, E_1(\xi)) - e^{i\xi} & -s_1(1, E_1(\xi)) \\ -c_2(1, E_1(\xi)) & c_1(1, E_1(\xi)) - e^{i\xi} \end{pmatrix}, \end{aligned}$$

where $\delta > 0$ is taken so that $e^{i\xi}$ is the only eigenvalue of $M(E_1(\xi))$ inside the circle $|z - e^{i\xi}| = \delta$. Since $s_2(1, \nu_{2n}) + 1 = c_1(1, \nu_{2n}) + 1 = s_1(1, \nu_{2n}) = c_2(1, \nu_{2n}) = 0$ (cf. [E, p.7 and p.29]), $\xi = \pi$ is a removable singularity of $p(\xi)$. We have $(p(\xi))_{11} \neq 0$ on $(0, 2\pi)$, since

$$(p(\pi))_{11} = (2i)^{-1} \partial_\xi (s_2(1, E_1(\xi)) - e^{i\xi})|_{\xi=\pi} = (2i)^{-1} (\partial_z s_2(1, \nu_{2n}) E_1'(\pi) + i) \neq 0.$$

Thus $p(\xi)$ is a real analytic rank one matrix on $(0, 2\pi)$. Note that the holomorphically extended $p(\zeta)$ to an open set containing $(0, 2\pi)$ is the eigenprojection for the eigenvalue $e^{i\zeta}$ of $M(E_1(\zeta))$. Thus the function $y_\zeta(x) := (p(\zeta))_{11} c_1(x, E_1(\zeta)) + (p(\zeta))_{21} s_1(x, E_1(\zeta))$ is a solution to (2) with z replaced by $E_1(\zeta)$; and so $u_\zeta(x) = e^{-i\zeta x} y_\zeta(x)$ is a $C(\mathbf{T})$ -valued holomorphic eigenfunction of $L(\zeta)$ corresponding to $E_1(\zeta)$ on an open set containing $(0, 2\pi)$. Thus as in the case (I), since $\zeta'(\lambda) > 0$ for $\lambda \in (\mu_{2n-1}, \mu_{2n})$, $y_{\zeta(\lambda+i\varepsilon)}(x)$ and $\overline{y_{\zeta(\lambda-i\varepsilon)}(x)}$ are linearly independent solutions to $Ly = (\lambda + i\varepsilon)y$. Hence, as in the proof of (I) we have

$$G_{\nu_{2n}+i0}(x, y) = \lim_{\varepsilon \downarrow 0} G_{\nu_{2n}+i\varepsilon}(x, y) = \frac{ie^{i(x-y)\pi}}{E_1'(\pi)} \frac{u_\pi(x) \overline{u_\pi(y)}}{\|u_\pi\|^2}, \quad y \leq x,$$

and for any integer $m \geq 1$,

$$\begin{aligned} G_{\nu_{2n}+i0}^{(m)}(x, y) &= \lim_{\varepsilon \downarrow 0} \left(\frac{d}{d\lambda}\right)^m G_{\nu_{2n}+i\varepsilon}(x, y) \\ &= \left(\frac{i}{E_1'(\pi)}\right)^{m+1} (x-y)^m e^{i(x-y)\pi} \frac{u_\pi(x) \overline{u_\pi(y)}}{\|u_\pi\|^2} (1 + O(|x-y|^{-1})), \quad y \leq x. \end{aligned}$$

Note that $E_1'(\pi) = \lambda'_{2n-1}(\pi-0)$. We have thus proved (III). (IV) is proved similarly. From the proof above it follows that the convergence $\lim_{\varepsilon \downarrow 0} \left(\frac{d}{d\lambda}\right)^m R(\lambda \pm i\varepsilon)f(x)$ is locally uniform with respect to λ . $\square$

The following is a direct consequence of Theorem 1.

Corollary 2. *Let λ be in the interior of $\sigma(L)$. Then $\left(\frac{d}{d\lambda}\right)^m R(\lambda \pm i0)$, $m \geq 0$, is bounded from $B_{\frac{1}{2}+m}$ to $B_{\frac{1}{2}+m}^*$.*

Proof. Let $f \in C_0^\infty(\mathbf{R})$. Since Theorem 1 yields that

$$\left|\left(\frac{d}{d\lambda}\right)^m R(\lambda + i0)f(x)\right| \leq C_m(1+|x|)^m \int_{\mathbf{R}} (1+|y|)^m |f(y)| dy \leq C_m(1+|x|)^m \|f\|_{B_{\frac{1}{2}+m}},$$

it follows that

$$\left\|\left(\frac{d}{d\lambda}\right)^m R(\lambda + i0)f(x)\right\|_{B_{\frac{1}{2}+m}^*} \leq C_m \|(1+|x|)^m\|_{B_{\frac{1}{2}+m}^*} \|f\|_{B_{\frac{1}{2}+m}} \leq C_m \|f\|_{B_{\frac{1}{2}+m}}.$$

$\square$

Next we study the case that the parameter $\lambda \in \mathbf{R}$ is in the resolvent set of L . This case is equivalent to $|D(\lambda)| > 2$. $D(\lambda) > 2$ if and only if $\lambda \in A_+ := (-\infty, \mu_1) \cup [\cup_{n=1}^\infty (\mu_{2n}, \mu_{2n+1})]$; and $D(\lambda) < -2$ if and only if $\lambda \in A_- := \cup_{n=1}^\infty (\nu_{2n-1}, \nu_{2n})$. Consider a function $e^\eta + e^{-\eta}$ on $(0, \infty)$, and solve the equation

$$e^\eta + e^{-\eta} = D(\lambda)$$

with respect to η , where $\lambda \in A_+$. By the implicit function theorem, we have a unique solution $\eta(\lambda)$ which is real analytic on A_+ . Similarly, define $\eta(\lambda)$ on A_- by $e^\eta + e^{-\eta} = -D(\lambda)$. Note that $\dim \text{Ker}(L(\pm i\eta(\lambda)) - \lambda) = 1$ for $\lambda \in A_+$ and $\dim \text{Ker}(L(\pi \pm i\eta(\lambda)) - \lambda) = 1$ for $\lambda \in A_-$ (cf. [E, p.6]).

Theorem 3. (i) Let $\lambda \in A_+$. Let u_λ and v_λ be real-valued eigenfunctions of $L(i\eta(\lambda))$ and $L(-i\eta(\lambda))$ corresponding to the eigenvalue λ , respectively.

Suppose $D'(\lambda) \neq 0$. Then $(u_\lambda, v_\lambda) \neq 0$ and

$$G_\lambda(x, y) = G_\lambda(y, x) = -\eta'(\lambda)e^{-\eta(\lambda)(x-y)} \frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)}, \quad y \leq x. \quad (4)$$

Suppose $D'(\lambda) = 0$. Then there exists a solution $\psi_{v_\lambda} \in H^1(\mathbf{T})$ of the equation $(L(-i\eta(\lambda)) - \lambda)\psi = v_\lambda$ such that $(u_\lambda, \psi_{v_\lambda}) \neq 0$, and

$$G_\lambda(x, y) = G_\lambda(y, x) = -\frac{\eta''(\lambda)}{2}e^{-\eta(\lambda)(x-y)} \frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})}, \quad y \leq x. \quad (5)$$

(ii) Let $\lambda \in A_-$. Let u_λ and v_λ be eigenfunctions of $L(\pi + i\eta(\lambda))$ and $L(\pi - i\eta(\lambda))$ corresponding to the eigenvalue λ , respectively.

Suppose $D'(\lambda) \neq 0$. Then $(u_\lambda, v_\lambda) \neq 0$ and

$$G_\lambda(x, y) = G_\lambda(y, x) = -\eta'(\lambda)e^{(i\pi - \eta(\lambda))(x-y)} \frac{u_\lambda(x)\overline{v_\lambda(y)}}{(u_\lambda, v_\lambda)}, \quad y \leq x.$$

Suppose $D'(\lambda) = 0$. Then there exists a solution $\psi_{v_\lambda} \in H^1(\mathbf{T})$ of the equation $(L(\pi - i\eta(\lambda)) - \lambda)\psi = v_\lambda$ such that $(u_\lambda, \psi_{v_\lambda}) \neq 0$, and

$$G_\lambda(x, y) = G_\lambda(y, x) = -\frac{\eta''(\lambda)}{2}e^{(i\pi - \eta(\lambda))(x-y)} \frac{u_\lambda(x)\overline{v_\lambda(y)}}{(u_\lambda, \psi_{v_\lambda})}, \quad y \leq x.$$

Proof. Let $\lambda \in A_+$. Since $c_1(1, \lambda) - e^{\pm\eta(\lambda)}$ and $s_2(1, \lambda) - e^{\pm\eta(\lambda)} = e^{\mp\eta(\lambda)} - c_1(1, \lambda)$ do not vanish simultaneously on a neighborhood of each $\lambda \in A_+$, there exist nonzero real analytic eigenvectors $\alpha_\pm(\lambda) = (\alpha_{\pm,1}(\lambda), \alpha_{\pm,2}(\lambda))$ of $M(\lambda)$ corresponding to the eigenvalues $e^{\eta(\lambda)}$ and $e^{-\eta(\lambda)}$, respectively. Then $y_\lambda(x) := \alpha_{-,1}(\lambda)c_1(x, \lambda) + \alpha_{-,2}(\lambda)s_1(x, \lambda)$ and $z_\lambda(x) := \alpha_{+,1}(\lambda)c_1(x, \lambda) + \alpha_{+,2}(\lambda)s_1(x, \lambda)$ are solutions to (2) with ζ replaced by $i\eta(\lambda)$ and $-i\eta(\lambda)$. Thus $u_\lambda(x) := e^{\eta(\lambda)x}y_\lambda(x)$ and $v_\lambda(x) := e^{-\eta(\lambda)x}z_\lambda(x)$ are $C(\mathbf{T})$ -valued real analytic eigenfunctions on A_+ of $L(i\eta(\lambda))$ and $L(i\eta(\lambda))^* = L(-i\eta(\lambda))$ corresponding to the eigenvalue λ , respectively. Hence $y_\lambda(x) = e^{-\eta(\lambda)x}u_\lambda(x)$ and $z_\lambda(x) = e^{\eta(\lambda)x}v_\lambda(x)$ are linearly independent solutions, and so

$$G_\lambda(x, y) = \begin{cases} y_\lambda(x)z_\lambda(y)/[y_\lambda, z_\lambda](0), & y \leq x, \\ y_\lambda(y)z_\lambda(x)/[y_\lambda, z_\lambda](0), & x \leq y. \end{cases}$$

Since $[y_\lambda, z_\lambda](x)$ is a constant independent of x , it follows that

$$[y_\lambda, z_\lambda](0) = \int_0^1 ([u_\lambda, v_\lambda](x) + 2\eta(\lambda)a(x)u_\lambda(x)v_\lambda(x))dx.$$

On the other hand, we have

$$\int_0^1 [a(x)(\frac{d}{dx} - \eta(\lambda))u_\lambda(x)(\frac{d}{dx} + \eta(\lambda))v_\lambda(x) + c(x)u_\lambda(x)v_\lambda(x)]dx = \lambda(u_\lambda, v_\lambda).$$

Differentiating both sides of this equation with respect to λ , we have

$$-\eta'(\lambda) \int_0^1 ([u_\lambda, v_\lambda](x) + 2\eta(\lambda)a(x)u_\lambda(x)v_\lambda(x))dx = (u_\lambda, v_\lambda).$$

Hence

$$-\eta'(\lambda)[y_\lambda, z_\lambda](0) = (u_\lambda, v_\lambda). \quad (6)$$

Suppose $D'(\lambda) \neq 0$. Then $\eta'(\lambda) = D'(\lambda)/(e^{\eta(\lambda)} - e^{-\eta(\lambda)}) \neq 0$ and

$$G_\lambda(x, y) = -\eta'(\lambda)e^{-\eta(\lambda)(x-y)}u_\lambda(x)v_\lambda(y)/(u_\lambda, v_\lambda), \quad y \leq x.$$

Suppose $D'(\lambda) = 0$. Then $\eta'(\lambda) = 0$ and $\eta''(\lambda) = D''(\lambda)/(e^{\eta(\lambda)} - e^{-\eta(\lambda)}) < 0$. Differentiating (6), we have

$$\eta''(\lambda)[y_\lambda, z_\lambda](0) = -(u_\lambda, v_\lambda)'. \quad (7)$$

Therefore

$$G_\lambda(x, y) = -\eta''(\lambda)e^{-\eta(\lambda)(x-y)}u_\lambda(x)v_\lambda(y)/(u_\lambda, v_\lambda)', \quad y \leq x.$$

By (6), $(u_\lambda, v_\lambda) = 0$. Moreover, since $\eta'(\lambda) = 0$,

$$(L(i\eta(\lambda)) - \lambda)\partial_\lambda u_\lambda = u_\lambda \text{ and } (L(-i\eta(\lambda)) - \lambda)\partial_\lambda v_\lambda = v_\lambda. \quad (8)$$

Put $\psi_{v_\lambda} = \partial_\lambda v_\lambda$. Then ψ_{v_λ} is a solution of $(L(-i\eta(\lambda)) - \lambda)\psi = v_\lambda$. By (8), we have

$$(\partial_\lambda u_\lambda, v_\lambda) = (\partial_\lambda u_\lambda, (L(-i\eta(\lambda)) - \lambda)\partial_\lambda v_\lambda) = ((L(i\eta(\lambda)) - \lambda)\partial_\lambda u_\lambda, \partial_\lambda v_\lambda) = (u_\lambda, \partial_\lambda v_\lambda).$$

Thus $(u_\lambda, v_\lambda)' = 2(u_\lambda, \psi_{v_\lambda})$, which together with (7) implies that $(u_\lambda, \psi_{v_\lambda}) \neq 0$. Therefore we have (5). The assertion (ii) is proved similarly. $\square$

We have seen that in the formula (4) and (5) the different factor $\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)}$ or $\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})}$ appears according to whether $D'(\lambda)$ does not vanish or not. This is related to the Laurent expansion of $(L(i\eta(\lambda)) - z)^{-1}$ with respect to z around λ .

Proposition 4. *Let $\lambda \in A_+$. If $D'(\lambda) \neq 0$, the eigenvalue λ of $L(i\eta(\lambda))$ is nondegenerate and its eigenprojection has the integral kernel $\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)}$; and if $D'(\lambda) = 0$, the eigenvalue λ of $L(i\eta(\lambda))$ is degenerate and its eigennilpotent has the integral kernel $\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})}$. Similar statement holds for $\lambda \in A_-$.*

Proof. We shall represent the integral kernel $R(\zeta, z; x, y)$ of the resolvent $R(\zeta, z) := (L(\zeta) - z)^{-1}$, by using $c_j(x, z)$ and $s_j(x, z)$. Let $(\zeta, z) \in \Gamma := \{(\zeta, z) \in \mathbb{C}^2; z \notin \sigma(L(\zeta))\}$. Put

$$k(z; x, y) := \begin{cases} c_1(x, z)s_1(y, z), & y \leq x, \\ s_1(x, z)c_1(y, z), & x \leq y. \end{cases}$$

For $f \in C_0^\infty(0, 1)$, put

$$K_z f(x) := \int k(z; x, y) f(y) dy.$$

Since $(L - z)K_z f(x) = f(x)$ and $(L - z)e^{ix\zeta} R(\zeta, z)e^{-ix\zeta} f(x) = f(x)$ on $(0, 1)$, $e^{ix\zeta} R(\zeta, z)e^{-ix\zeta} f(x) - K_z f(x)$ is a solution to $Ly = zy$. Thus

$$e^{ix\zeta} R(\zeta, z)e^{-ix\zeta} f(x) - K_z f(x) = \alpha c_1(x, z) + \beta s_1(x, z) \quad (9)$$

for some α and β . Since $R(\zeta, z)e^{-ix\zeta} f(x) \in D(L(\zeta))$ has the periodicity, we get

$$K_z f(x) + \alpha c_1(x, z) + \beta s_1(x, z) = e^{-i\zeta} (K_z f(x+1) + \alpha c_1(x+1, z) + \beta s_1(x+1, z)), \quad (10)$$

so putting $x = 0$, we have

$$\alpha = e^{-i\zeta} [c_1(1, z) \int_0^1 s_1(y, z) f(y) dy + \alpha c_1(1, z) + \beta s_1(1, z)]. \quad (11)$$

Differentiating both sides of (10) with respect to x and putting $x = 0$, we have

$$\int_0^1 c_1(y, z) f(y) dy + \beta = e^{-i\zeta} [c_2(1, z) \int_0^1 s_1(y, z) f(y) dy + \alpha c_2(1, z) + \beta s_2(1, z)]. \quad (12)$$

Note that $(\zeta, z) \in \Gamma$ if and only if $\delta(\zeta, z) := D(z) - e^{i\zeta} - e^{-i\zeta} \neq 0$. Solving (11) and (12) with respect to (α, β) , we have

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \delta(\zeta, z)^{-1} \int_0^1 \left[\begin{pmatrix} s_1(1, z) \\ e^{i\zeta} - c_1(1, z) \end{pmatrix} c_1(y, z) + \begin{pmatrix} e^{-i\zeta} - c_1(1, z) \\ -c_2(1, z) \end{pmatrix} s_1(y, z) \right] f(y) dy.$$

Combining this with (9), we obtain that

$$R(\zeta, z; x, y) = e^{i\zeta(y-x)} k(z; x, y) + \frac{e^{i\zeta(y-x)} s(\zeta, z; x, y)}{D(z) - e^{i\zeta} - e^{-i\zeta}},$$

where

$$\begin{aligned} s(\zeta, z; x, y) := & [s_1(1, z)c_1(x, z) + (e^{i\zeta} - c_1(1, z))s_1(x, z)]c_1(y, z) \\ & + [(e^{-i\zeta} - c_1(1, z))c_1(x, z) - c_2(1, z)s_1(x, z)]s_1(y, z). \end{aligned}$$

Suppose $D'(\lambda) \neq 0$. For z near λ , we have $D(z) - e^{\eta(\lambda)} - e^{-\eta(\lambda)} = (z - \lambda)F_\lambda(z)$ for some $F_\lambda(z)$ such that $F_\lambda(\lambda) = D'(\lambda) \neq 0$. Thus $R(i\eta(\lambda), z; x, y)$ has a pole λ of order one with the residue

$$r_1(\lambda; x, y) := D'(\lambda)^{-1} e^{(x-y)\eta(\lambda)} s(i\eta(\lambda), \lambda; x, y).$$

This implies that the eigenvalue λ of $L(i\eta(\lambda))$ is nondegenerate and its eigenprojection has the integral kernel $-r_1(\lambda; x, y)$. On the other hand, the eigenprojection and its adjoint are

projections onto the spaces $\text{Ker}(L(i\eta(\lambda)) - \lambda)$ and $\text{Ker}(L(-i\eta(\lambda)) - \lambda)$, respectively, so the eigenprojection has the integral kernel $\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)}$. Therefore $\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)} = -r_1(\lambda; x, y)$.

Let $\lambda_0 \in \mathbf{R}$ satisfy $D'(\lambda_0) = 0$. For z near λ_0 , we have $D(z) - e^{\eta(\lambda_0)} - e^{-\eta(\lambda_0)} = (z - \lambda_0)^2 H(z)$ for some $H(z)$ such that $H(\lambda_0) = D''(\lambda_0)/2 \neq 0$. Thus $R(i\eta(\lambda_0), z; x, y)$ has a pole λ_0 of order two:

$$R(i\eta(\lambda_0), z; x, y) = r_2(x, y)(z - \lambda_0)^{-2} + O((z - \lambda_0)^{-1}),$$

where

$$r_2(x, y) := 2D''(\lambda_0)^{-1}e^{(x-y)\eta(\lambda_0)}s(i\eta(\lambda_0), \lambda_0; x, y).$$

Hence the eigenvalue λ_0 of $L(i\eta(\lambda_0))$ is degenerate and its eigennilpotent has the integral kernel $-r_2(x, y)$. We shall show that $\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)} = -r_2(x, y)$ at $\lambda = \lambda_0$. Since

$$\begin{aligned}\partial_z c_1(x, z) &= \int_0^x (c_1(x, z)s_1(t, z) - s_1(x, z)c_1(t, z))c_1(t, z) dt, \\ \partial_z s_2(x, z) &= \int_0^x (c_2(x, z)s_1(t, z) - s_2(x, z)c_1(t, z))s_1(t, z) dt\end{aligned}$$

(cf. [E]), we have for $\lambda \in A_+$

$$\begin{aligned}D'(\lambda) &= \partial_\lambda c_1(1, \lambda) + \partial_\lambda s_2(1, \lambda) \\ &= \int_0^1 [c_2(1, \lambda)s_1(x, \lambda)^2 + (c_1(1, \lambda) - s_2(1, \lambda))c_1(x, \lambda)s_1(x, \lambda) - s_1(1, \lambda)c_1(x, \lambda)^2] dx \\ &= - \int_0^1 s(i\eta(\lambda), \lambda; x, x) dx.\end{aligned}$$

As eigenfunctions of $L(i\eta(\lambda))$ and $L(-i\eta(\lambda))$ for $\lambda \in A_+$ near λ_0 , we can choose u_λ and v_λ as follows: (i) when $c_1(1, \lambda_0) - e^{-\eta(\lambda_0)} \neq 0$,

$$\begin{aligned}u_\lambda(x) &:= e^{\eta(\lambda)x}[-s_1(1, \lambda)c_1(x, \lambda) + (c_1(1, \lambda) - e^{-\eta(\lambda)})s_1(x, \lambda)], \\ v_\lambda(x) &:= e^{-\eta(\lambda)x}[(c_1(1, \lambda) - e^{-\eta(\lambda)})c_1(x, \lambda) + c_2(1, \lambda)s_1(x, \lambda)];\end{aligned}$$

(ii) when $c_1(1, \lambda_0) - e^{\eta(\lambda_0)} \neq 0$,

$$\begin{aligned}u_\lambda(x) &:= e^{\eta(\lambda)x}[(c_1(1, \lambda) - e^{\eta(\lambda)})c_1(x, \lambda) + c_2(1, \lambda)s_1(x, \lambda)], \\ v_\lambda(x) &:= e^{-\eta(\lambda)x}[-s_1(1, \lambda)c_1(x, \lambda) + (c_1(1, \lambda) - e^{\eta(\lambda)})s_1(x, \lambda)].\end{aligned}$$

Let us treat the former case. (The latter is done similarly.) We have

$$\begin{aligned}s_1(1, \lambda)c_2(1, \lambda) &= c_1(1, \lambda)s_2(1, \lambda) - 1 \\ &= c_1(1, \lambda)(e^{\eta(\lambda)x} + e^{-\eta(\lambda)x} - c_1(1, \lambda)) - 1 = (e^{\eta(\lambda)x} - c_1(1, \lambda))(c_1(1, \lambda) - e^{-\eta(\lambda)x}).\end{aligned}$$

Thus

$$\begin{aligned} u_\lambda(x)v_\lambda(y) &= -e^{\eta(\lambda)(x-y)}(c_1(1, \lambda) - e^{-\eta(\lambda)})s(i\eta(\lambda), \lambda; x, y), \\ (u_\lambda, v_\lambda) &= (c_1(1, \lambda) - e^{-\eta(\lambda)})D'(\lambda). \end{aligned}$$

So $(u_\lambda, v_\lambda)' = (c_1(1, \lambda) - e^{-\eta(\lambda)})D''(\lambda)$ at $\lambda = \lambda_0$. Therefore

$$\frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, \psi_{v_\lambda})} = 2 \frac{u_\lambda(x)v_\lambda(y)}{(u_\lambda, v_\lambda)'} = -2 \frac{e^{\eta(\lambda)(x-y)}s(i\eta(\lambda), \lambda; x, y)}{D''(\lambda)} = -r_2(x, y)$$

at $\lambda = \lambda_0$. We have thus shown the proposition. $\square$

Finally, we give an asymptotic expansion of the Green function $G_z(x, y)$ as the spectral parameter z approaches one of edges of the spectrum of L . We show it in a direct and elementary way, although the expansion of resolvents for Schrödinger operators with periodic potentials is given by [G, Corollary 4.2]. Let $\Delta_+ := \mathbb{C} \setminus [0, \infty)$. We denote by $z^{\frac{1}{2}}$ a branch of the square root of $z \in \Delta_+$ such that $z^{\frac{1}{2}} = \sqrt{r}e^{i\theta/2}$ for $z = re^{i\theta}$, $0 < \theta < 2\pi$, $r > 0$. Note that λ is an edge of the spectrum of L if and only if $|D(\lambda)| = 2$ and $D'(\lambda) \neq 0$. If $D(\lambda) = 2$ and $D'(\lambda) \neq 0$, there exist real-valued linearly independent solutions u and ψ of $Ly = \lambda y$ such that u is a real-valued periodic function with period 1 and $\psi(x) = xu(x) + v(x)$ for some real-valued periodic function v with period 1; if $D(\lambda) = -2$ and $D'(\lambda) \neq 0$, there exist real-valued linearly independent solutions u and ψ of $Ly = \lambda y$ such that u is a real-valued semi-periodic function with semi-period 1, i.e., $u(x+1) = -u(x)$, and $\psi(x) = xu(x) + v(x)$ for some real-valued semi-periodic function v with semi-period 1 (cf. [E, p.7 and p.29]).

Theorem 5. *Assume that μ_{2n-1} is an edge of the spectrum of L . Then for any integer $m \geq -1$ one has the expansion for small $z - \mu_{2n-1} \in \Delta_+$*

$$G_z(x, y) = \sum_{j=-1}^m (z - \mu_{2n-1})^{\frac{j}{2}} q_j(x, y) + r_m(z; x, y),$$

where $r_m(z; x, y)$ satisfies the estimate: for any $0 \leq \theta \leq 1$

$$|r_m(z; x, y)| \leq C_m |z - \mu_{2n-1}|^{(m+\theta)/2} (|x - y| + 1)^{m+1+\theta}.$$

Furthermore, $q_j(x, y)$ is of the form

$$q_j(x, y) = q_j(y, x) = \sum_{k=0}^{j+1} (x - y)^k q_{j,k}(x, y), \quad y \leq x,$$

for some $q_{j,k}(x, y) \in C(\mathbb{T} \times \mathbb{T})$. In particular,

$$\begin{aligned} q_{-1}(x, y) &= \frac{i}{\sqrt{2\lambda''_{2n-1}(0)}} \frac{u(x)u(y)}{\|u\|^2}, \\ q_0(x, y) &= q_0(y, x) = \lambda''_{2n-1}(0)^{-1} (u(x)\psi(y) - \psi(x)u(y)) / \|u\|^2, \quad y \leq x, \end{aligned}$$

where $\lambda''_{2n-1}(0) > 0$, and u and ψ are real-valued linearly independent solutions of $Ly = \mu_{2n-1}y$ such that u is a periodic function with period 1 and $\psi(x) = xu(x) + v(x)$ for some periodic function v with period 1.

Remark 6. If ν_{2n-1} , ν_{2n} , or μ_{2n} is an edge of the spectrum, a similar expansion holds around it.

Proof. Since $D(\mu_{2n-1}) = 2$ and $D'(\mu_{2n-1}) < 0$, there exists a holomorphic inverse function D^{-1} of D near $D = 2$. Put $\lambda(\zeta) = D^{-1}(e^{i\zeta} + e^{-i\zeta})$ near $\zeta = 0$. Then $\lambda(\xi) = \lambda_{2n-1}(\xi) \geq \mu_{2n-1}$ for small $\xi \in \mathbf{R}$ and $\lambda'(0) = 0$. Furthermore, since $D(\lambda(\xi)) = 2 \cos \xi$, we have

$$D''(\lambda(\xi))\lambda'(\xi)^2 + D'(\lambda(\xi))\lambda''(\xi) = -2 \cos \xi.$$

This implies that $\lambda''(0) = -2/D'(\mu_{2n-1}) > 0$. Therefore we can choose a sufficiently small positive number R such that the set $\{\lambda(\zeta); \operatorname{Im} \zeta > 0, |\zeta| < R\}$ is a subdomain of $\mathbf{C} \setminus [\mu_{2n-1}, \infty)$. We have also that $s_1(1, \mu_{2n-1})$ and $c_2(1, \mu_{2n-1})$ are not both zero (cf. [E, p.29]). So we can choose a holomorphic eigenvector $(\alpha_1(\zeta), \alpha_2(\zeta))$ of $M(\lambda(\zeta))$ corresponding to the eigenvalue $e^{i\zeta}$ near $\zeta = 0$. Put $y_\zeta(x) := \alpha_1(\zeta)c_1(x, \lambda(\zeta)) + \alpha_2(\zeta)s_1(x, \lambda(\zeta))$. Then $u_\zeta(x) := e^{-i\zeta x}y_\zeta(x)$ is a holomorphic eigenfunction of $L(\zeta)$ corresponding to the eigenvalue $\lambda(\zeta)$ near $\zeta = 0$. Let $\mathbf{C}_+ := \{\zeta \in \mathbf{C}; \operatorname{Im} \zeta > 0\}$. For small $\zeta \in \mathbf{C}_+$, since $\overline{\lambda(\zeta)} = \lambda(\bar{\zeta})$, it follows that $y_\zeta = e^{i\zeta x}u_\zeta$ and $\overline{y_\zeta} = e^{-i\bar{\zeta}x}\overline{u_\zeta}$ are linearly independent solutions to $Ly = \lambda(\zeta)y$. Hence as in the proof of Theorem 1, since $i[y_\zeta, \overline{y_\zeta}](0) = \lambda'(\zeta)(u_\zeta, u_{\bar{\zeta}})$, we have for small $\zeta \in \mathbf{C}_+$

$$G_{\lambda(\zeta)}(x, y) = G_{\lambda(\zeta)}(y, x) = y_\zeta(x)\overline{y_\zeta(y)}/[y_\zeta, \overline{y_\zeta}](0) = i\lambda'(\zeta)^{-1}e^{i(x-y)\zeta}p_\zeta(x, y), \quad y \leq x, \quad (13)$$

where $p_\zeta(x, y) := u_\zeta(x)\overline{u_\zeta(y)}/(u_\zeta, u_{\bar{\zeta}})$ is a $C(\mathbf{T} \times \mathbf{T})$ -valued holomorphic function near $\zeta = 0$. Let $y \leq x$. We write the Taylor expansion of $e^{i(x-y)\zeta}p_\zeta(x, y)$ with respect to ζ as follows:

$$e^{i(x-y)\zeta}p_\zeta(x, y) = \sum_{j=0}^m \tilde{q}_j(x, y)\zeta^j + \tilde{r}_m(\zeta; x, y), \quad (14)$$

where

$$\tilde{q}_j(x, y) = \sum_{k=0}^j (x-y)^k \tilde{q}_{j,k}(x, y) \quad (15)$$

for some $\tilde{q}_{j,k}(x, y) \in C(\mathbf{T} \times \mathbf{T})$, and $\tilde{r}_m(\zeta; x, y)$ satisfies the estimate: for any $0 \leq \theta \leq 1$

$$|\tilde{r}_m(\zeta; x, y)| \leq C_m |\zeta|^{m+\theta} (|x-y|+1)^{m+\theta}. \quad (16)$$

Let us show this remainder estimate. We have

$$e^{i(x-y)\zeta} = \sum_{j=0}^m \frac{(i(x-y)\zeta)^j}{j!} + \frac{(i(x-y)\zeta)^{m+1}}{m!} \int_0^1 (1-t)^m e^{it(x-y)\zeta} dt.$$

Thus

$$\left| e^{i(x-y)\zeta} - \sum_{j=0}^m \frac{(i(x-y)\zeta)^j}{j!} \right| \leq \frac{(|x-y||\zeta|)^{m+1}}{(m+1)!},$$

since $\operatorname{Re}[it(x-y)\zeta] \leq 0$. This implies that

$$|\tilde{r}_m(\zeta; x, y)| \leq C_m |\zeta|^{m+1} (|x-y|+1)^{m+1}.$$

On the other hand, since

$$\tilde{r}_m(\zeta; x, y) = \tilde{r}_{m-1}(\zeta; x, y) - \tilde{q}_m(x, y)\zeta^m,$$

we have

$$|\tilde{r}_m(\zeta; x, y)| \leq C_m |\zeta|^m (|x - y| + 1)^m.$$

Hence we get the desired estimate (16). We see that $\tilde{q}_0(x, y) = p_0(x, y)$ and $\tilde{q}_1(x, y) = i(x - y)p_0(x, y) + \partial_\zeta p_\zeta(x, y)|_{\zeta=0}$. We shall show that $\tilde{q}_1(x, y) = i(\psi(x)u(y) - u(x)\psi(y))/\|u\|^2$, where $u(x)$ and $\psi(x) = xu(x) + v(x)$ are linearly independent solutions stated in the theorem. We have

$$\begin{aligned} \partial_\zeta y_\zeta|_{\zeta=0} &= \alpha'_1(0)c_1(x, \mu_{2n-1}) + \alpha'_2(0)s_1(x, \mu_{2n-1}) = ixu_0 + \partial_\zeta u_\zeta|_{\zeta=0}, \\ \partial_\zeta \bar{y}_\zeta|_{\zeta=0} &= \overline{\alpha'_1(0)}c_1(x, \mu_{2n-1}) + \overline{\alpha'_2(0)}s_1(x, \mu_{2n-1}) = -ix\bar{u}_0 + \partial_\zeta \bar{u}_\zeta|_{\zeta=0}. \end{aligned}$$

So $\partial_\zeta y_\zeta|_{\zeta=0}$ and $\partial_\zeta \bar{y}_\zeta|_{\zeta=0} = \overline{\partial_\zeta y_\zeta|_{\zeta=0}}$ are solutions of $Ly = \mu_{2n-1}y$, and we have $u_0 = cu$ and $\partial_\zeta y_\zeta|_{\zeta=0} = icv + c'u$ for some $c, c' \in \mathbb{C}$. Hence

$$\partial_\zeta u_\zeta|_{\zeta=0} = icv(x) + c'u(x), \quad \partial_\zeta \bar{u}_\zeta|_{\zeta=0} = -i\bar{c}v(x) + \bar{c}'u(x).$$

Using this we have

$$\begin{aligned} \tilde{q}_1(x, y) &= i(x - y)p_0(x, y) + \partial_\zeta p_\zeta(x, y)|_{\zeta=0} \\ &= i(x - y)p_0(x, y) + \frac{\partial_\zeta(u_\zeta(x)\bar{u}_\zeta(y))|_{\zeta=0}}{\|u_0\|^2} - p_0(x, y)\frac{(u_\zeta, u_\zeta)'|_{\zeta=0}}{\|u_0\|^2} \\ &= i(x - y)\frac{u(x)u(y)}{\|u\|^2} + \frac{(icv(x) + c'u(x))\bar{c}u(y) + cu(x)(-i\bar{c}v(y) + \bar{c}'u(y))}{|c|^2\|u\|^2} \\ &\quad - \frac{u(x)u(y)}{\|u\|^2} \frac{2\operatorname{Re}(icv + c'u, cu)}{|c|^2\|u\|^2} \\ &= i(x - y)u(x)u(y)/\|u\|^2 + i(v(x)u(y) - u(x)v(y))/\|u\|^2 \\ &= i(\psi(x)u(y) - u(x)\psi(y))/\|u\|^2. \end{aligned}$$

There exists an entire function $F(z)$ such that $F(\zeta^2) = e^{i\zeta} + e^{-i\zeta} - 2$; $F(z)$ is real for real z , $F(0) = 0$, and $F'(0) = -1$. So there exists an inverse function F^{-1} of F near the origin. Thus for $\delta > 0$ small, the map $z \in \{z \in \Delta_+ + \mu_{2n-1}; |z - \mu_{2n-1}| < \delta\} \mapsto \zeta(z) := (F^{-1}(D(z) - 2))^{\frac{1}{2}} \in \mathbb{C}_+$ is conformal from the disc with the cut to the intersection of a neighborhood of the origin and $\mathbb{C}_+$. Note that $\lambda(\zeta(z)) = z$. Noting that $D(z) - 2 = D'(\mu_{2n-1})(z - \mu_{2n-1}) + O((z - \mu_{2n-1})^2)$ and $F^{-1}(w) = -w + O(w^2)$, we have the Puiseux series

$$\zeta(z) = \sum_{j=0}^{\infty} a_j(z - \mu_{2n-1})^{j+\frac{1}{2}}, \quad (17)$$

where $a_0 = \sqrt{|D'(\mu_{2n-1})|} = \sqrt{2/\lambda''_{2n-1}(0)}$. Note that $\lambda'(\zeta(z))^{-1} = \zeta'(z)$. By (13), (14) and (17),

$$\begin{aligned} G_z(x, y) &= i\zeta'(z)e^{i(x-y)\zeta(z)}p_{\zeta(z)}(x, y) \\ &= i\left[\sum_{j=0}^{\infty} a_j\left(j + \frac{1}{2}\right)(z - \mu_{2n-1})^{j-1/2}\right]\left[\sum_{j=0}^m \tilde{q}_j(x, y)\zeta(z)^j + \tilde{r}_m(\zeta(z); x, y)\right] \\ &= \sum_{j=-1}^m (z - \mu_{2n-1})^{j/2} q_j(x, y) + r_m(z; x, y). \end{aligned}$$

This together with (15) and (16) yields the desired expansion. $\square$

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