

Simple setting for white noise calculus using Bargmann space

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§1. Notations

Let E_0 be a real separable Hilbert space with $\dim E_0 = \infty$ and $(\cdot, \cdot)_0$ be its inner product. Let D be a densely defined selfadjoint operator of E_0 such that $D > 1$ and D^{-1} is of Hilbert-Schmidt type. Further we assume that the eigen system of D^{-1} ;

$$\{(\lambda_j, \zeta_j)\}_{j=0}^{\infty} \quad \text{with} \quad D^{-1}\zeta_j = \lambda_j\zeta_j \quad (j = 0, 1, 2, \dots)$$

satisfies

$$1 > \lambda_j \geq \lambda_{j+1} \quad (j = 0, 1, \dots)$$

and that $\{\zeta_j; j = 0, 1, 2, \dots\}$ is an orthonormal basis of E_0 . The following constants t_0 , s_0 , and p_0 will appear frequently;

$$t_0 = -\log 2 / 2 \log \lambda_0, \text{ i.e., } 1/2 = \lambda_0^{2t_0},$$

$$s_0 = \inf\{s; \sum_{j=0}^{\infty} \lambda_j^{2s} < \infty\},$$

$$p_0 = \max(t_0, s_0).$$

Since $\|D^{-1}\|_{\text{HS}}^2 = \sum_{j=0}^{\infty} \lambda_j^2 < \infty$ is finite, s_0 is in $[0, 1]$.

For any real number $p > 0$ write E_p = the domain of D^p and define the inner product $(x, y)_p$ for $x, y \in E_p$ by

$$(x, y)_p = (D^p x, D^p y)_0.$$

Then $(E_p, (\cdot, \cdot)_p)$ is a Hilbert space. If $0 \leq q < p$, then $E_p \subset E_q$. Every E_p contains ζ_j 's, and so $E \equiv \bigcap_{p>0} E_p$ is not empty. Set $\|\xi\|_p = \sqrt{(\xi, \xi)_p}$ for $\xi \in E$. The system of norms $\{\|\xi\|_p; p \geq 0\}$ is compatible. Since D^{-1} is of Hilbert-Schmidt type, the space E equipped with the projective limit topology of

$\{(E_p, \|\cdot\|_p); p > 0\}$ is a nuclear space. We can easily see that $D^p(E_p) = E_0$ for $p > 0$. For $p > 0$, let E_{-p} be the completion of E_0 with respect to the norm $\|\cdot\|_{-p} \equiv \|D^{-p}\cdot\|_0$. Clearly, if $0 \leq q < p$, then $E_0 \subset E_{-q} \subset E_{-p}$. Let $E^* = \cup_{p>0} E_{-p}$ and let it be equipped with the inductive limit topology of $\{(E_{-p}, \|\cdot\|_{-p}); p > 0\}$. We have $E \subset E_0 \subset E^*$. Once the increasing family $\{E_p; p \in \mathbb{R}\}$ of Hilbert spaces is set, the operator D^q ($q \in \mathbb{R}$) acts naturally and isometrically as:

$$D^q: E_p \longrightarrow E_{p-q} \text{ (surjective) } (p \in \mathbb{R}),$$

and so it acts continuously on E^* with respect to the inductive limit topology. We can naturally identify the dual space of E_p with E_{-p} ($p \in \mathbb{R}$).

Let H_p be the complexification of E_p , i.e., $H_p = E_p + \sqrt{-1}E_p$. Then D^q extends to an isometry from H_p onto H_{p-q} naturally by setting

$$D^q(x + \sqrt{-1}y) = D^q x + \sqrt{-1}D^q y \quad \text{for } x, y \in E_p \text{ } (p, q \in \mathbb{R}).$$

Accordingly the real spaces E and E^* also have their complexifications H and H^* , respectively. The letters w and z are often used for elements of H^* or H_{-p} and letters x and y for ones of E^* or E_{-p} , where $p \geq 0$. Like in the real case, the operator D^q acts as an isometry from H_p onto H_{p-q} . Hence D^q acts on H^* continuously. Obviously we see that

$$\langle D^q w, \zeta \rangle = \langle w, D^q \zeta \rangle$$

holds for any $w \in H^*$ and any $\zeta \in H$. Where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form. That is, suppose that X is a locally convex topological vector space and X^* the dual of X . Then the value of x^* at x defined by each pair $(x, x^*) \in X \times X^*$ is always denoted by $\langle x^*, x \rangle$. This is linear in both arguments x and x^* .

Denote by $\mathcal{P}(X^*)$ the space of all polynomials in $\{\langle x^*, x \rangle; x$

$\in X\}$ over $\mathbb{C}$; that is,

$$\mathcal{P}(X^*) = \{\text{any finite sum of } c\Pi_j \langle x^*, x_j \rangle; x_j \in X, c \in \mathbb{C}\}.$$

If X is a nuclear space or Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, then the n -fold symmetric tensor product of X is denoted by $X^{\otimes n}$. If $x_1, x_2, \dots, x_n \in X$, then $\otimes_{j=1}^n x_j$ is the symmetrization of $x_1 \otimes x_2 \otimes \dots \otimes x_n$. In particular the n -fold tensor product of x is denoted by $x^{\otimes n}$.

The following notations on infinite-dimensional indices of nonnegative integers will be used.

$$\mathcal{N} = \{\text{all sequences of nonnegative integers}\}.$$

$$\mathcal{N}_0 = \{n = (n_0, n_1, n_2, \dots); n \in \mathcal{N}, n_j = 0 \text{ for almost all } j\}.$$

For $n, k \in \mathcal{N}_0$, write $n \geq k$ if and only if $n_j \geq k_j$ ($j \geq 0$). For $n, k \in \mathcal{N}_0$ and a nonnegative integer p , define

$$pn = (pn_0, pn_1, pn_2, \dots), \quad |n| = n_0 + n_1 + n_2 + \dots,$$

$$n \wedge k = (n_0 \wedge k_0, n_1 \wedge k_1, n_2 \wedge k_2, \dots),$$

$$n! = \prod_j n_j! \quad \text{and} \quad \binom{n}{k} = \prod_j \binom{n_j}{k_j}.$$

For $r \in \mathbb{R}$ and $n \in \mathcal{N}_0$ with $|n| = n$, the symbols λ^{rn} , $\zeta^{\otimes n}$, h_n and z^n are defined as follows:

$$\lambda^{rn} = \prod_j \lambda_j^{rn_j},$$

$$\zeta^{\otimes n} = \otimes_{n_j \neq 0} \zeta_j^{\otimes n_j} = \text{the symmetrization of } \otimes_{n_j \neq 0} \zeta_j^{\otimes n_j},$$

$$z^n = z^n(z) = (2^{n_n!})^{-1/2} \langle z^{\otimes n}, \zeta^{\otimes n} \rangle \text{ for } z \in H^*, \quad (1.1)$$

$$h_n = h_n(x) = (2^{n_n!})^{-1/2} \prod_j H_{n_j} \left(\langle x, \zeta_j \rangle / \sqrt{2} \right) \text{ for } x \in E^*, \quad (1.2)$$

where $\{(\lambda_j, \zeta_j)\}_{j=0}^\infty$ is the eigen system of D^{-1} and $H_n(u)$ is the Hermite polynomial of n degrees defined by

$$H_n(u) = (-1)^n \exp[u^2] (d/du)^n \exp[-u^2].$$

$\mathcal{B}$ is the smallest σ -algebra containing all cylindrical sets

of E^* . Here, cylindrical sets of E^* are subsets of E^* of the form:

$$\{x \in E^*; (\langle x, \xi_1 \rangle, \dots, \langle x, \xi_n \rangle) \in B_n\}$$

where n is any integer ≥ 1 , B_n is any n -dimensional Borel set, and $\xi_1, \dots, \xi_n$ are any elements of E .

§2. The space of white noise functionals (L^2), the Bargmann space ($\mathfrak{F}_0$) over a nuclear space, and Gauss transform G

The functional $C(\xi) = \exp[-\frac{1}{2}\|\xi\|_0^2]$ of ξ is positive definite and continuous on the nuclear space E . By Bochner-Minlos' theorem there exists a unique Gaussian probability measure μ in the measurable space $(E^*, \mathfrak{B})$ such that

$$\int_{E^*} \exp[\sqrt{-1}\langle x, \xi \rangle] d\mu(x) = C(\xi),$$

(Minlos [M]). Since D^{-s} for $s > s_0$ is of Hilbert-Schmidt type, $\mu(E_{-s}) = 1$ holds. Hence, when a functional is defined on E_{-s} for $s > s_0$, then we may consider that it is given μ -a.e. on E^* .

The space $L^2(E^*, \mathfrak{B}, \mu)$ is called the space of white noise functionals and denoted by (L^2) (Hida [H1], [H2]). Then $\mathcal{P}(E^*)$, the space of all polynomials in $\{\langle x, \xi \rangle; \xi \in E\}$ over $\mathbb{C}$, is dense in (L^2) . $\{h_m; m \in \mathcal{N}_0\}$ is a complete orthonormal system of (L^2) . From now on let CONS stand for complete orthonormal system.

Let us consider the product measure $\nu = \mu \times \mu$ in the space $H^* = E^* + \sqrt{-1}E^*$. Then the system $\{z^m; m \in \mathcal{N}_0\}$ of (1.1) is orthonormal in the space $L^2(H^*, \nu)$. A Bargmann space ($\mathfrak{F}_0$) is the closure of $\mathcal{P}(H^*)$ in $L^2(H^*, \nu)$, where $\mathcal{P}(H^*)$ is the space of all polynomials in $\{\langle z, \xi \rangle; \xi \in H\}$ over $\mathbb{C}$. It is well-known that the space of all entire functions, $\mathfrak{F}(\mathbb{C}^n)$, which are defined on $\mathbb{C}^n$ and square integrable with respect to

$$dg(z) = (2\pi)^{-n} \exp[-(z\bar{z})/2] (\sqrt{-1}/2)^n dzd\bar{z}$$

is closed in $L^2(\mathbb{C}^n, dg(z))$, (see Bargmann [B1]). $(\mathfrak{F}_0)$ is an analogue of $\mathfrak{F}(\mathbb{C}^n)$ in passing from $\mathbb{C}^n$ to the infinite dimensional space H^* . But the element of $(\mathfrak{F}_0)$ is in general not analytic in H^* . Nevertheless $(\mathfrak{F}_0)$ is isometrically isomorphic to a normed space consisting of specific analytic functionals in H_0 (see Kondrat'ev [K2]). If we introduce a nuclear rigging $(\mathfrak{F}) \subset (\mathfrak{F}_p) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}_{-p}) \subset (\mathfrak{F}')$, we can see this situation more clearly. The construction of the nuclear rigging and the problem of analytic functionals will be discussed in detail in §3, (see also Berezansky and Kondrat'ev [B-K]).

Now let us discuss the map G from $\mathcal{P}(E^*)$ onto $\mathcal{P}(H^*)$ defined as follows: every $\varphi(x) \in \mathcal{P}(E^*)$ can be naturally and analytically extended to $\varphi(z) \in \mathcal{P}(H^*)$ replacing $\langle x, \xi \rangle$ by $\langle z, \xi \rangle$. We can define a map G on $\mathcal{P}(E^*)$ by

$$G\varphi(w) = \int_{E^*} \varphi(x + w/\sqrt{2}) d\mu(x) \quad \text{for } \varphi \in \mathcal{P}(E^*). \quad (2.1)$$

Then obviously, $G\varphi$ belongs to $\mathcal{P}(H^*)$. Its inverse map G^{-1} is given by

$$G^{-1}f(x) = \int_{E^*} f(\sqrt{2}(x + \sqrt{-1}y)) d\mu(y) \quad \text{for } f \in \mathcal{P}(H^*). \quad (2.2)$$

Actually, we can see that

$$Gh_n = z^n \quad \text{and} \quad G^{-1}z^n = h_n. \quad (2.3)$$

Since $\{h_n; n \in \mathcal{N}_0\}$ and $\{z_n; n \in \mathcal{N}_0\}$ are CONS' in (L^2) and $(\mathfrak{F}_0)$ respectively, the map G extends to an isometry from (L^2) onto $(\mathfrak{F}_0)$:

$$\|G\varphi\|_{(\mathfrak{F}_0)} = \|\varphi\|_{(L^2)} \quad \text{for } \varphi \in (L^2). \quad (2.4)$$

The map G given by (2.1) is often called Gauss transform ([B&K],[H2],[K2]), so we also call this isometric isomorphism $G:\mathcal{P}(E^*) \longrightarrow \mathcal{P}(H^*)$ or its extension from (L^2) onto $(\mathfrak{F}_0)$ *Gauss*

transform. The integral expression (2. 1) of G (resp. (2. 2) of G^{-1}) is not valid on (L^2) (resp. on $(\mathfrak{F}_0)$). But we will show in a forthcoming paper that these expressions can extend to the ones between much wider spaces than $\mathcal{S}(E^*)$ and $\mathcal{S}(H^*)$.

§3. The Gel'fand triplet $(\mathfrak{F}) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}') \text{ rigged by the operator } \Lambda(D^p)$

Let D be the self-adjoint operator of H_0 introduced in §1. Since D^p act on H^* naturally and continuously, we can define operators $\Lambda(D^p)$ on $\mathcal{S}(H^*)$ by

$$\Lambda(D^p)f(z) = f(D^p z) \text{ for } f \in \mathcal{S}(H^*), \quad (3. 1)$$

where $z \in H^*$ and $p \in \mathbb{R}$. Let $f(z) = \prod_{j=1}^n \langle z, \xi_j \rangle \in \mathcal{S}(H^*)$. Then, by the relation

$$\Lambda(D^p)f(z) = \prod_{j=1}^n \langle D^p z, \xi_j \rangle = \prod_{j=1}^n \langle z, D^p \xi_j \rangle$$

we see that $\{(\lambda^{-p\mathfrak{n}}, z^{\mathfrak{n}}); \mathfrak{n} \in \mathcal{N}_0\}$ is an eigen system of $\Lambda(D^p)$:

$$\Lambda(D^p)z^{\mathfrak{n}}(z) = \left(\prod_j \lambda_j^{-p\mathfrak{n}_j} \right) z^{\mathfrak{n}}(z) = \lambda^{-p\mathfrak{n}} z^{\mathfrak{n}}(z). \quad (3. 2)$$

As is easily seen, $\mathcal{S}(H^*)$ is a pre-Hilbert space with the inner product

$$(\Lambda(D^p)f, \Lambda(D^p)g)_{(\mathfrak{F}_0)} = \int_{H^*} \left(\Lambda(D^p)f(z) \right) \overline{\Lambda(D^p)g(z)} d\nu(z). \quad (3. 3)$$

We will denote its completion by $(\mathfrak{F}_p)$ and the inner product by $(f, g)_{(\mathfrak{F}_p)}$. As well as in the case of D^q , we can see that the operator $\Lambda(D^q)$ is an isometry from the Hilbert space $(\mathfrak{F}_p)$ onto the Hilbert space $(\mathfrak{F}_{p-q})$. We can easily see the following:

PROPOSITION 3. 1. For any $p \in \mathbb{R}$, $\{\lambda^{p\mathfrak{n}} z^{\mathfrak{n}}; \mathfrak{n} \in \mathcal{N}_0\}$ is a CONS of $(\mathfrak{F}_p)$. And hence any $f \in (\mathfrak{F}_p)$ can be expressed in the form

$$f = \sum_{\mathfrak{n} \in \mathcal{N}_0} c_{\mathfrak{n}} z^{\mathfrak{n}} \quad (3. 4)$$

with coefficients $\{c_{\mathfrak{n}}; \mathfrak{n} \in \mathcal{N}_0\}$ satisfying

$$\|f\|_{(\mathfrak{F}_p)}^2 = \sum_{n \in \mathcal{N}_0} \lambda^{-2pn} |c_n|^2 < \infty. \quad (3.5)$$

Furthermore, we have that for $f \in (\mathfrak{F}_p)$ of the form (3.4)

$$\Lambda(D^q)f = \sum_{n \in \mathcal{N}_0} \lambda^{-qn} c_n z^n \in (\mathfrak{F}_{p-q}). \quad (3.6)$$

By the proposition, we can identify $(\mathfrak{F}_{-p})$ with the dual space of $(\mathfrak{F}_p)$ and get, for $p > q > 0$,

$$(\mathfrak{F}_p) \subset (\mathfrak{F}_q) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}_{-q}) \subset (\mathfrak{F}_{-p}).$$

Since D^{-1} is of Hilbert-Schmidt type, it follows that for any $p \in \mathbb{R}$ and for any $s > s_0$

$$\sum_{n \in \mathcal{N}_0} \|\lambda^{(p+s)n} z^n\|_{(\mathfrak{F}_p)}^2 = \prod_j (1 - \lambda_j^{2s})^{-1} < \infty. \quad (3.7)$$

This shows that the canonical injection from $(\mathfrak{F}_{p+s})$ into $(\mathfrak{F}_p)$ is also of Hilbert-Schmidt type. If we write

$$(\mathfrak{F}) = \cap_{p=0}^{\infty} (\mathfrak{F}_p) \quad \text{and} \quad (\mathfrak{F}') = \cup_{p=0}^{\infty} (\mathfrak{F}_{-p}), \quad (3.8)$$

then the dual space of $(\mathfrak{F})$ is $(\mathfrak{F}')$. Thus we obtain a Gel'fand triplet $(\mathfrak{F}) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}')$. The following is known: the triplet of this type has a "holomorphic realization" given by analytic functionals of at most order 2 (ref. [B-K],[K2]). Within our setting let us reform this as:

PROPOSITION 3.2. For any $p \in \mathbb{R}$, $f \in (\mathfrak{F}_p)$ with the expression (3.4),

$$\sum_{n \in \mathcal{N}_0} c_n z^n(z) \quad (3.9)$$

converges absolutely and uniformly to a functional $\tilde{f}(z)$ on any bounded set of H_{-p} . The limit functional $\tilde{f}(z)$ satisfies

$$|\tilde{f}(z)| \leq \exp\left[\frac{1}{4}\|z\|_{-p}^2\right] \|f\|_{(\mathfrak{F}_p)} \quad \text{for any } z \in H_{-p}. \quad (3.10)$$

Further $\tilde{f}(z)$ is not only continuous but analytic in H_{-p} in the sense of [H-P] (E. Hille & R. S. Phillips).

PROOF. By Schwarz' inequality and (1. 1), we have that for any $z \in H_{-p}$

$$\begin{aligned}
 & \sum_{n \in \mathcal{N}_0} |c_n z^n(z)| \\
 &= \sum_{n=0}^{\infty} \sum_{|n|=n} |c_n z^n(z)| \\
 &= \sum_{n=0}^{\infty} \sum_{|n|=n} |c_n \langle z^{\otimes n}, (2^n n!)^{-1/2} \zeta^{\otimes n} \rangle| \\
 &= \sum_{n=0}^{\infty} (2^n n!)^{-1/2} \sum_{|n|=n} |c_n| \lambda^{-pn} \left(\frac{n!}{n!} \right)^{1/2} |\langle z^{\otimes n}, \lambda^{pn} \zeta^{\otimes n} \rangle| \\
 &\leq \|f\|_{(\mathfrak{F}_p)} \exp\left[\frac{1}{4} \|z\|_{-p}^2\right].
 \end{aligned}$$

Therefore the series converges to a continuous functional $\tilde{f}$ on H_{-p} absolutely and uniformly on any bounded set of H_{-p} and $\tilde{f}$ satisfies (3. 9). The finite sums of the right hand side of (3. 4) are functionals analytic and locally uniformly bounded in H_{-p} in the sense of [H-P]. Applying Theorem 3. 18. 1 of [H-P], we have the analyticity of $\tilde{f}$ in H_{-p} . $\square$

PROPOSITION 3. 3. If $p > s_0$ and $f \in (\mathfrak{F}_p)$, then the functional $\tilde{f}$ in PROPOSITION 3. 2 is a unique continuous version of f in H_{-p} ; that is $\tilde{f}(z) = f(z)$ holds for ν -a.e. $z \in H^*$. Besides if $p > q + s_0$, then $\tilde{f}(D^q z)$ coincides with the continuous version of $\Lambda(D^q)f(z)$ in H_{-p+q} .

PROOF. f , as the L^2 -limit of (3. 4), is ν -a.e. defined and square-integrable in H^* . Since $\nu(H_{-p}) = 1$ for $p > s_0$, $\tilde{f}$ is equal to f ν -a.e. in H^* . Since every non void open set in H_{-p} has strictly positive ν -measure, the continuous version of f is uniquely given in H_{-p} . If $p > s_0 + q$ and $z \in H_{-p+q}$, then $D^q z \in H_{-p}$ and $p - q > s_0$. Therefore we see that

$$\begin{aligned}\tilde{f}(D^q z) &= \sum_{n \in \mathcal{N}_0} c_n z^n (D^q z) \\ &= \sum_{n \in \mathcal{N}_0}^{\infty} \lambda^{-qn} c_n z^n(z)\end{aligned}$$

converges uniformly on any bounded set in H_{-p+q} . Thus we have the last assertion. $\square$

For $p < s_0$ and $f \in (\mathfrak{F}_p)$, the functional $\tilde{f}$ analytic in H_{-p} does not mean a version in the sense of ν -a.e. because of $\nu(H_{-p}) = 0$. However, the version $\tilde{f}$ recovers f by means of Taylor coefficients (ref. [B-K], [K2]).

If $f \in (\mathfrak{F})$, then $\tilde{f}(z)$ can be defined on H_{-p} for any $p > 0$ and so $\tilde{f}(z)$ is defined in H^* . Moreover, if $p > q$, then the continuity of $\tilde{f}(z)$ in H_{-p} implies the one in H_{-q} . It follows from this that $\tilde{f}(z)$ is continuous in $z \in H^*$ with the inductive limit topology of $H^* = \varinjlim H_{-p}$. But we omit the proof. Besides we can say that $\tilde{f}(z)$ is not merely entire of at most order 2 on any H_{-p} ($p > 0$) but also of minimal type, as we see in the following as a corollary of PROPOSITION 3. 2 (ref. [B-K], [K2]).

COROLLARY 3. 1. *If $f \in (\mathfrak{F})$, then for any $p > 0$, any $k > 0$, and for any $z \in H_{-p}$ we have*

$$|\tilde{f}(z)| \leq \|f\|_{(\mathfrak{F}_{p+k})} \exp\left[\frac{1}{4}\lambda_0^{2k} \|z\|_{-p}^2\right]. \quad (3. 10)$$

PROOF. Let $z \in H_{-p}$. Then this is clear from (3. 10) and

$$\|z\|_{-(p+k)}^2 \leq \lambda_0^{2k} \|z\|_{-p}^2. \quad \square$$

§4. The triplet $(\varphi) \subset (L^2) \subset (\varphi')$ derived by Gauss transform from the triplet $(\mathfrak{F}) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}')$

In this section, we begin by reconsidering G as a map from

$\mathcal{D}(E^*)$ onto $\mathcal{D}(H^*)$. Next, we define operators $\{\Gamma(D^p) \equiv G^{-1}\Lambda(D^p)G; p \in \mathbb{R}\}$ which act on $\mathcal{D}(E^*)$. Using these operators we construct the nuclear rigging of white noise functionals:

$$(\mathcal{D}) \subset (\mathcal{D}_p) \subset (L^2) \subset (\mathcal{D}_{-p}) \subset (\mathcal{D}'). \quad (4.1)$$

It will turn out that the rigging (4.1) is obtained as the image of

$$(\mathfrak{F}) \subset (\mathfrak{F}_p) \subset (\mathfrak{F}_0) \subset (\mathfrak{F}_{-p}) \subset (\mathfrak{F}')$$

by the extended G^{-1} .

Let us define the operator $\Gamma(D^p)$ from $\mathcal{D}(E^*)$ onto itself. G is an isometry from $\mathcal{D}(E^*)$ onto $\mathcal{D}(H^*)$:

$$\left(\mathcal{D}(E^*), \|\cdot\|_{(L^2)}\right) \xrightarrow[\text{isometric}]{G} \left(\mathcal{D}(H^*), \|\cdot\|_{(\mathfrak{F}_0)}\right); \quad (4.2)$$

$\Lambda(D^p)$ maps $\mathcal{D}(H^*)$ onto $\mathcal{D}(H^*)$. Therefore we can define $\Gamma(D^p)$ for each $p \in \mathbb{R}$ by setting

$$\Gamma(D^p)\varphi = G^{-1}\Lambda(D^p)G\varphi \quad \text{for } \varphi \in \mathcal{D}(E^*). \quad (4.3)$$

Then, it is easy to see that $\mathcal{D}(E^*)$ is a pre-Hilbert space with the inner product

$$\left(\Gamma(D^p)\varphi, \Gamma(D^p)\psi\right)_{(L^2)} = \int_{E^*} \left(\Gamma(D^p)\varphi(x)\right) \overline{\Gamma(D^p)\psi(x)} d\mu(x). \quad (4.4)$$

Let us denote its completion by $(\mathcal{D}_p)$ and the inner product by $(\varphi, \psi)_{(\mathcal{D}_p)}$. We evidently see that $(\mathcal{D}_0) = (L^2)$. Corresponding to the eigen system of $\Lambda(D^p)$, $\Gamma(D^p)$ has the eigen system:

$$\Gamma(D^p)h_n(x) = \left(\prod_j \lambda_j^{-pn_j}\right)h_n(x) = \lambda^{-pn}h_n(x). \quad (4.5)$$

This follows from (2.3) and (3.2):

$$Gh_n = z^n, \quad G^{-1}z^n = h_n, \quad \text{and}$$

$$\Lambda(D^p)z^n(z) = \left(\prod_j \lambda_j^{-pn_j}\right)z^n(z) = \lambda^{-pn}z^n(z).$$

The system $\{h_n; n \in \mathcal{N}_0\}$ is a CONS of (L^2) , so we can easily see the following.

PROPOSITION 4. 1. For any $p \in \mathbb{R}$, $\{\lambda^{pn} h_n; n \in \mathcal{N}_0\}$ is a CONS of $(\mathcal{S}_p)$. And hence any $\varphi \in (\mathcal{S}_p)$ can be expressed in the form

$$\varphi = \sum_{n \in \mathcal{N}_0} c_n h_n \quad (4. 6)$$

with coefficients $\{c_n; n \in \mathcal{N}_0\}$ satisfying

$$\|\varphi\|_{(\mathcal{S}_p)}^2 = \sum_{n \in \mathcal{N}_0} \lambda^{-2pn} |c_n|^2 < \infty. \quad (4. 7)$$

Furthermore, for any p and $q \in \mathbb{R}$, $\Gamma(D^q)$ can extend its domain to $(\mathcal{S}_p)$ as an isometry from $(\mathcal{S}_p)$ to $(\mathcal{S}_{p-q})$ satisfying that for $\varphi \in (\mathcal{S}_p)$ of the form (4. 5)

$$\Gamma(D^q)\varphi = \sum_{n \in \mathcal{N}_0} \lambda^{-qn} c_n h_n \in (\mathcal{S}_{p-q}). \quad (4. 8)$$

By the proposition above we can identify the dual space of $(\mathcal{S}_p)$ with $(\mathcal{S}_{-p})$ for $p \in \mathbb{R}$. In fact, the bilinear form, $\langle \Psi, \phi \rangle$, of $(\phi, \Psi) \in (\mathcal{S}_p) \times (\mathcal{S}_{-p})$ is realized by

$$\langle \Psi, \phi \rangle = \int_{E^*} \left(\Gamma(D^{-p})\Psi(x) \right) \Gamma(D^p)\phi(x) d\mu(x). \quad (4. 9)$$

Let us write

$$(\mathcal{S}) = \cap_{p=0}^{\infty} (\mathcal{S}_p) \text{ and } (\mathcal{S}') = \cup_{p=0}^{\infty} (\mathcal{S}_{-p}). \quad (4. 10)$$

From (3. 7) it follows that, for any $p \in \mathbb{R}$ and any $s > s_0$,

$$\sum_{n \in \mathcal{N}_0} \|\lambda^{(p+s)n} h_n\|_{(\mathcal{S}_p)}^2 = \prod_j (1 - \lambda_j^{2s})^{-1} < \infty.$$

Thus we have a nuclear rigging

$$(\mathcal{S}) \subset (\mathcal{S}_p) \subset (L^2) \subset (\mathcal{S}_{-p}) \subset (\mathcal{S}'), \quad p > 0. \quad (4. 11)$$

Clearly $(\mathcal{S}')$ is a dual space of $(\mathcal{S})$. We call $(\mathcal{S})$ the space of test white noise functionals and $(\mathcal{S}')$ the space of generalized white noise functionals, as usual.

Let $p \in \mathbb{R}$. It follows from (4. 2) that for any $f \in \mathcal{S}(H^*)$

$$\|G^{-1}f\|_{(\mathcal{S}_p)} = \|\Gamma(D^p)G^{-1}f\|_{(L^2)} = \|\Lambda(D^p)f\|_{(\mathfrak{F}_0)} = \|f\|_{(\mathfrak{F}_p)}.$$

Therefore G^{-1} can extend uniquely to an isometric operator G_p^{-1}

from $(\mathfrak{F}_p)$ onto $(\mathscr{P}_p)$. The extensions $\{G_p^{-1}; p \in \mathbb{R}\}$ are consistent. That is, if $p < q$, then G_p^{-1} coincides with G_q^{-1} on $(\mathfrak{F}_q)$. So we have a unique continuous extension from $(\mathfrak{F}')$ onto $(\mathscr{P}')$, which we denote by the same symbol G^{-1} . It satisfies that for any $f, g \in (\mathfrak{F}_p)$ and any $p \in \mathbb{R}$

$$(G^{-1}f, G^{-1}g)_{(\mathscr{P}_p)} = (f, g)_{(\mathfrak{F}_p)}. \quad (4.12)$$

Moreover, we can easily see that for $F \in (\mathfrak{F}_{-p})$ and $f \in (\mathfrak{F}_p)$

$$\langle G^{-1}F, G^{-1}f \rangle = \langle F, f \rangle. \quad (4.13)$$

The above nuclear rigging is the same as the usual rigging of white noise calculus, as we see in the following. Let us put for $n = 0, 1, 2, \dots$

$$\mathscr{P}_n(E^*) = \{\varphi; \varphi \in \mathscr{P}(E^*), \text{ the degree of } \varphi \leq n\},$$

$$\overline{\mathscr{P}}_n = (L^2)\text{-closure of } \mathscr{P}_n(E^*),$$

$$\mathscr{K}_n = \overline{\mathscr{P}}_n \ominus \overline{\mathscr{P}}_{(n-1)} \quad (n \geq 1), \text{ and } \mathscr{K}_0 = \mathbb{C},$$

where $\mathscr{P}_0(E^*) = \overline{\mathscr{P}}_0 = \mathbb{C}$. Then $\langle x^{\widehat{\otimes} n}, f_n \rangle$ is well-defined as an element of $\overline{\mathscr{P}}_n$ for any $f_n \in H_0^{\widehat{\otimes} n}$ and $\{h_n; n \in \mathcal{N}_0 \text{ and } |n| = n\}$ is an orthonormal basis of $\mathscr{K}_n$ for each $n \geq 0$. It is well-known that (L^2) has Wiener-Itô decomposition

$$(L^2) = \sum_{n=0}^{\infty} \oplus \mathscr{K}_n,$$

Let us put

$$F_n(H) = \{\text{any finite sum of } \widehat{\otimes}_{j=1}^n \xi_j; \xi_j \in H \ (1 \leq j \leq n)\},$$

$$Z_n = \{\text{any linear combination of } \zeta^{\widehat{\otimes} n} / \sqrt{n!}; |n| = n\},$$

$$\Phi(H) = \{(f_n)_{n=0}^{\infty}; f_n = 0 \text{ for almost all } n \geq 0 \text{ and } f_n \in F_n(H)\},$$

and

$$\Phi = \{(f_n)_{n=0}^{\infty}; f_n = 0 \text{ for almost all } n \geq 0 \text{ and } f_n \in Z_n\}.$$

Clearly $\Phi \subset \Phi(H)$ and Φ is dense in Φ_0 . Let Φ_p ($p \in \mathbb{R}$) be the

Fock space defined as a direct sum of Hilbert spaces $H_p^{\otimes n}$ with weights $\sqrt{n!}$ ($n \geq 0$). That is,

$$\Phi_p = \sum_{n=0}^{\infty} \oplus \sqrt{n!} H_p^{\otimes n}.$$

Then it is easy to see that the Fock space Φ_p ($p \in \mathbb{R}$) coincides with the completion of $\Phi(H)$ by the inner product

$$\begin{aligned} (\vec{f}, \vec{g})_{\Phi_p} &= \sum_{n=0}^{\infty} n! (f_n, g_n)_{H_p^{\otimes n}} \\ &= \sum_{n=0}^{\infty} n! \left((D^p)^{\otimes n} f_n, (D^p)^{\otimes n} g_n \right)_{H_0^{\otimes n}} \end{aligned} \quad (4.14)$$

where $\vec{f} = (f_n)_{n=0}^{\infty}$ and $\vec{g} = (g_n)_{n=0}^{\infty} \in \Phi(H)$.

The Segal isomorphism I_S from Φ_0 to (L^2) is defined by

$$I_S((f_n)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} : \langle x^{\otimes n}, f_n \rangle :, \quad (4.15)$$

where $: \langle x^{\otimes n}, f_n \rangle :$ denotes the orthogonal projection of $\langle x^{\otimes n}, f_n \rangle$ to the space $\mathcal{H}_n$ for each $f_n \in H_0^{\otimes n}$. Let us assure ourselves that I_S is well-defined. First we note that the right hand side of (4.15) is a finite sum if $\vec{f} = (f_n)_{n=0}^{\infty}$ is an element of Φ or $\Phi(H)$. If we apply the formula

$$(2u)^n = \sum_{2k \leq n} \binom{n}{2k} \frac{(2k)!}{k!} H_{n-2k}(u)$$

to $\langle x^{\otimes n}, \zeta^{\otimes n} / \sqrt{n!} \rangle$, then we have

$$\langle x^{\otimes n}, \zeta^{\otimes n} / \sqrt{n!} \rangle = \sum_{2k \leq n} \binom{n}{2k} \frac{(2k)!}{k!} 2^{-k} ((n-2k)! / n!)^{1/2} h_{n-2k}(x).$$

But $\{h_n; |n| = n\}$ is an orthonormal basis of $\mathcal{H}_n$ for each $n \geq 0$ and these bases are mutually orthogonal for different $n \geq 0$; hence we have

$$: \langle x^{\otimes n}, \zeta^{\otimes n} / \sqrt{n!} \rangle : = h_n(x) \quad (n \in \mathcal{N}_0),$$

$$\text{i.e.,} \quad I_S \left((0, \dots, 0, \zeta^{\otimes n} / \sqrt{n!}, 0, \dots) \right) = h_n. \quad (4.16)$$

In addition we have

$$\|(0, \dots, 0, \zeta^{\otimes n}/\sqrt{n!}, 0, \dots)\|_{\Phi_0} = \|h_n\|_{(L^2)}. \quad (4.17)$$

This means that I_S can be well-defined as an isometry from Φ_0 to (L^2) .

PROPOSITION 4. 2. For $p \in \mathbb{R}$, the Hilbert space $(\mathscr{Y}_p)$ coincides with the completion of $\{I_S(\vec{f}); \vec{f} \in \Phi(H)\}$ by the inner product

$$(I_S(\vec{f}), I_S(\vec{g}))_p = (\vec{f}, \vec{g})_{\Phi_p}. \quad (4.18)$$

PROOF. By PROPOSITION 4. 1, the set

$$\mathscr{Y} \equiv \{\sum_{n \in J} c_n h_n; J \text{ (finite subset)} \subset \mathcal{N}_0, c_n \in \mathbb{C}\}$$

is dense in $(\mathscr{Y}_p)$. Since $\{\lambda^{pn}(n!/n!)^{1/2}\zeta^{\otimes n}; |n| = n\}$ is a CONS of $H_p^{\otimes n}$ for each $n \geq 0$, the set $\{I_S(\vec{f}); \vec{f} \in \Phi\}$ is dense in the completion of $\{I_S(\vec{f}); \vec{f} \in F(H)\}$ completed by (4.17). Further by (4.16) we have $I_S(\Phi) = \mathscr{Y}$. Therefore the assertion is true. $\square$

COROLLARY 4. 1. If $p \geq 0$ and $\vec{f} \in \Phi_p$, then $I_S(\vec{f})$ is defined as an element of (L^2) . Hence the space $(\mathscr{Y}_p)$ coincides with the totality of $\{I_S(\vec{f}); \vec{f} \in \Phi_p\}$.

PROOF. This is clear from $\Phi_p \subset \Phi_0$ for $p \geq 0$. $\square$

In the following we consider several properties of white noise functionals. We will see that our setting makes the computations easy and helps us obtain the sharper inequalities.

THEOREM 4. 1. Let $p > p_0$. For any $\varphi \in (\mathscr{Y}_p)$ with the expression (4.6), the functional of $z \in H_{-p}$

$$\sum_{n \in \mathcal{N}_0} c_n h_n(z) \quad (4.19)$$

converges absolutely and uniformly to a functional $\tilde{\varphi}(z)$ on any bounded set of H_{-p} . The limit functional $\tilde{\varphi}(z)$ is continuous on H_{-p} , (is called the continuous continuation of φ in H_{-p}) and satisfies, for any $z = x + \sqrt{-1}y \in H_{-p}$ ($x, y \in E_{-p}$),

$$\begin{aligned} & |\tilde{\varphi}(z)| \\ & \leq \sqrt{\alpha_{2p}} \exp\left[\frac{1}{2} \sum_{j=0}^{\infty} \left\{ \frac{\lambda_j^{2p}}{1+\lambda_j^{2p}} |\langle x, \zeta_j \rangle|^2 + \frac{\lambda_j^{2p}}{1-\lambda_j^{2p}} |\langle y, \zeta_j \rangle|^2 \right\}\right] \|\varphi\|_{(\mathcal{F}_p)} \\ & \leq \sqrt{\alpha_{2p}} \exp[\|z\|_{-p}^2] \|\varphi\|_{(\mathcal{F}_p)}. \end{aligned} \quad (4.20)$$

where $\alpha_p = \prod_{j=0}^{\infty} (1 - \lambda_j^{2p})^{-1/2}$. Therefore $\tilde{\varphi}(z)$ is analytic in H_{-p} in the sense of [H-P] (E. Hille & R. S. Phillips).

PROOF. Let $p > p_0$ and $\varphi \in (\mathcal{F}_p)$ which has the expression

$$\varphi = \sum_{n \in \mathcal{N}_0} c_n h_n \quad \text{with} \quad \sum_{n \in \mathcal{N}_0} \lambda_j^{-2pn} |c_n|^2 < \infty.$$

By Schwartz' inequality and Mehler's formula: for $|s| < 1$

$$\begin{aligned} & \sum_{n=0}^{\infty} s^n (2^n n!)^{-1} H_n(u) H_n(v) \\ & = (1-s^2)^{-1/2} \exp[(1-s^2)^{-1} \{2suv - s^2(u^2+v^2)\}], \end{aligned} \quad (4.21)$$

we have

$$\begin{aligned} & |\sum_{n \in \mathcal{N}_0} c_n h_n(z)| \\ & \leq \left(\sum_{n \in \mathcal{N}_0} \lambda_j^{-2pn} |c_n|^2 \right)^{1/2} \left(\sum_{n \in \mathcal{N}_0} \lambda_j^{2pn} |h_n(z)|^2 \right)^{1/2} \\ & = \|\varphi\|_{(\mathcal{F}_p)} \left(\sum_{n \in \mathcal{N}_0} \lambda_j^{2pn} h_n(z) h_n(\bar{z}) \right)^{1/2} \\ & = \|\varphi\|_{(\mathcal{F}_p)} \left(\prod_j \sum_{n=0}^{\infty} \lambda_j^{2pn} (2^n n!)^{-1} H_n\left(\frac{\langle z, \zeta_j \rangle}{\sqrt{2}}\right) H_n\left(\frac{\langle \bar{z}, \zeta_j \rangle}{\sqrt{2}}\right) \right)^{1/2} \\ & = \|\varphi\|_{(\mathcal{F}_p)} \sqrt{\alpha_{2p}} \exp\left[\frac{1}{2} \sum_{j=0}^{\infty} \left\{ \frac{\lambda_j^{2p}}{1+\lambda_j^{2p}} |\langle x, \zeta_j \rangle|^2 + \frac{\lambda_j^{2p}}{1-\lambda_j^{2p}} |\langle y, \zeta_j \rangle|^2 \right\}\right]. \end{aligned}$$

If $0 < u < 1/2$, then $u/(1-u) < 2u$. So we have, putting $u = \lambda_j^{2p}$,

$$|\sum_{n \in \mathcal{N}_0} c_n h_n(z)| \leq \|\varphi\|_{(\mathcal{F}_p)} \sqrt{\alpha_{2p}} \exp[\|z\|_{-p}^2], \quad (4.22)$$

and (4. 19) converges absolutely and uniformly to a functional $\tilde{\varphi}(x)$ on any bounded set of H_{-p} . This inequality gives the locally uniformly boundedness of every finite sum of (4. 19) and it follows from this that $\tilde{\varphi}(z)$ is analytic in H_{-p} (in the sense of [H-P]). $\square$

COROLLARY 4. 2. *Especially, if $x \in E_{-p}$ and $\varphi \in (\mathcal{V}_p)$ for $p > s_0$, then we have*

$$\begin{aligned} |\tilde{\varphi}(x)| &\leq \sqrt{\alpha_{2p}} \exp\left[\frac{1}{2} \sum_{j=0}^{\infty} \frac{\lambda_j^{2p}}{1+\lambda_j^{2p}} |\langle x, \zeta_j \rangle|^2\right] \|\varphi\|_{(\mathcal{V}_p)} \\ &\leq \sqrt{\alpha_{2p}} \exp\left[\frac{1}{2} \|x\|_{-p}^2\right] \|\varphi\|_{(\mathcal{V}_p)} \end{aligned} \quad (4. 23)$$

and $\varphi(x) = \tilde{\varphi}(x)$ μ -a.e. $x \in E^*$. ($\tilde{\varphi}(x)$ is called a continuous version of φ .)

PROOF. Let $y = 0$ in (4. 20). Then for $p > s_0$ we can see that our assertion is true. $\square$

§5. Other properties of two triplets

THEOREM 5. 1. *Let $0 \leq q < p - p_0$. Then the functional $\exp[\frac{1}{2} \|x\|_{-p}^2]$ defined in E_{-p} belongs to $(\mathcal{V}_q)$. Actually, the $(\mathcal{V}_q)$ -norm is evaluated as*

$$\|\exp[\frac{1}{2} \|\cdot\|_{-p}^2]\|_{(\mathcal{V}_q)} = \prod_j \left((1 - \lambda_j^{2p})^2 - \lambda_j^{4(p-q)} \right)^{-1/4}.$$

PROOF. By a direct computation we can see that if $p > p_0$, then the functional $\exp[\frac{1}{2} \|x\|_{-p}^2]$ belongs to $(L^2) = (\mathcal{V}_0)$. So it is expanded into a Fourier series. Let us compute the Fourier coefficients

$$c_n = \int_{E^*} \exp[\frac{1}{2} \|x\|_{-p}^2] h_n(x) d\mu(x) \quad (5. 1)$$

with respect to the CONS $\{h_n(x); n \in \mathcal{N}_0\}$ of (L^2) . To get the values c_n , if we note the equality

$$\exp[\frac{1}{2}\|x\|_{-p}^2] = \prod_{j=0}^{\infty} \exp[\frac{1}{2}\lambda_j^{2p}\langle x, \zeta_j \rangle^2]$$

and independentness of $\langle x, \zeta_j \rangle$'s, we have only to calculate the integrals

$$\int_{E^*} \exp[\frac{1}{2}\lambda_j^{2p}\langle x, \zeta_j \rangle^2] H_n(\langle x, \zeta_j \rangle/\sqrt{2}) d\mu(x).$$

But if n is odd, then the integral is equal to zero and if n is even, say $n = 2k$, then it is equal to

$$(1 - \lambda_j^{2p})^{-1/2} \frac{n!}{k!} \left(\lambda_j^{2p} / (1 - \lambda_j^{2p}) \right)^k.$$

So we have for $n = 2k = (2k_0, 2k_1, 2k_2, \dots)$

$$c_n = \alpha_p (2^{n!})^{-1/2} \frac{n!}{k!} \prod_j \left(\lambda_j^{2p} / (1 - \lambda_j^{2p}) \right)^{k_j}$$

else $c_n = 0$, where

$$\alpha_p = \prod_{j=0}^{\infty} (1 - \lambda_j^{2p})^{-1/2}.$$

Therefore

$$\begin{aligned} & \|\exp[\frac{1}{2}\|\cdot\|_{-p}^2]\|_{(\mathcal{F}_Q)}^2 \\ &= \sum_{n \in \mathcal{N}_0} \lambda^{-2qn} |c_n|^2 \\ &= \alpha_p^2 \sum_{k \in \mathcal{N}_0} 2^{-2k} \binom{2k}{k} \prod_j \left(\lambda_j^{4(p-q)} / (1 - \lambda_j^{2p})^2 \right)^{k_j}. \end{aligned} \tag{5. 2}$$

If we recall the definition of the constant p_0 and the formula

$$2^{-2k} \binom{2k}{k} = (-1)^k \binom{-1/2}{k}$$

(5. 2) is followed by

$$\alpha_p^2 \sum_{k \in \mathcal{N}_0} \prod_j \binom{-1/2}{k_j} \left(-\lambda_j^{2(p-q)} / (1 - \lambda_j^{2p}) \right)^{2k_j}.$$

But $0 \leq q < p - p_0$ implies that $\lambda_j^{2(p-q)} / (1 - \lambda_j^{2p}) < 1$ and so this infinite sum of the finite product is equal to

$$\begin{aligned}
& \alpha_p^2 \prod_j \sum_{k=0}^{\infty} \binom{-1/2}{k} \left(-\lambda_j^{2(p-q)} / (1 - \lambda_j^{2p}) \right)^{2k} \\
& = \alpha_p^2 \prod_j \left(1 - \lambda_j^{4(p-q)} / (1 - \lambda_j^{2p})^2 \right)^{-1/2} \\
& = \prod_j \left((1 - \lambda_j^{2p})^2 - \lambda_j^{4(p-q)} \right)^{-1/2} < \infty. \quad \square
\end{aligned}$$

THEOREM 5. 2 Let $s_0 < s$ and $p_0 < p$. Then we have

$$(\mathfrak{F}_{S+p}) \cdot (\mathfrak{F}_{S+p}) \subset (\mathfrak{F}_S)$$

and for $f, g \in (\mathfrak{F}_{S+p})$

$$\|f \cdot g\|_{(\mathfrak{F}_S)} \leq \gamma_p^2 \|f\|_{(\mathfrak{F}_{S+p})} \|g\|_{(\mathfrak{F}_{S+p})} \quad (5. 3)$$

where γ_p is given in LEMMA 5. 1. Hence $(\mathfrak{F})$ is an algebra.

PROOF. First we note that for $m, n \in \mathcal{N}_0$

$$\binom{m+n}{n} \leq 2^{|m|+|n|} \quad \text{and} \quad z^m(z) \cdot z^n(z) = \binom{m+n}{n}^{1/2} z^{m+n}(z).$$

Let $c_n = (f, z^n)_{(\mathfrak{F}_0)}$ and $d_n = (g, z^n)_{(\mathfrak{F}_0)}$. Then we have

$$\tilde{f}(z) = \sum_{n \in \mathcal{N}_0} c_n z^n(z) \quad \text{and} \quad \tilde{g}(z) = \sum_{n \in \mathcal{N}_0} d_n z^n(z).$$

By PROPOSITION 3. 1 these two series are absolutely convergent on H_{-s-p} . Therefore we have

$$\tilde{f}(z) \cdot \tilde{g}(z) = \sum_{m, n \in \mathcal{N}_0} c_m d_n \binom{m+n}{n}^{1/2} z^{m+n}(z)$$

and so, using the triangle inequality and Schwarz' one,

$$\begin{aligned}
& \|f \cdot g\|_{(\mathfrak{F}_S)} \\
& \leq \sum_{m, n \in \mathcal{N}_0} |c_m| |d_n| 2^{(|m|+|n|)/2} \lambda^{-s(m+n)} \\
& \leq \left(\sum_{m \in \mathcal{N}_0} |c_m| \lambda^{-(s+p)m} 2^{|m|/2} \lambda^{pm} \right) \left(\sum_{n \in \mathcal{N}_0} |d_n| \lambda^{-(s+p)n} 2^{|n|/2} \lambda^{pn} \right) \\
& \leq \|f\|_{(\mathfrak{F}_{S+p})} \|g\|_{(\mathfrak{F}_{S+p})} \prod_j \sum_{n=0}^{\infty} \left(2\lambda_j^{2p} \right)^n \\
& = \|f\|_{(\mathfrak{F}_{S+p})} \|g\|_{(\mathfrak{F}_{S+p})} \prod_j (1 - 2\lambda_j^{2p})^{-1}. \quad \square
\end{aligned}$$

Let us mention the fact that $(\mathcal{V})$ is an algebra. How to conclude this result was shown in [Ku-T2]. But our setting described above makes some computations a little bit simple. A rewritten form of this theorem within our framework is:

PROPOSITION 5. 1 *Let $s_0 < s$ and $2p_0 < p$. If the functionals φ and ψ are in $(\mathcal{V}_{s+p})$, then $\varphi \cdot \psi$ belongs to $(\mathcal{V}_s)$ and*

$$\|\varphi \cdot \psi\|_{(\mathcal{V}_s)} \leq \beta_s \kappa_p \|\varphi\|_{(\mathcal{V}_{s+p})} \|\psi\|_{(\mathcal{V}_{s+p})}$$

where $\beta_s = \prod_j (1 - \lambda_j^{4s}/4)^{-1/2}$ and $\kappa_p = \prod_j (1 - 4\lambda_j^{2p})^{-1}$.

PROOF. Let $\varphi, \psi \in (\mathcal{V}_{s+p})$. Suppose that φ and ψ have the expansions as elements of $(\mathcal{V}_s)$:

$$\varphi = \sum_{n \in \mathcal{N}_0} c_n h_n \quad \text{with} \quad \sum_{n \in \mathcal{N}_0} \lambda^{-2sn} |c_n|^2 < \infty$$

and

$$\psi = \sum_{n \in \mathcal{N}_0} d_n h_n \quad \text{with} \quad \sum_{n \in \mathcal{N}_0} \lambda^{-2sn} |d_n|^2 < \infty.$$

The absolute convergence for $x \in E_{-s}$ of the series

$$\tilde{\varphi}(x) = \sum_{n \in \mathcal{N}_0} c_n h_n(x) \quad \text{and} \quad \tilde{\psi}(x) = \sum_{n \in \mathcal{N}_0} d_n h_n(x)$$

implies the absolute convergence of

$$\tilde{\varphi}(x) \cdot \tilde{\psi}(x) = \sum_{m, n \in \mathcal{N}_0} c_m d_n h_m(x) h_n(x).$$

Therefore we have

$$\begin{aligned} \|\varphi \psi\|_{(\mathcal{V}_s)} &\leq \sum_{m, n \in \mathcal{N}_0} |c_m d_n| \|h_m h_n\|_{(\mathcal{V}_s)} \\ &\leq \sum_{m, n \in \mathcal{N}_0} \lambda^{-(s+p)(m+n)} |c_m d_n| \|\lambda^{sm} h_m \lambda^{sn} h_n\|_{(\mathcal{V}_s)} \lambda^{p(m+n)}. \end{aligned}$$

But if we apply the formula

$$H_m(u) H_n(u) = \sum_{k=0}^{m \wedge n} 2^k k! \binom{m}{k} \binom{n}{k} H_{m+n-2k}(u),$$

the fact that $\{(\lambda^{-sn}, h_n); n \in \mathcal{N}_0\}$ is an eigen system of $\Lambda(D^S)$,

and

the inequality $\binom{m}{k} \leq 2^m$

to the norm $\|\lambda^{sm} h_m \cdot \lambda^{sn} h_n\|_{(\mathcal{F}_S)}$, we have

$$\|\lambda^{sm} h_m \lambda^{sn} h_n\|_{(\mathcal{F}_S)}^2 = \sum_{k \leq m \wedge n} \binom{m}{k} \binom{n}{k} \binom{m+n-2k}{n-k} \lambda^{4sk} \leq \beta_S^2 4^{m+n}.$$

After all we obtain

$$\begin{aligned} & \|\varphi\psi\|_{(\mathcal{F}_S)} \\ & \leq \beta_S \sum_{m,n \in \mathcal{N}_0} \lambda^{-(s+p)(m+n)} |c_m d_n| (2\lambda^p)^{(m+n)} \\ & = \beta_S \|\varphi\|_{(\mathcal{F}_{S+p})} \|\psi\|_{(\mathcal{F}_{S+p})} \sum_{n \in \mathcal{N}_0} (2\lambda^p)^{2n} \\ & = \beta_S \kappa_p \|\varphi\|_{(\mathcal{F}_{S+p})} \|\psi\|_{(\mathcal{F}_{S+p})}. \quad \square \end{aligned}$$

From this proposition we can easily conclude that $(\mathcal{F})$ is an algebra (cf. [L],[Y]).

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