

Upper triangular operators, SVEP and Browder, Weyl theorems

B. P. Duggal

Abstract

We show the important role played by SVEP, the *single-valued extension property*, and the *polaroid property* in relating the spectrum, and certain distinguished parts thereof, of the operators $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and M_0 for some Banach space operators A, B and $C \in B(\mathcal{X})$.

1. Results

Let $B(\mathcal{X})$ denote the algebra of operators (equivalently, bounded linear transformations) on a Banach space $\mathcal{X}$. For $A, B, C \in B(\mathcal{X})$, let M_C denote the upper triangular operator $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and let $M_0 = A \oplus B$. The spectrum, and certain distinguished parts thereof, of the operators M_C and M_0 has been studied by a number of authors in the recent past; see references. Of particular interest to us is the relationship between the spectral, the Fredholm, the Browder and the Weyl properties.

Most of our notation is standard. For an operator $T \in B(\mathcal{X})$:

$$\begin{aligned} \sigma(T) &= \text{spectrum of } T, \\ \sigma_a(T) &= \text{approximate point spectrum of } T, \\ \Phi_+(\mathcal{X}) &= \{T \in B(\mathcal{X}) : T \text{ is upper semi-Fredholm}\}, \text{ and} \\ \Phi_-(\mathcal{X}) &= \{T \in B(\mathcal{X}) : T \text{ is lower semi-Fredholm}\}. \end{aligned}$$

The Browder, the Weyl, the upper semi-Fredholm, the lower semi-Fredholm spectrum of T are the sets

$$\begin{aligned} \sigma_b(T) &= \{\lambda \in \sigma(T) : T - \lambda \text{ is not Fredholm or } \text{asc}(T - \lambda) \neq \text{dsc}(T - \lambda)\}, \\ \sigma_w(T) &= \{\lambda \in \sigma(T) : T - \lambda \text{ is not Fredholm or } \text{ind}(T - \lambda) \not\leq 0\}, \\ \sigma_{SF_+}(T) &= \{\lambda \in \sigma(T) : T - \lambda \notin \Phi_+(\mathcal{X})\}, \text{ and} \\ \sigma_{SF_-}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_-(\mathcal{X})\}, \end{aligned}$$

AMS(MOS) subject classification (2000). Primary 47B47, 47A10, 47A11

Key words and phrases. Banach space, single valued extension property, Browder and Weyl spectra, Browder and Weyl theorems.

respectively. The Browder essential approximate point spectrum $\sigma_{ab}(T)$, and the Weyl essential approximate point spectrum $\sigma_{aw}(T)$, of T are the sets

$$\begin{aligned}\sigma_{ab}(T) &= \{\lambda \in \sigma_a(T) : T - \lambda \notin \Phi_+(\mathcal{X}) \text{ or } \text{asc}(T - \lambda) = \infty\}, \text{ and} \\ \sigma_{aw}(T) &= \{\lambda \in \sigma_a(T) : T - \lambda \notin \Phi_+(\mathcal{X}) \text{ or } \text{ind}(T - \lambda) \not\leq 0\}.\end{aligned}$$

Let

$$\begin{aligned}\Xi(T) &= \{\lambda \in \mathbb{C} : T \text{ does not have SVEP at } \lambda\}, \\ \Xi_+(T) &= \{T - \lambda \in \Phi_+(\mathcal{X}) : \lambda \in \Xi(T)\}, \text{ and} \\ \Xi_+^*(T) &= \{T - \lambda \in \Phi_+(\mathcal{X}) : \lambda \in \Xi(T^*)\}.\end{aligned}$$

The following inclusions/equalities are either well known or are easily proved:

$$\begin{aligned}\sigma(M_C) &\subseteq \sigma(A) \cup \sigma(B) = \sigma(M_0) = \sigma(M_C) \cup \{\Xi(A^*) \cup \Xi(B)\}, \\ \sigma_b(M_C) &\subseteq \sigma_b(A) \cup \sigma_b(B) = \sigma_b(M_0), \text{ and} \\ \sigma_w(M_C) &\subseteq \sigma_w(M_0) \subseteq \sigma_w(A) \cup \sigma_w(B).\end{aligned}$$

Furthermore, if we let $P = A$ and $Q = B$ or $P = A^*$ and $Q = B^*$, then:

$$\begin{aligned}\sigma_b(M_0) &= \sigma_b(M_C) \cup \{\Xi(A^*) \cup \Xi(B)\}, \text{ and} \\ \sigma_w(A) \cup \sigma_w(B) &\subseteq \sigma_w(M_C) \cup \{\Xi(P) \cup \Xi(Q)\}.\end{aligned}$$

Consequently, if $\Xi(P) \cup \Xi(Q) = \emptyset$, then

$$\sigma_b(M_0) = \sigma_w(M_0) = \sigma_b(M_C) = \sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B).$$

For the spectra σ_{ab} and σ_{aw} , one has:

$$\begin{aligned}\sigma_{ab}(M_C) &\subseteq \sigma_{ab}(M_0) \subseteq \sigma_{ab}(M_C) \cup \{\Xi_+^*(A) \cup \Xi_+(B)\}, \text{ and} \\ \sigma_{aw}(M_0) &\subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B) \subseteq \sigma_{ab}(A) \cup \sigma_{ab}(B) \\ &= \sigma_{ab}(M_0) \subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B) \cup \{\Xi_+(A) \cup \Xi_+(B)\}.\end{aligned}$$

Recall that a necessary and sufficient condition for M_0 to satisfy Browder's theorem (or, Bt), $\text{acc}\sigma(M_0) \subseteq \sigma_w(M_0) \iff \sigma_w(M_0) = \sigma_b(M_0)$ (resp., a -Browder's theorem (or, $a - Bt$), $\text{acc}\sigma_a(M_0) \subseteq \sigma_{aw}(M_0) \iff \sigma_{aw}(M_0) = \sigma_{ab}(M_0)$), is that A and B have SVEP at points $\lambda \in \sigma(M_0)$ such that $A - \lambda$, $B - \lambda$ are Fredholm and $\text{ind}(A - \lambda) + \text{ind}(B - \lambda) = 0$ (resp., A and B have SVEP at points $\lambda \in \sigma_a(M_0)$ such that $A - \lambda$, $B - \lambda$ are upper semi-Fredholm and $\text{ind}(A - \lambda) + \text{ind}(B - \lambda) \leq 0$). Similarly, a necessary and sufficient condition for M_C to satisfy Bt , $\text{acc}\sigma(M_C) \subseteq \sigma_w(M_C) \iff \sigma_w(M_C) = \sigma_b(M_C)$ (resp., $a - Bt$, $\text{acc}\sigma_a(M_0) \subseteq \sigma_{aw}(M_0) \iff \sigma_{ab}(M_C) = \sigma_{aw}(M_C)$), is that M_C has SVEP at points $\lambda \in \sigma(M_0) \setminus \sigma_{aw}(M_C)$ (resp., at points $\lambda \in \sigma_a(M_C) \setminus \sigma_{aw}(M_C)$). It is easily verified that if M_0 satisfies Bt then $\sigma_w(M_0) = \sigma_w(A) \cup \sigma_w(B)$, and that if M_0 satisfies $a - Bt$ then $\sigma_{aw}(M_0) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. A similar result, however, fails for the operator M_C , as follows from a consideration of the operator

$\begin{pmatrix} U & 1 - UU^* \\ 0 & U^* \end{pmatrix}$, where U is the forward unilateral shift on a Hilbert space. Evidently, the complement of $\sigma_w(M_C)$ in the complex plane $\mathbb{C}$ is the union of the complement of $\sigma_w(M_0)$ in $\mathbb{C}$ with $\sigma_w(M_0) \setminus \sigma_w(M_C)$, and the complement of $\sigma_{aw}(M_C)$ in the complex plane $\mathbb{C}$ is the union of the complement of $\sigma_{aw}(M_0)$ in $\mathbb{C}$ with $\sigma_{aw}(M_0) \setminus \sigma_{aw}(M_C)$; thus, if M_0 satisfies Bt (resp., $a - Bt$) and M_C has SVEP on $\sigma_w(M_0) \setminus \sigma_w(M_C)$ (resp., $\sigma_{aw}(M_0) \setminus \sigma_{aw}(M_C)$), then M_C satisfies Bt (resp., $a - Bt$). This may be achieved in a number of ways.

Theorem 1.1 (a). If either (i) A has SVEP at points $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF_-}(A)$ and B has SVEP at points $\mu \in \sigma_w(M_0) \setminus \sigma_{SF_-}(B)$, or (ii) both A and A^* have SVEP at points $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF_+}(A)$, or (iii) A^* has SVEP at points $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF_+}(A)$ and B^* has SVEP at points $\mu \in \sigma_w(M_0) \setminus \sigma_{SF_-}(B)$, then M_0 satisfies Bt implies M_C satisfies Bt .

(b). If (i) A has SVEP on $\lambda \in \sigma_{aw}(M_0) \setminus \sigma_{SF_+}(A)$ and A^* has SVEP on $\mu \in \sigma_w(M_0) \setminus \sigma_{SF_+}(A)$, or (ii) A^* has SVEP on $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF_+}(A)$ and B^* has SVEP on $\mu \in \sigma_w(M_0) \setminus \sigma_{SF_+}(B)$, then M_0 satisfies $a - Bt$ implies M_C satisfies $a - Bt$.

As a consequence one has:

Corollary 1.2 (a) [4, Proposition 4.1] If $\{\Xi(A) \cap \Xi(B^*)\} \cup \Xi(A^*) = \emptyset$, then M_0 satisfies Bt (resp., $a - Bt$) implies M_C satisfies Bt (resp., $a - Bt$).

(b) [2, Theorem 3.2] If either $\sigma_{aw}(A) = \sigma_{SF_+}(B)$ or $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B) = \emptyset$, then M_0 satisfies Bt (resp., $a - Bt$) implies M_C satisfies Bt (resp., $a - Bt$).

Both Theorem 1.1 and Corollary 1.2 are a particular case of the following theorem.

Theorem 1.3 (i) If $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then M_0 satisfies Bt if and only if M_C satisfies Bt .

(ii) If $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, then M_0 satisfies $a - Bt$ implies M_C satisfies $a - Bt$. If in addition either A^* or B has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$, then M_C satisfies $a - Bt$ if and only if M_0 satisfies $a - Bt$.

Here, we observe that if $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then $\sigma_w(M_0) = \sigma_w(M_C)$ and $\sigma(M_C) = \sigma(M_0)$; the hypothesis $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ implies that $\sigma_{aw}(M_C) = \sigma_{aw}(M_0)$ (and $\sigma(M_C) = \sigma(M_0)$). Observe also that if B has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$ and M_C satisfies $a - Bt$, then A and B have SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$; if A^* has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$, then $\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B)$.

Let $\pi_0(M_C) = \{\lambda \in \text{iso}\sigma(M_C) : 0 < \dim \ker(M_C - \lambda)^{-1}(0) < \infty\}$, and let $\pi_0^a(M_C) = \{\lambda \in \text{iso}\sigma_a(M_C) : 0 < \dim \ker(M_C - \lambda)^{-1}(0) < \infty\}$. An operator T is said to be *polaroid* (resp., *a-polaroid*) at a points $\lambda \in \text{iso}\sigma(T)$ (resp., $\lambda \in \text{iso}\sigma_a(T)$) if λ is a pole of the resolvent of T (resp., $(T - \lambda)\mathcal{X}$ is closed and $\text{asc}(T - \lambda) < \infty$). Let $\mathcal{R}_0(T) = \{\lambda \in \text{iso}\sigma(T) : \lambda \text{ is a finite rank pole of the resolvent of } T\}$ and $\mathcal{R}_0^a(T) = \{\lambda \in \text{iso}\sigma_a(T) : T - \lambda \in \Phi_+(\mathcal{X}), \text{asc}(T - \lambda) < \infty\}$.

In common with current terminology, we say that T satisfies *Weyl's theorem*, or *Wt* (resp., *a-Weyl's theorem*, or *a - Wt*) if $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$ (resp., $\sigma_a(T) \setminus \sigma_{aw}(T) = \pi_0^a(T)$).

The following result is not difficult to prove:

Theorem 1.4 (a). M_C satisfies *Wt* if and only if M_C has SVEP at $\lambda \notin \sigma_w(M_C)$ and M_C is polaroid at points $\mu \in \pi_0(M_C)$.

(b). M_C satisfies *a - Wt* if and only if M_C has SVEP at $\lambda \notin \sigma_{aw}(M_C)$ and M_C is a-polaroid at points $\mu \in \pi_0^a(M_C)$.

Combining Theorems 1.1 and 1.4, an additional well known argument, see [5] and [6], implies the following:

Theorem 1.5 [5, Theorem 3.7] If either of the SVEP hypotheses (i), (ii) and (iii) of Theorem 1.1(a) is satisfied, then M_C satisfies *Wt* for every $C \in B(\mathcal{X})$ if and only if M_0 satisfies *Wt* and A is polaroid at $\lambda \in \pi_0(M_C)$.

Theorem 1.5 implies the following:

Corollary 1.6 (a) [4, Theorem 4.2] If $\{\Xi(A) \cap \Xi(B^*)\} \cup \Xi(A^*) = \emptyset$, A is polaroid at $\lambda \in \pi_0(M_C)$ (or A is isoloid and satisfies *Wt*) and M_0 satisfies *Wt*, then M_C satisfies *Wt*.

(b) [2, Theorem 3.3] If $\sigma_{aw}(A) = \sigma_{SF_+}(B)$ or $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B) = \emptyset$, A is polaroid at $\lambda \in \pi_0(M_C)$ (or A is isoloid and satisfies *Wt*) and M_0 satisfies *Wt*, then M_C satisfies *Wt*.

Both Theorem 1.5 and Corollary 1.6 are subsumed by the following general result.

Theorem 1.7 *If $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then M_0 satisfies $Wt \iff M_C$ satisfies Wt if and only if $\mathcal{R}_0(M_0) = \pi_0(M_C)$.*

The proof of the theorem is a straightforward consequence of the facts that $Wt \implies Bt$, and $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B) \implies \sigma_w(M_0) = \sigma_w(M_C)$ and $\sigma(M_C) = \sigma(M_0)$.

The result corresponding to Theorem 1.7 for $a - Wt$ is the following:

Theorem 1.8 (i). *If $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$, then M_0 satisfies $a - Wt$ implies M_C satisfies $a - Wt$ if and only if $\pi_0^a(M_C) \subseteq \pi_0^a(M_0)$.*
(ii). *Conversely, if $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$ and A^* has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$, then M_C satisfies $a - Wt$ implies M_0 satisfies $a - Wt$ if and only if $\pi_0^a(M_0) \subseteq \pi_0^a(M_C)$.*

Theorem 1.8 implies in particular that:

Corollary 1.9 [5, Theorem 3.11] *If $\Xi(A^*) \cup \Xi(B^*) = \emptyset$, A is polaroid at $\lambda \in \pi_0^a(M_C)$ (or, A is isoloid and satisfies Wt) and B is polaroid at $\mu \in \pi_0^a(B)$, then M_C satisfies $a - Wt$.*

We note here that the hypothesis $\Xi(A^*) \cup \Xi(B^*) = \emptyset$ implies that a -poles of A and B are indeed poles of their respective resolvents.

It is well known that if a Banach space operator T is such that T^* has SVEP, then T satisfies Wt if and only if T satisfies $a - Wt$. Observe that a sufficient condition for M_0^* and M_C^* to have SVEP is that both A^* and B^* have SVEP. More generally:

Theorem 1.10 *Let $M_X = M_0$ or M_C . If A^* has SVEP on $\sigma(A) \setminus \sigma_{SF_+}(A)$ and B^* has SVEP on $\sigma(B) \setminus \sigma_{SF_+}(B)$, then M_X satisfies $Wt \iff M_X$ satisfies $a - Wt$.*

Although T has SVEP and satisfies Wt does not guarantee T^* satisfies $a - Wt$, we do have that if T has SVEP and is polaroid, then T satisfies Wt and T^* satisfies $a - Wt$. This leads us to: *if A and B have SVEP and are polaroid, then M_C satisfies Wt and M_C^* satisfies $a - Wt$.*

Acknowledgements

Paper presented at RIMS 2008, Research Institute for Mathematical Sciences, Kyoto University, October 29 – October 31. It is my pleasure to thank the organisers of the Conference, especially Professors M. Fujii and Muneo Chō, for making it possible for me to partake. Many thanks for the hospitality, which is highly appreciated.

References

- [1] Bruce A. Barnes, *Riesz points of upper triangular operator matrices*, Proc. Amer. Math. Soc. **133**(2005), 1343-1347.
- [2] Xiaohong Cao and Bin Meng, *Essential approximate point spectra and Weyl's theorem for operator matrices*, J. Math. Anal. Appl. **304**(2005), 759-771.
- [3] Dragan Djordjević, *Perturbation of spectra of operator matrices*, J. Oper. Th. **48**(2002), 467-486.
- [4] Slavisa V. Djordjević and Hassan Zguitti, *Essential point spectra of operator matrices through local spectral theory*, J. Math. Anal. Appl. **338**(2008), 285-291.

Upper triangular operator matrices

- [5] B. P. Duggal, *Upper triangular operator matrices, SVEP and Browder, Weyl theorems*, Integr. Equat. Op. Th.
- [6] Woo Young Lee, *Weyl's theorem for operator matrices*, Integr. Equat. Op. Th. **32**(1998), 319-331.
- [7] Woo Young Lee, *Weyl spectra of operator matrices*, Proc. Amer. Math. Soc. **129**(2001), 131-138.

8 Redwood Grove, Northfield Avenue
Ealing, London W5 4SZ, United Kingdom.
e-mail: bpduggal@yahoo.co.uk