

AN APPLICATION OF MELLIN-BARNES TYPE INTEGRALS TO THE MEAN SQUARES OF DIRICHLET-HURWITZ-LERCH L-FUNCTIONS

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ABSTRACT. Complete asymptotic expansions associated with the mean squares, in the discrete and continuous forms, of Dirichlet-Hurwitz-Lerch L -functions are presented (Theorems 1 and 2), together with their outlined proofs.

1. INTRODUCTION

Throughout the paper, $s = \sigma + it$, u and v are complex variables, α and λ real parameters with $\alpha \geq 0$, χ any Dirichlet character modulo (arbitrary) $q \geq 1$, and $\bar{\chi}$ the complex conjugate of χ . We frequently use the notation $e(s) = e^{2\pi is}$, $e_q(s) = e(s/q) = e^{2\pi is/q}$, denote by ι the principal character modulo $q \geq 1$, and write $X_c(l) = X(c+l)$ ($c, l \in \mathbb{Z}$) for any Dirichlet character X .

The Dirichlet-Hurwitz-Lerch L -function $L_{\chi_c}(s, \alpha, \lambda)$ is defined by

$$(1.1) \quad L_{\chi_c}(s, \alpha, \lambda) = \sum'_{l=0}^{\infty} \frac{\chi_c(l) e_f\{(\alpha + l)\lambda\}}{(\alpha + l)^s} \quad (\operatorname{Re}(s) = \sigma > 1),$$

and its meromorphic continuation over the whole s -plane. The primed summation symbols hereafter indicate omission of the impossible terms of the form $1/0^s$ (if they occur). This reduces if $(q, \chi) = (1, \iota)$ to the Lerch zeta-function $\psi(s, \alpha, \lambda) = e(\alpha\lambda)\phi(s, \alpha, \lambda)$, and further if $(q, \lambda) = (1, 0)$ to the Hurwitz zeta-function $\zeta(s, \alpha)$, while if $(q, \alpha) = (1, 0)$ to the exponential zeta-function $\zeta_\lambda(s)$, if $(\alpha, \lambda) = (0, 0)$ to the (shifted) Dirichlet L -function $L_{\chi_c}(s)$, and hence if $(q, \chi) = (1, \lambda)$ and $(\alpha, \lambda) = (0, 0)$ to the Riemann zeta-function $\zeta(s)$.

A more flexible definition of the Dirichlet-Hurwitz-Lerch L -function, for any real α and λ , and for any integer c , asserts

$$(1.2) \quad L_{\chi_c}^*(s, \alpha, \lambda) = \sum_{-\alpha < l \in \mathbb{Z}} \frac{\chi_c(l) e_q\{(\alpha + l)\lambda\}}{(\alpha + l)^s} \quad (\operatorname{Re}(s) = \sigma > 1),$$

for which several results have recently been shown by Noda and the author [13]. Let $\Gamma(s)$ denote the gamma function, and $G_\chi = \sum_{h=0}^{q-1} \chi(h) e_q(h)$ Gauß' sum. We can show:

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Theorem -2 ([13, Theorem 5]). *For any real α and λ , any integer c , and any primitive character χ modulo $q \geq 1$, we have*

$$L_{\chi^c}^*(s, \alpha, \lambda) = e_q\{\lambda(\alpha - c)\} \frac{G_\chi \Gamma(s)}{q^s (2\pi)^{1-s}} \left\{ \chi(-1) e^{\pi i(1-s)/2} L_{\bar{\chi}}^*(1-s, \lambda, -(\alpha - c)) \right. \\ \left. + e^{-\pi i(1-s)/2} L_{\bar{\chi}}^*(1-s, -\lambda, \alpha - c) \right\}.$$

Next let B_k ($k = 0, 1, \dots$) be the Bernoulli numbers (cf [3, p.35, 1.13.(1)]), and τ a complex parameter in the sector $|\arg \tau| < \pi/2$. Then the celebrated formulae of Euler and Ramanujan for specific values of $\zeta(s)$ assert respectively that

$$\zeta(2k) = \frac{(-1)^{k-1} (2\pi)^{2k}}{2(2k)!} B_{2k} \quad (k = 1, 2, \dots),$$

and for any integer $k \neq 0$,

$$\zeta(2k+1) + 2 \sum_{l=1}^{\infty} \frac{l^{-2k-1} e^{-2\pi l \tau}}{1 - e^{-2\pi l \tau}} + (2\pi)^{2k+1} \sum_{j=0}^{k+1} \frac{(-1)^j B_{2j} B_{2k+2-2j}}{(2j)!(2k+2-2j)!} \tau^{2k+1-2j} \\ = (-1)^k \tau^{2k} \left\{ \zeta(2k+1) + 2 \sum_{l=1}^{\infty} \frac{l^{-2k-1} e^{-2\pi l/\tau}}{1 - e^{-2\pi l/\tau}} \right\}.$$

Let $L_{\chi_a}^*(s, \alpha, \mu)$ and $L_{\psi_b}^*(s, \beta, \nu)$ for any real α, β, μ and ν , and any integers a and b be the Dirichlet-Hurwitz-Lerch L -functions (defined by (1.2)), attached to any (shifted) primitive characters χ_a and ψ_b modulo $f \geq 1$ and $g \geq 1$ respectively. Then we can further show:

Theorem -1 ([13, Theorem 4]). *There exist various character analogues of Euler's formula for $L_{\chi_a}^*(s, \alpha, \mu)$, as well as of Ramanujan's formula connecting specific values of $L_{\chi_a}^*(s, \alpha, \mu)$ and $L_{\psi_b}^*(s, \beta, \nu)$ with any primitive characters of (possibly) different moduli.*

The observation above suggests that the following empirical 'theorem' seems to be true!

Theorem 0. *It is worth pursuing the functional (or arithmetical) nature of a class of Dirichlet-Hurwitz-Lerch L -functions.*

2. ASYMPTOTICS FOR THE DISCRETE MEAN SQUARE

Let $\varphi(n)$ denote Euler's totient function, $\mu(n)$ Möbius' function, and write, for any $n \in \mathbb{Z}$, the shifted factorial of s as

$$(s)_n = \frac{\Gamma(s+n)}{\Gamma(s)} = \begin{cases} s(s+1) \cdots (s+n-1) & \text{if } n \geq 0, \\ \frac{1}{(s-1)(s-2) \cdots (s-|n|)} & \text{if } n < 0. \end{cases}$$

The chief concern in this section is the asymptotic expansions for the discrete mean square

$$(2.1) \quad \varphi(q)^{-1} \sum_{\chi \pmod{q}} |L_{\chi^c}(\sigma + it, \alpha, \lambda)|^2,$$

averaged over all characters χ modulo $q \geq 1$.

We give here an overview of the results related to (2.1). Atkinson [1] first established a precise asymptotic formula for the error term $E(T)$ of the mean square $\int_0^T |\zeta(1/2 + it)|^2 dt$ in terms of an innovative dissection method applied to the product $\zeta(u)\zeta(v)$. Heath-Brown [5] derived, irrelevant to [1], an asymptotic series for $\sum_{\chi(\bmod q)} |L_\chi(1/2)|^2$ (at the central point) as $q \rightarrow +\infty$. Motohashi [16] obtained, when $q = p$ is a prime, an asymptotic formula for $(p-1)^{-1} \sum_{\chi(\bmod p)} |L_\chi(1/2 + it)|^2$ as $p \rightarrow +\infty$ with the error term $O(p^{-3/2})$, based Atkinson's dissection method. Matsumoto and the author [10] established a (ramified) asymptotic expansion for $\varphi(q)^{-1} \sum_{\chi(\bmod q)} |L_\chi(\sigma + it)|^2$ as $q \rightarrow +\infty$, in the stripe $0 < \sigma < 1$, which further implies, when $q = p$ is a prime, a complete asymptotic expansion for $(p-1)^{-1} \sum_{\chi(\bmod p)} |L_\chi(\sigma + it)|^2$ as $p \rightarrow +\infty$ through the set of primes, in the same region of σ above, based on Atkinson's dissection method. They [11] derived, taking the limit $\sigma + it \rightarrow 1^-$ of the result above, a complete asymptotic expansion for $(p-1)^{-1} \sum_{\chi(\bmod p), \chi \neq 1} |L_\chi(1)|^2$ as $p \rightarrow +\infty$ through the set of primes. The author [7] gave a quite transparent treatment of the same discrete mean squares by joining Atkinson's dissection method to the Mellin-Barnes type integrals, which appropriate to the relevant settings. The reader is to be referred, e.g. to [9, Sect. 2] for a more detailed history.

We now proceed to state our first main result. For this, let $\langle x \rangle = x - [x]$ denote the fractional part of $x \in \mathbb{R}$, and define the (exceptional) set $E \subset \mathbb{C}$ as

$$(2.2) \quad E = \{s \in \mathbb{C} \mid \operatorname{Re} s = 1 - n/2 \text{ or } s = 1 - n \quad (n = 0, 1, \dots)\}.$$

Theorem 1. *Let $c, q \in \mathbb{Z}$ and $\alpha, \lambda \in \mathbb{R}$ be arbitrary with $q \geq 1$ and $\alpha \geq 0$. Then for any integer $N \geq 0$, in the region $-N + 1 < \sigma < N + 1$ except the points $\sigma + it \in E$, we have*

$$(2.3) \quad \begin{aligned} & \varphi(q)^{-1} \sum_{\chi(\bmod q)} |L_{\chi_c}(\sigma + it, \alpha, \lambda)|^2 \\ &= q^{-2\sigma} \sum_{k|q} \mu\left(\frac{q}{k}\right) k^{2\sigma} \zeta\left(2\sigma, \frac{\alpha k}{q} + \left\langle -\frac{ck}{q} \right\rangle\right) \\ & \quad + 2q^{-2\sigma} \varphi(q) \Gamma(2\sigma - 1) \operatorname{Re} \left\{ \zeta_\lambda(2\sigma - 1) \frac{\Gamma(1 - \sigma + it)}{\Gamma(\sigma + it)} \right\} \\ & \quad + 2q^{-2\sigma} \sum_{k|q} \mu\left(\frac{q}{k}\right) \operatorname{Re} \{ S_{c,q}(\sigma + it, \sigma - it; \alpha, \lambda; k) \}, \end{aligned}$$

where k runs through all positive divisors of q , and $S_{c,q}$ is given by

$$\begin{aligned} S_{c,q}(u; v; \alpha, \lambda; k) &= \sum_{n=0}^{N-1} \frac{(-1)^n (u)_n}{n!} \zeta_\lambda(u + n) \zeta\left(v - n, \frac{\alpha k}{q} + \left\langle -\frac{ck}{q} \right\rangle\right) k^{v-n} \\ & \quad + T_{c,q,N}(u, v; \alpha, \lambda; k). \end{aligned}$$

Here $T_{c,q,N}$ is the reminder expressed by the Mellin-Barnes type integral in (2.9) below, and bounded above as

$$\begin{aligned} & T_{c,q,N}(\sigma + it; \sigma - it; \alpha, \lambda; k) \\ &= \begin{cases} O\{k^{\sigma-N}(|t| + 1)^{2N+1/2-\sigma}\} & \text{if } -N + 1 < \sigma < N, \\ O\{k^{\sigma-N}(|t| + 1)^{(3N+1-\sigma)/2+\varepsilon}\} & \text{if } N \leq \sigma < N + 1 \end{cases} \end{aligned}$$

for any $\varepsilon > 0$, where the implied O -constants depend at most on σ, q, N and ε .

Remark. The presence of the error bounds above is reasonable, since the n -th indexed term in the asymptotic series is of order

$$\ll \begin{cases} k^{\sigma-n}(|t|+1)^{2n+1/2-\sigma} & \text{if } -n+1 < \sigma < n, \\ k^{\sigma-n}(|t|+1)^{(3n+1-\sigma)/2+\varepsilon} & \text{if } n \leq \sigma < n+1. \end{cases}$$

Let $\delta(x)$ is equal to 1 or 0 according to $x \in \mathbb{Z}$ or otherwise, and $\gamma_j(\alpha, \lambda)$ ($j = 0, 1, \dots$) the j -th generalized Euler-Stieltjes constants defined by

$$\psi(s, \alpha, \lambda) = \frac{\delta(\lambda)}{s-1} + \sum_{j=0}^{\infty} \gamma_j(\alpha, \lambda)(s-1)^j$$

centered at $s = 1$, where $\gamma_j = \gamma_j(0, 0) = \gamma_j(1, 0)$ ($j = 0, 1, \dots$) are the classical Euler-Stieltjes constant (cf. [3, p.34, 1.12.(17)]). The asymptotic expansions on the exceptional set E (see (2.2)) can then be deduced from Theorem 1 by taking appropriate limits, e.g., the following formulae are valid.

Corollary 1.1. *Under the same settings as in Theorem 1, we have:*

i) letting $\sigma \rightarrow 1/2$,

$$\begin{aligned} & \varphi(q)^{-1} \sum_{\chi(\bmod q)} \left| L_{\chi_c} \left(\frac{1}{2} + it, \alpha, \lambda \right) \right|^2 \\ &= q^{-1} \sum_{k|q} \mu \left(\frac{q}{k} \right) k \gamma_0 \left(\frac{\alpha k}{q} + \left\langle -\frac{ck}{q} \right\rangle, 0 \right) \\ &+ \frac{\varphi(q)}{q} \left[\log q + \sum_{p|q} \frac{\log p}{p-1} + \operatorname{Re} \left\{ \zeta'_\lambda(0) - \zeta_\lambda(0) \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} + it \right) \right\} + \gamma_0 \right] \\ &+ 2q^{-1} \sum_{k|q} \mu \left(\frac{q}{k} \right) \operatorname{Re} \left\{ S_{c,q} \left(\frac{1}{2} + it, \frac{1}{2} - it; \alpha, \lambda; k \right) \right\}; \end{aligned}$$

ii) letting $\sigma \rightarrow 1$,

$$\begin{aligned} & \varphi(q)^{-1} \sum_{\chi(\bmod q)} \left| L_{\chi_c} (1 + it, \alpha, \lambda) \right|^2 \\ &= q^{-2} \sum_{k|q} \mu \left(\frac{q}{k} \right) k^2 \zeta \left(2, \frac{\alpha k}{q} + \left\langle -\frac{ck}{q} \right\rangle \right) \\ &+ \frac{\varphi(q)}{q^2} \left[-\delta(\lambda) \left\{ 2 \operatorname{Re} \frac{1}{it} \frac{\Gamma'}{\Gamma} (1 + it) + \frac{1}{t^2} \right\} + \frac{\operatorname{Im} \gamma_0(0, \lambda)}{t} \right] \\ &+ 2q^{-2} \sum_{k|q} \mu \left(\frac{q}{k} \right) \operatorname{Re} \left\{ S_{c,q} (1 + it, 1 - it; \alpha, \lambda; k) \right\}. \end{aligned}$$

Here k runs through all positive divisors of q , p through all prime divisors of q , and $S_{c,q}$ gives an asymptotic series as in Theorem 1.

The formula in Theorem 1 does not asserts (in a strict sense) a complete asymptotic expansion in the descending order of q itself; however it gives, if $q = p$ is a prime, a (true)

complete asymptotic expansion in the descending order of p , since $S_{c,p}(u, v; \alpha, \lambda; 1)$ can be computed explicitly.

Corollary 1.2. *Under the same settings as in Theorem 1, in the region $-N + 1 < \sigma < N + 1$ except the points $\sigma + it \in E$, we have*

$$\begin{aligned}
 (2.4) \quad & (p-1)^{-1} \sum_{\chi(\bmod p)} |L_{\chi_c}(\sigma + it, \alpha, \lambda)|^2 \\
 &= (1 + p^{-2\sigma}) \zeta(2\sigma, \alpha) - p^{-2\sigma} \zeta\left(2\sigma, \frac{\alpha}{p} + \left\langle -\frac{c}{p} \right\rangle\right) - p^{-2\sigma} |\psi(\sigma + it, \alpha, \lambda)|^2 \\
 &\quad + 2p^{1-2\sigma} \Gamma(2\sigma - 1) \operatorname{Re} \left\{ \zeta_\lambda(2\sigma - 1) \frac{\Gamma(1 - \sigma + it)}{\Gamma(\sigma + it)} \right\} \\
 &\quad + 2p^{-2\sigma} \operatorname{Re} \{ S_{c,p}(\sigma + it, \sigma - it; \alpha, \lambda; p) \},
 \end{aligned}$$

whose limiting case $\sigma \rightarrow 1/2$ asserts

$$\begin{aligned}
 (2.5) \quad & (p-1)^{-1} \sum_{\chi(\bmod p)} \left| L_{\chi_c} \left(\frac{1}{2} + it, \alpha, \lambda \right) \right|^2 \\
 &= (1 + p^{-1}) \gamma_0(\alpha, 0) - p^{-1} \gamma_0 \left(\frac{\alpha}{p} + \left\langle -\frac{c}{p} \right\rangle, 0 \right) \\
 &\quad - 2 \operatorname{Re} \left\{ \zeta'_\lambda(0) + \zeta_\lambda(0) \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} + it \right) \right\} + \gamma_0 + \log p - p^{-1} \left| \psi \left(\frac{1}{2} + it, \alpha, \lambda \right) \right|^2 \\
 &\quad + 2p^{-1} \operatorname{Re} \left\{ S_{c,p} \left(\frac{1}{2} + it; \frac{1}{2} - it; \alpha, \lambda; p \right) \right\}.
 \end{aligned}$$

Here the term $S_{c,p}$, both in (2.4) and (2.5), gives a complete asymptotic expansion in the descending order of p as $p \rightarrow +\infty$ through the set of primes.

We now proceed to outline of the proof Theorem 1. For this, we set

$$R(u, v; \lambda) = \Gamma(u + v - 1) \zeta_\lambda(u + v - 1) \frac{\Gamma(1 - v)}{\Gamma(u)},$$

use a (modified) Möbius' inversion

$$\sum'_{\substack{l=0 \\ (c+l, q)=1}}^{\infty} \frac{1}{(\alpha + l)^s} = q^{-s} \sum_{k|q} \mu \left(\frac{q}{k} \right) k^s \zeta \left(s, \frac{\alpha k}{q} + \left\langle -\frac{ck}{q} \right\rangle \right) \quad (\sigma > 1),$$

and write

$$\Gamma \left(\frac{\alpha_1, \dots, \alpha_m}{\beta_1, \dots, \beta_n} \right) = \frac{\prod_{h=1}^m \Gamma(\alpha_h)}{\prod_{k=1}^n \Gamma(\beta_k)}$$

for $\alpha_h, \beta_k \in \mathbb{C}$ ($h = 1, \dots, m; k = 1, \dots, n$). The dissection formula, for $\operatorname{Re}(u) > 1$ and $\operatorname{Re}(v) > 1$,

$$\begin{aligned} & \varphi(q)^{-1} \sum_{\chi \pmod{q}} L_{\chi_c}(u, \alpha, \lambda) L_{\bar{\chi}_c}(v, \alpha, -\lambda) \\ &= q^{-u-v} \sum_{k|q} \mu\left(\frac{q}{k}\right) k^{u+v} \zeta\left(u+v, \frac{\alpha k}{q} + \left\langle -\frac{ck}{q} \right\rangle\right) + f_{c,q}(u, v; \alpha, \lambda) \\ & \quad + f_{c,q}(v, u; \alpha, -\lambda), \end{aligned}$$

is crucial in proving Theorem 1, where $f_{c,q}$ is a variant of Euler's double zeta-function (see [1] for the case of $\zeta(u)\zeta(v)$). This further splits into

$$f_{c,q}(u, v; \alpha, \lambda) = q^{-u-v} \varphi(q) R(u, v; \lambda) + g_{c,q}(u, v; \alpha, \lambda),$$

where $g_{c,q}$ is given by

$$g_{c,q}(u, v; \alpha, \lambda) = q^{-u-v} \sum_{k|q} \mu\left(\frac{q}{k}\right) S_{c,q}(u, v; \alpha, \lambda; k)$$

with $S_{c,q}$ being expressed as the Mellin-Barnes type integral of the form

$$\begin{aligned} (2.6) \quad S_{c,q}(u, v; \alpha, \lambda; k) &= \frac{1}{2\pi i} \int_{\mathcal{C}} \Gamma\left(\begin{matrix} u+s, -s \\ u \end{matrix}\right) \zeta_{\lambda}(-s) \\ & \quad \times \zeta\left(u+v+s, \frac{\alpha k}{q} + \left\langle -\frac{ck}{q} \right\rangle\right) k^{u+v+s} ds, \end{aligned}$$

where the path $\mathcal{C}$ separates the poles of the integrand at $s = 1 - u - v$ and $s = -1 + m$ ($m = 0, 1, \dots$) from those at $s = -u - n$ ($n = 0, 1, \dots$).

We proceed further along the lines above, moving appropriately the path $\mathcal{C}$ to the left, and eventually obtain the formula (2.8) below, which yields, upon setting $(u, v) = (\sigma + it, \sigma - it)$, various *complete* asymptotic expansions for the discrete mean square (2.1). Let (σ) for any $\sigma \in \mathbb{R}$ denote the vertical straight path from $\sigma - i\infty$ to $\sigma + i\infty$, and define the (exceptional) set $\tilde{E} \subset \mathbb{C}^2$ as

$$(2.7) \quad \tilde{E} = \left\{ (u, v) \in \mathbb{C}^2 \mid u+v = 2-n \text{ or } u = 1-n \text{ or } v = 1-n \ (n = 0, 1, 2, \dots) \right\}.$$

We can then show the following formula. For any integer $N \geq 0$, in the region $-N+1 < \operatorname{Re}(u) < N+1$ and $-N+1 < \operatorname{Re}(v) < N+1$ except the points $(u, v) \in \tilde{E}$, we have

$$\begin{aligned} (2.8) \quad & \varphi(q)^{-1} \sum_{\chi \pmod{q}} L_{\chi_c}(u, \alpha, \lambda) L_{\bar{\chi}_c}(v, \alpha, -\lambda) \\ &= q^{-u-v} \sum_{k|q} \mu\left(\frac{q}{k}\right) k^{u+v} \zeta\left(u+v, \frac{\alpha k}{q} + \left\langle -\frac{ck}{q} \right\rangle\right) + q^{-u-v} \varphi(q) \{ R(u, v; \lambda) \\ & \quad + R(v, u; -\lambda) \} + q^{-u-v} \sum_{k|q} \mu\left(\frac{q}{k}\right) \{ S_{c,q}(u, v; \alpha, \lambda; k) + S_{c,q}(v, u; \alpha, -\lambda; k) \}, \end{aligned}$$

where $S_{c,q}$ is expressed as

$$S_{c,q}(u, v; \alpha, \lambda; k) = \sum_{n=0}^{N-1} \frac{(-1)^n (u)_n}{n!} \zeta_\lambda(u+n) \zeta\left(v-n, \frac{\alpha k}{q} + \left\langle \frac{ck}{q} \right\rangle\right) k^{v-n} \\ + T_{c,q,N}(u, v; \alpha, \lambda; k),$$

and $T_{c,q,N}$ is given by the Mellin-Barnes type integral

$$(2.9) \quad T_{c,q,N}(u, v; \alpha, \lambda; k) = \frac{1}{2\pi i} \int_{(\sigma_N)} \Gamma\left(u+s, -s\right)_u \zeta_\lambda(-s) \\ \times \zeta\left(u+v+s, \frac{\alpha k}{q} + \left\langle -\frac{ck}{q} \right\rangle\right) k^{u+v+s} ds$$

with σ_N satisfying $-\operatorname{Re}(u)-N < \sigma_N < \min(-\operatorname{Re}(u)-N+1, -1, -\operatorname{Re}(u+v))$. Theorem 1 is in fact a direct consequence of (2.8) upon setting $(u, v) = (\sigma + it, \sigma - it)$.

3. ASYMPTOTIC EXPANSIONS FOR THE CONTINUOUS MEAN SQUARE

The chief concern in this section is the asymptotic expansions for the continuous mean square

$$(3.1) \quad \int_0^1 |L_{\chi_c}(\sigma + it, \alpha + q\xi, \lambda)|^2 d\xi.$$

We give here an overview of the results on (3.1), mainly when $(q, \chi) = (1, \iota)$, $(\alpha, \lambda) = (1, 0)$, i.e. the case of the continuous mean square of $\zeta(s, 1 + \xi)$. Koksma-Lekkerkerker [14] initiated the study into the direction to obtain the asymptotic bound $O(\log t)$ as $t \rightarrow +\infty$ on the critical line $\sigma = 1/2$. Subsequent research are made by Gallagher [4], Balasubramanian [2], Rane [17], Klush [15], Zhang [18][19]. Matsumoto and the author [11] established, for the case above, a complete asymptotic expansion in the descending order of $\operatorname{Im} s = t$ as $t \rightarrow +\infty$, by means of Atkinson's dissection method. The author [6] derived a complete asymptotic expansion when $(q, \chi) = (1, \iota)$ and $\alpha = 1$, i.e. for the case of the continuous mean square of $\phi(s, 1 + \xi, \lambda)$, by means of Atkinson's dissection method enhanced by Mellin-Barnes type integrals. The author [8] established a complete asymptotic expansion, when $(q, \chi) = (1, \iota)$, for the multiple mean square of the form

$$\int_0^1 \cdots \int_0^1 |\phi(s, \alpha + \xi_1 + \cdots + \xi_m, \lambda)|^2 d\xi_1 \cdots d\xi_m \quad (m = 1, 2, \dots)$$

in the descending order of $\operatorname{Im} s = t$ as $t \rightarrow \pm\infty$, by means of Atkinson's dissection method, enhanced by Mellin-Barnes type integrals. These are further manipulated with several properties of (generalized) hypergeometric functions. The reader is to be referred, e.g. to [9, Sect. 3] for a more detailed history.

We proceed to state our second main result. We set, for any $(u, v) \in \mathbb{C}^2 \setminus \widetilde{E}$ (see (2.7)),

$$\mathcal{R}_{\chi_c}(u, v; \lambda) = \Gamma(u+v-1) \frac{\Gamma(1-v)}{\Gamma(u)} \sum_{a,b=0}^{q-1} \chi_c(1+a+b) \overline{\chi_c}(b) \psi\left(u+v-1, \frac{1+a}{q}, \lambda\right).$$

Theorem 2. Let $c, q \in \mathbb{Z}$ and $\alpha, \lambda \in \mathbb{R}$ be arbitrary with $q \geq 1$ and $\alpha \geq 0$. Then for any integer $N \geq 0$, in the region $-N+1 < \sigma < N+1$ except the points $\sigma + it \in E$ (see (2.2)), we have

$$(3.2) \quad \int_0^1 |L_{\chi_c}(\sigma + it, \alpha + q\xi, \lambda)|^2 d\xi \\ = -\frac{q^{-2\sigma}}{1-2\sigma} \sum_{k|q} \mu\left(\frac{q}{k}\right) k^{2\sigma} \left(\frac{\alpha k}{q} + \left\langle -\frac{ck}{q} \right\rangle\right)^{1-2\sigma} + 2q^{-2\sigma} \operatorname{Re}\{\mathcal{R}_{\chi_c}(\sigma + it, \sigma - it; \lambda)\} \\ - 2q^{-2\sigma} \operatorname{Re}\{\mathcal{S}_{\chi_c, N}(\sigma + it, \sigma - it; \alpha, \lambda) + \mathcal{T}_{\chi_c, N}(\sigma + it, \sigma - it; \alpha, \lambda)\},$$

where $\mathcal{S}_{\chi_c, N}$ and $\mathcal{T}_{\chi_c, N}$ are given by

$$\mathcal{S}_{\chi_c, N}(u, v; \alpha, \lambda) = \sum_{a, b=0}^{q-1} \chi_c(1+a+b) \bar{\chi}_c(b) \mathcal{S}_N\left(u, v; \frac{1+a}{q}, \frac{\alpha+b}{q}, \lambda\right), \\ \mathcal{T}_{\chi_c, N}(u, v; \alpha, \lambda) = \sum_{a, b=0}^{q-1} \chi_c(1+a+b) \bar{\chi}_c(b) \mathcal{T}_N\left(u, v; \frac{1+a}{q}, \frac{\alpha+b}{q}, \lambda\right)$$

with

$$(3.3) \quad \mathcal{S}_N(u, v; x, y, \lambda) = \sum_{n=0}^{N-1} \frac{(u)_n y^{n+1-v}}{(1-v)_{n+1}} e(-y\lambda) \psi(u+n; x+y, \lambda),$$

$$(3.4) \quad \mathcal{T}_N(u, v; x, y, \lambda) = \frac{(u)_N y^{N+1-v}}{(1-v)_N} \sum_{l=0}^{\infty} \frac{e\{(x+l)\lambda\}}{(x+l)^{u+v-1}} \int_{x+l}^{\infty} \frac{\eta^{u+v-2}}{(y+\eta)^{u+N-1}} d\eta.$$

Note that the last expression converges absolutely for $\operatorname{Re}(u) > -N+1$ and $\operatorname{Re}(v) < N+1$. This further asserts

$$(3.5) \quad \mathcal{T}_N(u, v; x, y, \lambda) \\ = y^{N+1-v} \left[\sum_{k=1}^K \frac{(-1)^{k-1} (2-u+v)_{k-1} (u)_{N-k}}{(1-v)_N} \sum_{l=0}^{\infty} \frac{e\{(x+l)\lambda\}}{(x+l)^{k+1-u+v} (x+y+l)^{u+N-k}} \right. \\ \left. + \frac{(-1)^K (2-u+v)_K (u)_{N-K}}{(1-v)_N} \sum_{l=0}^{\infty} \frac{e\{(x+l)\lambda\}}{(x+l)^{u+v-1}} \int_{x+l}^{\infty} \frac{y^{u+v-K-2}}{(y+\eta)^{u+N-K}} d\eta \right],$$

which gives upon $(u, v) = (\sigma + it, \sigma - it)$ the asymptotic expansion in the descending order of $\operatorname{Im} s = t$ as $t \rightarrow \pm\infty$.

We proceed to outline the proof of Theorem 2. The dissection formula, for $\operatorname{Re}(u) > 1$ and $\operatorname{Re}(v) > 1$,

$$L_{\chi_c}(u, \alpha, \lambda) L_{\bar{\chi}_c}(v, \alpha, -\lambda) \\ = q^{-u-v} \sum_{k|q} \mu\left(\frac{q}{k}\right) k^{u+v} \zeta\left(u+v, \frac{\alpha k}{q} + \left\langle -\frac{ck}{q} \right\rangle\right) + f_{\chi_c}(u, v; \alpha, \lambda) + f_{\bar{\chi}_c}(v, u; \alpha, -\lambda)$$

is crucial in proving Theorem 2, where f_{χ_c} (or $f_{\bar{\chi}_c}$) is a variant of Euler's double zeta-function (see [1] for the case of $\zeta(u)\zeta(v)$). This further splits into

$$f_{\chi_c}(u, v; \alpha, \lambda) = q^{-u-v} \mathcal{R}_{\chi_c}(u, v; \lambda) + g_{\chi_c}(u, v; \alpha, \lambda),$$

where

$$g_{\chi_c}(u, v; \alpha, \lambda) = q^{-u-v} \sum_{a,b=0}^{q-1} \chi_c(1+a+b) \bar{\chi}_c(b) g\left(u, v; \frac{1+a}{q}, \frac{\alpha+b}{q}, \lambda\right)$$

with

$$g(u, v; x, y, \lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}} \Gamma\left(\begin{matrix} u+s, -s \\ u \end{matrix}\right) \psi(-s, x, \lambda) \zeta(u+v+s, y) ds,$$

where $\mathcal{C}$ is the same contour as in (2.6).

We suppose now that $\operatorname{Re}(u) > 1$ and $\operatorname{Re}(v) < 1$, under which the path $\mathcal{C}$ can be taken as a straight line $\mathcal{C} = (\sigma_0)$ with σ_0 satisfying $-\operatorname{Re}(u) < \sigma_0 < \min(-1, 1 - \operatorname{Re}(u+v))$. It suffices, for the treatment of the continuous mean square (3.1), to evaluate the integral

$$\int_0^1 g_{\chi_c}(u, v; \alpha + q\xi, \lambda) d\xi = -q^{-u-v} \sum_{a,b=0}^{q-1} \chi_c(1+a+b) \bar{\chi}_c(b) \tilde{g}\left(u, v; \frac{1+a}{q}, \frac{\alpha+b}{q}, \lambda\right),$$

say, where the relation, for any complex $s \neq 1$,

$$\int_0^1 \zeta(s, y + \xi) d\xi = -\frac{y^{1-s}}{1-s}$$

(cf. [8, Lemma 2]) is used to integrate the Mellin-Barnes type expression of $g(u, v; x, y, \lambda)$. This gives

$$\tilde{g}(u, v; x, y, \lambda) = \frac{1}{2\pi i} \int_{(\sigma_0)} \Gamma\left(\begin{matrix} u+s, -s \\ u \end{matrix}\right) \psi(-s, x, \lambda) \frac{y^{1-u-v-s}}{1-u-v-s} ds.$$

Note further that the Mellin-Barnes formula, for $0 < \operatorname{Re}(z) < \operatorname{Re}(w)$,

$$\frac{1}{w-z} = \frac{1}{2\pi i} \int_{(\rho)} \Gamma\left(\begin{matrix} z+r, w, 1+r, -r \\ z, w+1+r \end{matrix}\right) e^{\pi i r} dr$$

holds with a constant ρ satisfying $\max(-\operatorname{Re} z, -1) < \rho < 0$ (cf. [6, Lemma 3]). This upon $z = u + s$ and $w = 1 - v$ is substituted into the denominator factor $1/(1-u-v-s)$ above to transform the s -integral expression of $\tilde{g}$ as

$$\tilde{g}(u, v; x, y, \lambda) = \frac{1}{2\pi i} \int_{(\rho_0)} \Gamma\left(\begin{matrix} u+r, 1-v, 1+r, -r \\ u, 2-v+r \end{matrix}\right) e^{\pi i r} e(-y\lambda) \psi(u+r, x, \lambda) y^{1-v+r} dr$$

with ρ_0 satisfying $\max(-\operatorname{Re}(u), -\sigma_0, -1) < \rho_0 < 0$. Moving the path (ρ_0) to the right appropriately, we obtain

$$\tilde{g}(u, v; x, y, \lambda) = \mathcal{S}_N(u, v; x, y, \lambda) + \mathcal{T}_N(u, v; x, y, \lambda)$$

with the expression in (3.3) and

$$(3.6) \quad \mathcal{T}_N(u, v; x, y, \lambda) = \frac{(u)_N y^{N+1-v}}{(1-v)_{N+1}} \sum_{l=0}^{\infty} \frac{e\{(x+l)\lambda\}}{(x+y+l)^{u+N}} {}_2F_1\left(\begin{matrix} u+N, 1 \\ N+2-v \end{matrix}; \frac{y}{x+y+l}\right),$$

where ${}_2F_1$ denotes Gauß' hypergeometric function (cf. [3, p.59, 2.1.1(12)]). This is further transformed, through Euler's formula for ${}_2F_1$ (cf. [3, p.59, 2.1.3.(10)]), to imply (3.4). The asymptotic expansion in (3.5) for $\mathcal{T}_N$ is obtained by substituting the formula, coming from a repeated use of a contiguity relation of ${}_2F_1$ (cf. [3, p.103, 2.8.(37)]),

$$\begin{aligned} \frac{(u)_N}{(1-v)_{N+1}} {}_2F_1\left(\begin{matrix} u+N, 1 \\ N+2-v \end{matrix}; Z\right) &= \sum_{k=1}^K \frac{(-1)^{k-1} (2-u-v)_{k-1} (u)_{N-k}}{(1-v)_{N+1}} (1-Z)^{-k} \\ &+ \frac{(-1)^K (2-u-v)_K (u)_{N-K}}{(1-v)_{N+1}} (1-Z)^{-K} {}_2F_1\left(\begin{matrix} u+N-K, 1 \\ N+2-v \end{matrix}; Z\right) \end{aligned}$$

with $Z = y/(x+y+l)$ into each term of the series expression of $\mathcal{T}_N$ in (3.6) to yield (3.5).

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