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THE RUMIN COMPLEX ON CR MANIFOLDS

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ABSTRACT. Building on work of Rumin, Akahori, and Miyajima, we introduce a bigraded differential complex on hypersurface-type CR manifolds, whose cohomology reproduces Kohn-Rossi cohomology. Our main result is a Hodge theorem for this complex in the strictly pseudoconvex case.

1. BACKGROUND

Let M be a $(2n+1)$-dimensional nondegenerate CR manifold of hypersurface type. The principal cohomological invariants of M are its Kohn–Rossi cohomology groups $H^{p,q}(M)$. In seeking to connect $H^{p,q}(M)$ with deRham cohomology and with CR deformation theory, T. Akahori and K. Miyajima [AM] defined a new double complex $F^{p,q}$, with differential operators $d'\colon F^{p,q}\to F^{p+1,q}$ and $d''\colon F^{p,q}\to F^{p,q+1}$, and showed that the complex $(F^{p,q},d'')$ reproduces Kohn-Rossi cohomology when $p+q>n+1$. In a recent unpublished preprint [A], Akahori also claimed that subellipticity does not work when $q=n-1$ or $q=n$. In fact, it is easily seen that $\Box''$ cannot be subelliptic when $q=n$, because $H^{p,q}(M)$ is infinite-dimensional for all p in that case. We give a correct statement in Theorem 3 below; see [GL] for details.)

Recently M. Rumin [R] introduced a new way to compute deRham cohomology on a contact manifold. If θ is a contact form and $\mathcal{I}$ is the ideal in Λ^*M generated by θ and $d\theta$, set

$$F^k=\mathcal{I}^\perp\cap\Lambda^kM,\qquad E^k=\Lambda^kM/(\mathcal{I}\cap\Lambda^kM). \tag{1}$$

(Here $\mathcal{I}^\perp$ represents the annihilator of $\mathcal{I}$ with respect to the wedge product.) Since $\mathcal{I}$ is a differential ideal, it is easy to check that d maps F^k into F^{k+1}, and descends to a map, which we also call d, from E^k to E^{k+1}. However, $F^k=0$ when $k\le n$ and $E^k=0$ when $k\ge n+1$, so each of these complexes is nontrivial only in half of the degrees on a given manifold.

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Rumin constructed the following sequence:

$$0 \to \mathbf{R} \hookrightarrow E^0 \xrightarrow{d} E^1 \xrightarrow{d} \cdots \xrightarrow{d} E^n \xrightarrow{D} F^{n+1} \xrightarrow{d} \cdots \xrightarrow{d} F^{n+1} \to 0.$$

Here D is defined by $D[\omega] = d(\omega + \theta \wedge \beta)$ for $[\omega] \in E^n$, with β chosen so that $d(\omega + \theta \wedge \beta) \in F^{n+1}$. Rumin showed that D is well defined, and the above sequence is a complex, which we call the *Rumin complex*. He showed moreover that it is an acyclic resolution of the constant sheaf $\mathbf{R}$, so that its cohomology is isomorphic to deRham cohomology; and that M can be endowed with a Riemannian metric in such a way that each deRham cohomology class has a unique "harmonic" representative $\omega \in \Gamma(E^k)$ or $\Gamma(F^k)$ satisfying

$$\begin{aligned} d\omega = d^*\omega = 0 \quad & \text{in } E^k,\ k < n; \\ D\omega = d^*\omega = 0 \quad & \text{in } E^k,\ k = n; \\ d\omega = D^*\omega = 0 \quad & \text{in } F^k,\ k = n+1; \\ d\omega = d^*\omega = 0 \quad & \text{in } F^k,\ k > n+1. \end{aligned}$$

2. A BIGRADING OF THE RUMIN COMPLEX

We wish to bring together the constructions of Rumin and of Akahori and Miyajima by introducing a bigrading of Rumin's complex. To fix our notation, let $T^{1,0} \subset \mathbf{C}TM$ denote the sub-bundle defining the CR structure, with $T^{0,1} = \overline{T^{1,0}}$, $T^{1,0} \cap T^{0,1} = \{0\}$, and $[\Gamma(T^{0,1}), \Gamma(T^{0,1})] \subset \Gamma(T^{0,1})$ (the "integrability condition"). If we set $H = \operatorname{Re}(T^{1,0} \oplus T^{0,1}) \subset TM$, then H has real dimension $2n$ and carries a complex structure.

Let $\mathbf{C}E^k$ and $\mathbf{C}F^k$ denote the complexified Rumin bundles, obtained by replacing $\Lambda^k M$ in (1) by the bundle $\mathbf{C}\Lambda^k M$ of complex-valued forms, and $\mathfrak{J}$ by its complexification. Define subspaces $E^{p,q} \subset \mathbf{C}E^{p+q}$ and $F^{p,q} \subset \mathbf{C}F^{p+q}$ by

$$\begin{aligned} E^{p,q} &= \left\{ [\omega] \in \mathbf{C}E^{p+q} : \gamma|_H \text{ is of type } (p,q) \text{ for some } \gamma \in [\omega] \right\}, \\ F^{p,q} &= \left\{ \omega \in \mathbf{C}F^{p+q} : (X \lrcorner \omega)|_H \text{ is of type } (p-1,q) \text{ for any } X \notin H \right\}, \end{aligned}$$

where a complex-valued $(p+q)$-form on H is said to be of type (p,q) if it gives zero whenever it acts on more than p vectors from $T^{1,0}$ or more than q vectors from $T^{0,1}$.

It is straightforward to check that

$$\mathbf{C}E^k = \bigoplus_{p+q=k} E^{p,q}; \qquad \mathbf{C}F^k = \bigoplus_{p+q=k} F^{p,q}.$$

The integrability condition then implies that

$$\begin{aligned} d\colon E^{p,q} &\to E^{p,q+1} \oplus E^{p+1,q}, \\ d\colon F^{p,q} &\to F^{p,q+1} \oplus F^{p+1,q}. \end{aligned}$$

Write the resulting operators (d followed by projection) as d'', d'. (This coincides with Akahori and Miyajima's definition on $F^{p,q}$.) From $d^2 = (d''+d')^2 = 0$, therefore, it follows that

$$d'd' = d''d'' = d'd'' + d''d' = 0.$$

When $p+q = n$, things are not quite so nice. The best we can say is

$$D\colon E^{p,q} \to F^{p,q+1} \oplus F^{p+1,q} \oplus F^{p+2,q-1}.$$

Writing the three resulting operators as D'', D', and D^+, and decomposing $Dd = dD = 0$ into types, we obtain

$$D''d'' = d''D'' = d'D'' + d''D' = D'd'' + D''d' = 0.$$

Theorem 1. *Consider the sequence*

$$0 \to \mathcal{K}^p \hookrightarrow E^{p,0} \xrightarrow{d''} E^{p,1} \xrightarrow{d''} \cdots \xrightarrow{d''} E^{p,n-p} \xrightarrow{D''} F^{p,n-p+1} \xrightarrow{d''} \cdots \xrightarrow{d''} F^{p,2n+1-p} \to 0$$

where $\mathcal{K}^p := \ker d''\colon E^{p,0} \to E^{p,1}$ *for* $p < n$, $\mathcal{K}^n := \ker D''\colon E^{n,0} \to F^{n,1}$, *and* $\mathcal{K}^{n+1} := \ker d''\colon F^{n+1,0} \to F^{n+1,1}$. *This is an acyclic resolution of the sheaf* $\mathcal{K}^p$, *which is isomorphic to the sheaf* $\mathcal{O}^p$ *of CR-holomorphic* $(p,0)$ *forms. Therefore the* q^{th} *cohomology group of the complex above is isomorphic to Kohn-Rossi cohomology* $H^{p,q}(M)$ *for all* p,q.

Unfortunately, this does not fit together into a double complex because $D'd'$ and $d'D'$ are not zero in general. Instead, we have:

Theorem 2. *The filtration of the Rumin complex R defined by*

$$\mathcal{F}^p R^{p+q} = R^{p,q} + R^{p+1,q-1} + \cdots + R^{n+1,p+q-n-1}$$

(where $R = E$ or F as appropriate) induces a CR-invariant spectral sequence $R_r^{p,q}$, *whose* $r = 1$ *term is Kohn-Rossi cohomology and which converges to the graded group associated with the induced filtration of deRham cohomology.*

The proofs of these two theorems will be given in [GL].

3. Hodge Theory

By themselves the results of the preceding section don't tell us much that is new about CR manifolds. Tanaka [T] studied a similar spectral sequence defined in terms of the deRham complex, and it can be shown that our spectral sequence is isomorphic to his for $r \geq 1$. Our hope is that the real utility of this point of view will be proved by applying Hodge theory.

As above, let $R^{p,q}$ denote $E^{p,q}$ when $p+q \leq n$, and $F^{p,q}$ when $p+q \geq n+1$. Let θ be a fixed choice of contact form on M, and T its *characteristic vector field*: this is the vector field uniquely determined by $T \lrcorner\, \theta = 1$, $T \lrcorner\, d\theta = 0$. The *Webster metric* g_θ is the Riemannian metric defined by using the Levi form on H and declaring T to be orthonormal to H (see [W]). Using this metric, we can identify the quotient bundles $E^{p,q}$ with honest bundles of $(p+q)$-forms by noting that $E^{p,q} \cong E_\theta^{p,q} := \{\omega \in \mathbb{C}\Lambda^{p+q}M \ : \ \omega|_H$ is of type (p,q) and $\omega \perp \mathfrak{I}$ with respect to $g_\theta\}$, and then we can define adjoint operators d''^* of d'' and D''^* of D''. We say a section u of $R^{p,q}$ is *harmonic* if $d''u = d''^*u = 0$, with d'' replaced by D'' when $p+q=n$, and d''^* by D''^* when $p+q=n+1$. Our main result is the following:

Theorem 3. (A Hodge theorem for the bigraded Rumin complex) *Let M be a compact, strictly pseudoconvex CR manifold of dimension $2n+1$, $n \geq 2$. For each (p,q), $H^{p,q}$ is isomorphic to the space of harmonic sections of $R^{p,q}$.*

Proof. We only sketch the proof here. Complete details will appear in [GL].

Let $\Box'' \colon R^{p,q} \to R^{p,q}$ be defined as follows:

$$\Box'' = \begin{cases} d''d''^* + d''^*d'', & p+q \neq n, n+1; \\ (d''d''^*)^2 + D''^*D'', & p+q = n; \\ D''D''^* + (d''^*d'')^2, & p+q = n+1. \end{cases}$$

The main part of the proof is showing that $\Box''$ is subelliptic on $R^{p,q}$ for $0 < q < n$. Akahori [A] proved this when $p+q > n+1$ and $2 \leq p, q \leq n-2$ (see the remark above). Here is a somewhat simplified version of his proof, which works also in the case $(p,q) = (n+1, 1)$.

Our choice of contact form θ determines a canonical connection ∇, the *pseudohermitian connection* [W, T], which is compatible with H and its complex structure, and with respect to which θ and $d\theta$ are parallel. For any tensor field u on M, the total covariant derivative ∇u can be decomposed as

$$\nabla u = \nabla' u + \nabla'' u + \nabla_T u \otimes \theta,$$

where $\nabla' u$ involves derivatives only with respect to $(1,0)$ vector fields, and $\nabla'' u$ only with respect to $(0,1)$ vector fields. Writing $\nabla_H u = \nabla' u + \nabla'' u$, the *Folland-Stein norms* $\|\cdot\|_k$ are defined by

$$\|u\|_k^2 = \sum_{j=0}^{k} \|\nabla_H^j u\|^2,$$

where $\|\cdot\|$ denotes the L^2 norm.

Define operators

$$\partial' u = (-1)^{p+q}\,\mathrm{Alt}(\nabla' u), \qquad \partial'' u = (-1)^{p+q}\,\mathrm{Alt}(\nabla'' u) \tag{2}$$

acting on complex-valued forms of any degree. A computation shows that $d' = \partial'$ and $d'' = \partial''$ on $F^{p,q}$. By commuting covariant derivatives, we obtain the following Bochner identity for sections of $F^{p,q}$:

$$\partial''^*\partial'' + \partial''\partial''^* = \frac{n-q}{n}\nabla''^*\nabla'' + \frac{q}{n}\nabla'^*\nabla' + \mathcal{O}_0, \tag{3}$$

where ∂''^* is the adjoint of ∂'' acting on *all* forms (not just sections of $F^{p,q}$), and $\mathcal{O}_0$ represents an operator of order zero. It follows by conjugation (noting that conjugation takes $F^{p,q}$ to $F^{q+1,p-1}$) that

$$\partial'^*\partial' + \partial'\partial'^* = \frac{n+1-p}{n}\nabla'^*\nabla' + \frac{p-1}{n}\nabla''^*\nabla'' + \mathcal{O}_0.$$

By integration, therefore, we obtain the following L^2 identities:

$$\|\partial'' u\|^2 + \|\partial''^* u\|^2 = \frac{n-q}{n}\|\nabla'' u\|^2 + \frac{q}{n}\|\nabla' u\|^2 + \mathcal{O}(\|u\|^2), \tag{4}$$

$$\|\partial' u\|^2 + \|\partial'^* u\|^2 = \frac{n+1-p}{n}\|\nabla' u\|^2 + \frac{p-1}{n}\|\nabla'' u\|^2 + \mathcal{O}(\|u\|^2). \tag{5}$$

(One can check that ∂'' agrees with the Kohn-Rossi operator $\overline{\partial}_b$ up to an operator of order zero, and that (4), which actually holds in much greater generality, gives an easy proof of the subellipticity of the Kohn Laplacian $\Box_b$ on (p,q)-forms when $0 < q < n$.)

The adjoint $d''^* : F^{p,q+1} \to F^{p,q}$ is given by ∂''^* followed by projection onto $F^{p,q}$. A straightforward computation yields for $u \in \Gamma(F^{p,q})$, $p+q > n+1$,

$$\begin{aligned}(u, \Box'' u) &= \|d'' u\|^2 + \|d''^* u\|^2 \\ &= \|\partial'' u\|^2 + \|\partial''^* u\|^2 - \frac{1}{p+q-n}\|\partial' u\|^2.\end{aligned} \tag{6}$$

Following Akahori, we use (4), (5), and (6) to obtain

(7)

$$
\begin{aligned}
(u,\Box''u) &\geq \|\bar\partial''u\|^2+\|\bar\partial''^*u\|^2-\frac{1}{p+q-n}(\|\bar\partial'u\|^2+\|\bar\partial'^*u\|^2)\\
&\geq \frac{n-q}{n}\|\nabla''u\|^2+\frac{q}{n}\|\nabla'u\|^2\\
&\quad -\frac{1}{p+q-n}\left(\frac{n+1-p}{n}\|\nabla'u\|^2+\frac{p-1}{n}\|\nabla''u\|^2\right)-C\|u\|^2\\
&= \frac{(n-q)(p+q-n)-(p-1)}{n(p+q-n)}\|\nabla''u\|^2\\
&\quad +\frac{q(p+q-n)-(n+1-p)}{n(p+q-n)}\|\nabla'u\|^2-C\|u\|^2.
\end{aligned}
$$

When $0<q<n-1$ and $p+q>n+1$, both coefficients on the right-hand side above are strictly positive, so we obtain the following Gårding-type inequality:

(8)
$$(u,\Box''u)\geq c\|u\|_1^2-C\|u\|^2.$$

The subellipticity of $\Box''$ then follows by standard arguments.

When $q=n-1$ the coefficient of $\|\nabla'u\|^2$ in (7) is positive, but that of $\|\nabla''u\|^2$ is zero. To handle this case, we observe that (2) implies $\|\bar\partial'u\|^2\leq K\|\nabla'u\|^2$. (The constant factor K arises because there are two different norms in use—the Hodge norm for $(p+q+1)$-forms, and the Hodge norm for (covector-valued) $(p+q)$-forms.) Thus (6) and (4) in the case $q=n-1$ give

(9)
$$(u,\Box''u)\geq\frac{1}{n}\|\nabla''u\|^2+\frac{n-1}{n}\|\nabla'u\|^2-\frac{K}{p-1}\|\nabla'u\|^2-C\|u\|^2.$$

Adding ε times (9) plus $(1-\varepsilon)$ times (7), for suitably small ε, yields (8).

For the case $p+q<n$, we could proceed in a similar manner. However, it is easier to note that this case is equivalent to the case $p+q>n+1$, as follows.

Let $*$ be the Hodge star operator determined by g_θ and the orientation $\theta\wedge(d\theta)^n$, and let $\bar *$ denote $*$ followed by conjugation. These operators have the following mapping properties:

$$
\begin{aligned}
*&: E_\theta^{p,q}\to F^{n+1-q,n-p};\\
*&: F^{p,q}\to E_\theta^{n-q,n+1-p};\\
\bar*&: E_\theta^{p,q}\to F^{n+1-p,n-q};\\
\bar*&: F^{p,q}\to E_\theta^{n+1-p,n-q}.
\end{aligned}
$$

Moreover, the adjoint operator $d''^{*}\colon E^{p,q} \longrightarrow E^{p,q-1}$ is $(-1)^{p+q} * d' * = (-1)^{p+q} \bar{*} d'' \bar{*}$; therefore, $\bar{*}\Box'' = \Box''\bar{*}$. It follows immediately that subellipticity of $\Box''$ on $F^{p,q}$ for $p+q > n+1$ and $0 < q < n$ implies subellipticity on $E^{p,q}$ for $p+q < n$, $0 < q < n$. (This also gives a new proof of Serre duality for Kohn-Rossi cohomology, $H^{p,q}(M) \cong H^{n+1-p,n-q}(M)$, originally due to Tanaka [T].)

The remaining cases ($p+q = n$, $n+1$) are more difficult, since then $\Box''$ is a fourth-order operator. Consider first the case $p+q = n+1$, $0 < q < n$. The key observation is that the Hodge star operator $*\colon F^{p,q} \longrightarrow E_{\theta}^{p-1,q}$ has a particularly simple expression: it is just $*u = cT \lrcorner u$ for some constant c of modulus one. Using this, we can show by a laborious computation that for $u \in \Gamma(F^{p,q})$,

$$\begin{aligned}(u, \Box'' u) &= \|D''^{*}u\|^2 + \|d''^{*}d''u\|^2 \\ &= \frac{3}{4}\|(\partial'\partial''^{*} - \partial'\partial'^{*})u\|^2 + \frac{1}{4}\|(\partial''^{*}\partial' + \partial'\partial''^{*})u\|^2 + O(\|u\|_1^2).\end{aligned}$$

Throwing away the first term and writing $\Delta'' = \partial''^{*}\partial' + \partial''\partial''^{*}$, we obtain

$$(u, \Box'' u) \geq \frac{1}{4}\|\Delta'' u\|^2 - C\|u\|_1^2.$$

Combined with the fact that Δ'' is subelliptic on $F^{p,q}$ when $0 < q < n$ (which follows from (4)), this easily yields subellipticity of $\Box''$. The argument for the case $p+q = n$ can be carried out similarly; alternatively that case can be deduced from the $p+q = n+1$ case by means of the Hodge star operator.

Finally, we prove that every cohomology class has a unique harmonic representative (i.e., a representative in $\operatorname{Ker}\Box''$) when $n \geq 2$. When $0 < q < n$, this follows directly from the fact that $\Box''$ is subelliptic, hence Fredholm. When $q = 0$, it is true for the trivial reason that d''^{*} is the zero operator, so $H^{p,0}(M) = \operatorname{Ker} d'' = \operatorname{Ker}\Box''$. (Replace d''^{*} by D''^{*} when $p = n+1$, and d'' by D'' when $p = n$.) On the other hand, when $q = n$, we argue as follows. Assume first that $p \geq 2$. Any cohomology class in $H^{p,n}(M)$ is represented by a section u of $F^{p,n}$ (with $d''u = 0$ trivially). Since $\Box''$ is Fredholm on $F^{p,n-1}$, we can write

$$d''^{*}u = \Box'' v + w = d''^{*}d''v + d''d''^{*}v + w,$$

where $\Box'' w = 0$. Since $\operatorname{Im} d''^{*}$, $\operatorname{Im} d''$, and $\operatorname{Ker}\Box''$ are all mutually orthogonal, we must have $d''^{*}u - d''^{*}d''v = d''d''^{*}v = w = 0$. In particular, $u - d''v \in \operatorname{Ker} d''^{*} = \operatorname{Ker}\Box''$, and we are done. The $p = 1$ and $p = 0$ cases are virtually identical, with $E^{p,q}$, D'', D''^{*} inserted in place of $F^{p,q}$, d'', and d''^{*} as appropriate. $\Box$

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