

Formal Solutions and True Solutions with Gevery Type Asymptotic Expansion for Some Nonlinear Partial Differential Equations in the Complex Domain

By

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Abstract

$\bar{\text{O}}uchi$ [3] showed that for some linear partial differential equations in a complex domain, there exists a true solution $u_S(t, x)$ which is a holomorphic function in a sector S , and has an asymptotic expansion as $t \rightarrow 0$ in S . In this paper, we extend these results for nonlinear equations, and give another construction of such a solution.

§ 1. Introduction

Let $\mathbb{C}$ be the complex plane or the set of all complex numbers, t a coordinate of $\mathbb{C}_t$, and $x = (x_1, \dots, x_n)$ coordinates of $\mathbb{C}_x^n = \mathbb{C}_{x_1} \times \dots \times \mathbb{C}_{x_n}$. Set $\mathbb{N} := \{0, 1, 2, \dots\}$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we set $|\alpha| := \alpha_1 + \dots + \alpha_n$, and $(\partial/\partial x)^\alpha := (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$. Set $|x| := \max_{1 \leq i \leq n} \{|x_i|\}$, $D_R := \{x \in \mathbb{C}_x^n; |x| < R\}$ and

$$S_\theta(T) := \{t \in \mathbb{C}_t; 0 < |t| < T, |\arg t| < \theta\}.$$

We denote by $\mathcal{O}(D_R)$ (resp. $\mathcal{O}(S_\theta(T) \times D_R)$) the set of all holomorphic functions defined on D_R (resp. $S_\theta(T) \times D_R$).

In this paper, we consider the following equation:

$$(1.1) \quad D(u(t, x)) = f(t, x).$$

Here $D(u(t, x))$ is a nonlinear partial differential operator with coefficients in holomorphic functions on a neighborhood of the origin for an unknown function $u(t, x)$, and $f(t, x)$ is a given function.

Received March 20, 2007, Accepted November 4, 2007.

2000 Mathematics Subject Classification(s): 34G20, 35A07, 35C10.

Key Words: true solutions, asymptotic expansion.

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Problem 1. Suppose that $f(t, x)$ is a holomorphic function. Can we construct a true solution?

Then the answer is “Yes” (Ōuchi [1], [2], Yamazawa [4]).

Problem 2. Suppose that $f(t, x)$ is a function with Gevrey type asymptotic expansion. Can we construct a true solution of the same type?

In this paper we consider Problem 2; we construct a true solution to (1.1), and moreover, we prove that the solution has the same Gevrey type asymptotic expansion as $f(t, x)$.

§ 2. Solvability in $\text{Asy}_{\{\gamma\}}^0$

We define some function spaces which will be used in this paper.

Definition 2.1. Let $\gamma > 0$. Then we define a subspace $\text{Asy}_{\{\gamma\}}^0(S_\theta(T) \times D_R)$ of $\mathcal{O}(S_\theta(T) \times D_R)$ as follows: $f(t, x) \in \text{Asy}_{\{\gamma\}}^0(S_\theta(T) \times D_R)$ if for any $S_0 = S_{\theta_0}(T_0)$ with $0 < \theta_0 < \theta$ and $0 < T_0 < T$ (which we denote by $S_0 \Subset S$), there exist $f_k(x) \in \mathcal{O}(D_R)$ and C and $c_0 > 0$ such that

$$|f(t, x)| \leq C \exp(-c_0|t|^{-\gamma})$$

holds in S_0 .

Definition 2.2. Let $\gamma > 0$. Then we define a subspace $\text{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R)$ of $\mathcal{O}(S_\theta(T) \times D_R)$ as follows: $f(t, x) \in \text{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R)$ if for any $S_0 \Subset S_{\theta_0}(T_0)$, there exist $f_k(x) \in \mathcal{O}(D_R)$ and $A_0, B_0 > 0$ such that for any $N \in \mathbb{N} \setminus \{0\}$

$$(2.1) \quad \left| f(t, x) - \sum_{k=0}^{N-1} f_k(x)t^k \right| \leq A_0 B_0^N |t|^N \Gamma\left(\frac{N}{\gamma} + 1\right)$$

holds in S_0 . If the condition (2.1) is satisfied, then we write

$$f(t, x) \sim_\gamma \tilde{f}(t, x) = \sum_{k \geq 0} f_k(x)t^k \quad \text{in } S_\theta(T).$$

We call $\tilde{f}(t, x)$ an Gevrey type asymptotic expansion with index γ for $f(t, x)$.

We consider the following operator $D(u)$:

$$(2.2) \quad D(u(t, x)) = F(t, x, \{(\partial/\partial t)^j (\partial/\partial x)^\alpha u(t, x)\}_{j+|\alpha| \leq m}).$$

We assume that $F(t, x, Z)$ ($Z = \{Z_{j,\alpha}\}_{j+|\alpha| \leq m}$) admits an expansion which is a convergent power series with respect to Z :

$$(2.3) \quad F(t, x, Z) = \sum_{|q| \geq 1} a_q(t, x) \prod_{j+|\alpha| \leq m} \{Z_{j,\alpha}\}^{q_{j,\alpha}},$$

here $q_{j,\alpha} \in \mathbb{N}$, and we set $q := \{q_{j,\alpha} \in \mathbb{N}; j + |\alpha| \leq m\}$, $|q| := \sum_{j+|\alpha| \leq m} q_{j,\alpha}$, and we assume that $a_q(t, x) \in \text{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R)$. Then we consider the following equation (E_0) .

$$(E_0) \quad D(u(t, x)) = f(t, x) \in \text{Asy}_{\{\gamma\}}^0(S_\theta(T) \times D_R).$$

For the equation (E_0) we introduce Newton polygon due to Ōuchi [2], [3]. We write each coefficient $a_q(t, x) \in \text{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R)$ as

$$a_q(t, x) = t^{\sigma_q} b_q(t, x) \quad (b_q(0, x) \neq 0, \sigma_q \in \mathbb{N} \setminus \{0\}).$$

We set

$$\Pi(a, b) := \{(x, y) \in \mathbb{R}^2; x \leq a \text{ and } y \geq b\}.$$

Moreover, set $l_q := \max\{j + |\alpha|; q_{j,\alpha} \in q, q_{j,\alpha} \neq 0\}$ and $e_q := \sigma_q - \sum_{q_{j,\alpha} \in q} j q_{j,\alpha}$. Then we define Newton polygon $NP_1(D)$ for the linear part of the operator $D(\cdot)$ by

$$NP_1(D) := CH \left\{ \bigcup_{|q|=1} \Pi(l_q, e_q); b_q(t, x) \neq 0 \right\},$$

where $CH\{\cdot\}$ is the convex hull of a set.

The boundary of Newton polygon $NP_1(D)$ consists of a vertical half line Σ_p^* , a horizontal half line Σ_p^* and segments Σ_i^* ($1 \leq i \leq p-1$). Let γ_i^* be the slope of Σ_i^* for $i = 0, \dots, p$. Then we have $0 = \gamma_p^* < \gamma_{p-1}^* < \dots < \gamma_0^* = \infty$. Further Newton polygon $NP_1(D)$ has p -point vertices which we denote by (l_i^*, e_i^*) with $l_{p-1}^* < l_{p-2}^* < \dots < l_0^* = m$.

Next let us define an operator $\mathcal{L}_i$ with respect to Σ_i^* for $i = 1, \dots, p-1$. We set

$$I_i := \{q \subset \mathbb{N}; e_i^* - e_q = \gamma_i^*(l_i^* - l_q) \text{ and } |q| = 1\}.$$

Then we set

$$\begin{aligned} \mathcal{L}_i u(t, x) &:= \sum_{q \in I_i} t^{\sigma_q} b_q(t, x) \prod_{j+|\alpha| \leq m} \left\{ \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u(t, x) \right\}^{q_{j,\alpha}} \\ &= \sum_{(j,\alpha) \in J_i} t^{\sigma_{j,\alpha}} b_{j,\alpha}(t, x) \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u(t, x), \end{aligned}$$

where

$$J_i := \{j + |\alpha| \leq m; e_i^* - (\sigma_{j,\alpha} - j) = \gamma_i^*(l_i^* - j - |\alpha|)\}.$$

Let m_i^* be the order with respect to $\partial/\partial x$ of $\mathcal{L}_i$.

We assume the following conditions for the equation (E_0) :

$$(C_1) \quad D(u) \text{ has a linear part of order } m,$$

$(C_2)_i$ the operator $\mathcal{L}_i$ satisfies

- (1) if $j + |\alpha| < l_i^*$ then $|\alpha| < m_i^*$ and
- (2) $\sum_{\substack{j+|\alpha|=l_i^* \\ |\alpha|=m_i^*}} b_{j,\alpha}(0,0)\widehat{\xi}^\alpha \neq 0$ with $\widehat{\xi} := (1, 0, \dots, 0)$.

If we assume that the equation (E_0) satisfies the condition (C_1) , then it is sufficient to define Newton polygon for the only linear part (see Ōuchi [2, Proposition 1.7]).

We have an existence theorem concerning exponential decay solutions.

Theorem 2.3. *Let $\gamma_{s+1}^* \leq \gamma < \gamma_s^*$ and $f(t, x) \in \text{Asy}_{\{\gamma\}}^0(S_\theta(T) \times D_R)$ ($|\theta| < \pi/2\gamma$). Suppose the conditions (C_1) and $(C_2)_i$ ($i = 0, \dots, s$). Then for any $0 < r < R$, there exists a solution $u(t, x) \in \text{Asy}_{\{\gamma\}}^0(S_{\theta'}(T) \times D_r)$ ($|\theta'| < \pi/2\gamma_1^*$) to the equation (E_0) .*

In the case where $\gamma = \gamma_{s+1}^*$, Theorem 2.3 was obtained in Ōuchi [2] under the condition that $F(t, x, Z)$ is a polynomial in Z , and this condition was removed in Yamazawa [4]. If $\gamma > \gamma_{s+1}^*$, we can prove this theorem as same way as in the case where $\gamma = \gamma_{s+1}^*$.

§ 3. Construction of True Solution

We consider the following equation:

$$(E_1) \quad D(u(t, x)) = f(t, x),$$

where each $a_q(t, x)$ in (2.3) is holomorphic in a neighborhood of the origin, and $f(t, x)$ belongs to $\text{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R)$.

For the equation (E_1) we want to construct a solution that belongs to the same class as $f(t, x)$. In a linear case we shall recall Ōuchi's result in [3].

Let us consider the following condition $(C'_2)_i$ for the operator $\mathcal{L}_i$:

- $(C'_2)_i$ (1) $\sigma_{j,\alpha} = 0$ for any (j, α) with $j + |\alpha| = l_i^*$ and $|\alpha| = m_i^*$.
- (2) $\sum_{\substack{j+|\alpha|=l_i^* \\ |\alpha|=m_i^*}} b_{j,\alpha}(0,0)\widehat{\xi}^\alpha \neq 0$.

Then we have the following theorem:

Theorem 3.1 (Ōuchi [3]). *Assume that $D(u)$ satisfies $(C'_2)_i$ ($i = 1, \dots, s$). Let $\gamma \geq \gamma_{s+1}^*$. Then for any $S_0 \Subset S$ with $0 < \theta_0 < \pi/(2\gamma_1^*)$ there exists a solution $u(t, x) \in \text{Asy}_{\{\gamma\}}(S_0 \times D_r)$ to (E_1) .*

Remark. In [3], Ōuchi constructed a solution to (E_1) by using an integral kernel G and the solution is expressed as follows:

$$\int_c G(t, x; w) f(w) dw,$$

but it is very complicated to construct $G(t, x; w)$.

Next let us consider a nonlinear case. We shall get the same result as in a linear case, and give another construction of a true solution.

Let s be a nonnegative integer with $\gamma_{s+1}^* \leq \gamma < \gamma_s^*$, and we set $k_s^* = l_s^* - m_s^*$. We will give a condition for nonlinear terms.

For all nonlinear term, let us assume the following condition:

(A₁) For any (l_q, e_q) with $|q| \geq 2$, there exist $J_q^- > 0$ and $J_q^+ \geq 0$ such that

$$l_q = \begin{cases} -\frac{e^* - e_q}{\gamma_{s+1}^*} + l_s^* - J_q^- & \text{for } e^* \geq e_q, \\ \frac{e_q - e^*}{\gamma_s^*} + l_s^* - J_q^+ & \text{for } e_q > e^*. \end{cases}$$

Further we assume

$$\left[\frac{m_s}{\gamma_{s+1}^*} \right]_0 < J_q \quad \text{if } e^* > e_q \text{ and } |q| \geq 2,$$

where $[a]_0$ is the decimal part of a number a .

Then we have the following result for a formal solution.

Theorem 3.2. Suppose the conditions (A₁) and $(C'_2)_s$ for (E_1) . Then for $|\theta| < \pi/(2\gamma)$, we can construct a formal power series $\tilde{u}(t, x) = \sum_{h \leq 0} u_{(h)}(t, x)$ that formally satisfies (E_1) . Further $u_{(h)}(t, x)$ satisfies the following estimate:

$$(3.1) \quad |u_{(h)}(t, x)| \leq \tau^{-m_s} \tilde{U}_{(h)} B^h \Gamma\left(-\frac{h}{\gamma_s^*} + 1\right) \Gamma\left(\frac{k_s^*}{\gamma} + 1\right) |t|^{k_s^* - h} \text{ in } S_0 \Subset S_\theta(T),$$

and $\sum_{h \leq 0} \tilde{U}_{(h)} t^{-h}$ is a convergent power series in a neighborhood of the origin.

We shall give a sketch of proof of Theorem 3.2 in Section 5.

As for the formal solution $\tilde{u}(t, x)$ we have the following fact.

Lemma 3.3. There exists a function $u_{S_0}(t, x) \in \text{Asy}_{\{\gamma_s^*\}}(S_0 \times D_r)$ with $0 < \theta_0 < \pi/(2\gamma_s^*)$ such that for any $S_1 \Subset S_0$

$$|u_{S_0}(t, x) - \sum_{h=0}^{-N+1} u_{(h)}(t, x)| \leq \tilde{U}'_{(h)} |t|^N \Gamma\left(\frac{N}{\gamma_s^*} + 1\right) \text{ for } t \in S_1,$$

where $\sum_{n \geq 0} \tilde{U}'_{(h)} t^N$ is a convergent series.

Proof. Put

$$\begin{aligned} \hat{u}_{(h)}(t, x; \xi) &= \frac{u_{(h)}(t, x)}{t^{k_s - h + \gamma_s^*}} \frac{\xi^{-h/\gamma_s^*}}{\Gamma(-h/\gamma_s^* + 1)}, \quad \hat{u}_H(t, x; \xi) = \sum_{h \leq -H-1} \hat{u}_{(h)}(x, t, \xi), \\ \hat{u}(t, x; \xi) &= \sum_{h \leq 0} \hat{u}_{(h)}(x, t, \xi). \end{aligned}$$

It follows from Theorem 3.2 that $\hat{u}_H(t, x; \xi)$ and $\hat{u}(t, x; \xi)$ converge on $D_R \times S_0 \times \{|\xi| \leq \hat{\xi}_0\}$ for some $\hat{\xi}_0 > 0$. Then there exist $\hat{\xi}$ with $0 < \hat{\xi} < \hat{\xi}_0$ and B_0 such that

$$\begin{aligned} |\hat{u}_H(t, x; \xi)| &\leq AB_0^{H+1} |t|^{-\gamma_s^*} |\xi|^{(H+1)/\gamma_s^*} \quad \text{on } D_R \times S_0 \times \{|\xi| \leq \hat{\xi}\}, \\ \sum_{h=0}^{-H} |\hat{u}_{(h)}(t, x; \xi)| &\leq AB_0^{H+1} |t|^{-\gamma_s^*} |\xi|^{(H+1)/\gamma_s^*} \quad \text{on } D_R \times S_0 \times \{|\xi| \geq \hat{\xi}\}. \end{aligned}$$

Define

$$u_{S_0}(t, x) = t^{k_s^*} \int_0^{\hat{\xi}} \exp(-\xi t^{-\gamma_0^*}) \hat{u}(t, x; \xi) d\xi.$$

By the estimates above we can see that $u_{S_0}(t, x)$ is the desired function. $\square$

We can construct a true solution as $u(t, x) = u_{S_0}(t, x) + v(t, x)$, where $v(t, x)$ is an unknown function. Set $D^{u_{S_0}}(v) := D(u_{S_0} + v) - D(u_{S_0})$. Then we have the following theorem.

Theorem 3.4. *Suppose that for the differential equation $D^{u_{S_0}}(v)$ with respect to $v(t, x)$, the conditions $(C_2)_i$ are satisfied ($i = 0, \dots, s-1$). Then for any $S_1 \Subset S_0$ with $0 < \theta_1 < \pi/(2\gamma_1^*)$, there exists a solution $u_{S_1}(t, x) \in \text{Asy}_{\{\gamma\}}(S_1 \times D_r)$ to (E_1) .*

Proof. If the condition (C_1) is satisfied for the equation (E_1) , then the same condition is also satisfied for the equation $D^{u_{S_0}}(v)$. As the proof of in [2, Lemma 4.4] we can prove that $f(t, x) - D(u_{S_0}) \in \text{Asy}_{\{\gamma_s^*\}}^0(S_0 \times D_r)$. Therefore we can adapt Theorem 2.3 to an equation $D^{u_{S_0}}(v) = f(t, x) - D(u_{S_0})$, and this equation has a solution $v_{S_1}(x) \in \text{Asy}_{\{\gamma_s^*\}}^0(S_1 \times D_r)$. Then $u_{S_0}(t, x) + v_{S_1}(t, x)$ is a solution to the equation (E_1) . $\square$

Remark. We can prove Theorem 3.4 if we replace the condition $(C_2)_i$ by $(C'_2)_i$.

§ 4. Majorant Function

First let us introduce a majorant function in [2]. Set

$$\theta(t) = \sum_{n \geq 0} \frac{ct^n}{(n+1)^{m+2}} \quad \text{and} \quad \theta^{(k)}(t) = \left(\frac{\partial}{\partial t}\right)^k \theta(t),$$

where $m \in \mathbb{N}$.

Lemma 4.1. *There exists a positive constant c such that for $0 \leq k' \leq k \leq m$*

$$(4.1) \quad \theta^{(k)}(t) \theta^{(k')}(t) \ll \theta^{(k)}(t).$$

We fix $c > 0$ so that (4.1) holds.

Lemma 4.2. (1) *There exists a constant $C > 0$ such that the following holds for any k :*

$$\theta^{(k)}(t) \ll \frac{C}{k+1} \theta^{(k+1)}(t).$$

(2) *Let $0 \leq k' \leq k \leq m$. Then*

$$\sum_{i=0}^n \frac{n!}{i!(n-i)!} \theta^{(n-i+k+p)}(t) \theta^{(i+k'+p')}(t) \ll \frac{p!p'!}{(p+p')!} \theta^{(n+k+p+p')}(t).$$

Set $\Psi_R(t) = \theta(t/R)$ and $\Psi_R^{(k)}(t) = (\partial/\partial t)^k \Psi_R(t)$ for $0 < R < 1$. By Lemma 4.2 we can prove the following proposition:

Proposition 4.3. (1) *There exists a constant $C > 0$ such that the following holds for any k :*

$$\Psi_R^{(k)}(t) \ll \frac{C}{k+1} R \Psi_R^{(k+1)}(t).$$

(2) *Let $0 \leq k' \leq k \leq m$. Then for $\delta \geq 1$*

$$\sum_{i=0}^n \frac{n!}{i!(n-i)!} \Psi_R^{([(n-i)/\delta] + k + p)}(t) \Psi_R^{([i/\delta] + k' + p')}(t) \ll \frac{p!p'!}{(p+p')!} R^{-k'} \Psi_R^{([n/\delta] + k + p + p')}(t),$$

where $[a]$ denotes the integer part of a number a .

§ 5. Sketch of Proof of Theorem 3.2

First we study a solvability for an operator with respect to a vertex of Newton polygon.

We set

$$\Delta(s) := \{q; l_q = l_s^* \text{ and } e_q = e_s^*\}.$$

Then we define an operator by

$$Su(t, x) = \sum_{q \in \Delta(s)} a_q(t, x) \prod_{j+|\alpha| \leq m} \left\{ \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u \right\}^{q_{j,\alpha}}.$$

Under the condition (A_1) , the operator S is a linear operator. So we can write:

$$S = \sum_{(j,\alpha) \in \Delta(s)} t^{\sigma_{j,\alpha}} b_{j,\alpha}(t, x) \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha,$$

where

$$\Delta(s) := \{(j, \alpha); j + |\alpha| = l_s^* \text{ and } \sigma_{j,\alpha} - j = e_s^*\}.$$

We fix s and set $\Delta^* = \Delta(s)$, $k^* = k_s^*$ and $m^* = m_s^*$ and $(l^*, e^*) = (l_s^*, e_s^*)$ for short.

Let us consider the following equation:

$$(E_S) \quad Su(t, x) = F(t, x),$$

where $F(t, x)$ satisfies the following estimate as a formal power series in (t, x) for some $F > 0$:

$$F(t, x) \ll F \sum_{k \geq m_h} \frac{1}{k!} \Psi_R^{([k/\delta] + m^* + c_h)}(\chi) \left(\frac{t}{\zeta} \right)^k,$$

where $\chi = \tau x_1 + \sum_{i=2}^n x_i$, $\tau, \zeta > 0$.

Lemma 5.1. *Under the conditions $(C'_2)_s$ and (A_1) , the equation (E_S) has a solution $u(t, x)$ satisfying*

$$u(t, x) \ll \tau^{-m^*} \zeta^{-e^*} c(\tau) F \sum_{k \geq m_h + k^*} \frac{1}{k!} \Psi_R^{([k/\delta] + c_h)}(\chi) \left(\frac{t}{\zeta} \right)^k,$$

where $c(\tau)$ is a positive and bounded function of $\tau > 0$.

We will construct a formal solution $u(t, x) = \sum_{g \geq 0} u^g(t, x)$ to the equation (E_1) : We choose the sequence $\{u^g(t, x)\}$ satisfying the following equations:

$$(E^g) \quad \left\{ \begin{array}{l} Su^0(t, x) + \sum_{e_q \leq e^*} t^{\sigma_q} b_q(t, x) \prod_{j+|\alpha| \leq m} \left\{ \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u^0(t, x) \right\}^{q_{j,\alpha}} = f(t, x), \\ Su^g(t, x) + \sum_{\substack{e_q \leq e^* \\ |g'| = g}} t^{\sigma_q} b_q(t, x) \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u^{g_i}(t, x) \\ = - \sum_{\substack{e_q > e^* \\ |g'| = g - (e_q - e^*)}} t^{\sigma_q} b_q(t, x) \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u^{g_i}(t, x) \quad (g \geq 1). \end{array} \right.$$

Here we set $|g'| := \sum_{j+|\alpha| \leq m} \sum_{i=1}^{q_{j,\alpha}} g_i$.

We solve the equation (E^g) inductively: First let us solve the equation (E^0) . We construct a solution $u^0(t, x) = \sum_{h \geq 0} u_h^0(t, x)$ as

$$\begin{aligned} Su_0^0(t, x) &= f(t, x), \\ Su_h^0(t, x) &:= W_h^0(t, x) \\ &= - \sum_{\substack{e_q < e^* \\ |h'| = h - (e^* - e_q)}} t^{\sigma_q} b_q(t, x) \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^0(t, x) \\ &\quad - \sum_{\substack{e_q = e^* \\ |h'| = h - 1}} t^{\sigma_q} b_q(t, x) \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^0(t, x) \quad (h \geq 1). \end{aligned}$$

Here we set $|h'| := \sum_{j+|\alpha| \leq m} \sum_{i=1}^{q_{j,\alpha}} h_i$.

Since the function $f(t, x)$ belongs to $\text{Asy}_{\{\gamma\}}(S_\theta(T) \times D_R)$ we can assume that $f(t, x)$ satisfies

$$f(t, x) \ll F \sum_{k \geq 0} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta] + m^*)}(\chi) \left(\frac{t}{\zeta} \right)^k,$$

where $\frac{1}{\delta} = 1 + \frac{1}{\gamma}$, and each $b_q(t, x)$ satisfies for some $B_q > 0$:

$$b_q(t, x) \ll B_q \sum_{k \geq 0} \frac{1}{k!} \Psi_R^{(k)}(\chi) \left(\frac{t}{\zeta} \right)^k.$$

Then by Lemma 5.1, we get

$$u_0^0(t, x) \ll \tau^{-m^*} U_0^0 \sum_{k \geq k^*} \frac{1}{k!} \Psi_R^{([k/\delta])}(\chi) \left(\frac{t}{\zeta} \right)^k,$$

where $U_0^0 := \zeta^{-e^*} c(\tau) F$. By induction on $h > 0$, we can obtain $u_h^0(t, x)$ by Lemma 5.1. So we will inductively give an estimate for $u_h^0(t, x)$.

Proposition 5.2. *Under the conditions $(C'_2)_s$ and (A_1) , for any $h \in \mathbb{N}$ there exists $U_h^0 > 0$ such that*

$$(5.1) \quad u_h^0(t, x) \ll \tau^{-m^*} U_h^0 \sum_{k \geq k^*} \frac{1}{k!} \Psi_R^{([k/\delta])}(\chi) \left(\frac{t}{\zeta} \right)^k,$$

and that $\sum_{h \geq 0} U_h^0$ converges for a sufficiently large $\tau > 0$ and a sufficiently small $R > 0$.

Proof. We set $\gamma_0 = \gamma_{s+1}^*$ for short. We already obtain the estimate (5.1) for $h = 0$. Let us assume the following estimate for $h_i < h$:

$$\left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u_{h_i}^0(t, x) \ll \tau^{\alpha_1 - m^*} \zeta^{-j} U_{h_i}^0 \sum_{k \geq (k^* - j)_+} \frac{1}{k!} \Psi_R^{([(k+j)/\delta] + |\alpha|)}(\chi) \left(\frac{t}{\zeta}\right)^k,$$

where $(a)_+ = a$ if $a \geq 0$ and $(a)_+ = 0$ if $a < 0$ for $a \in \mathbb{R}$. By Proposition 4.3, we have

$$\begin{aligned} t^{\sigma_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u_{h_i}^0(t, x) \\ \ll C^{\sigma_q} \zeta^{e_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} (\tau^{\alpha_1 - m^*} U_{h_i}^0) \sum_{k \geq 0} \frac{1}{k!} \Psi_R^{([k/\delta - e_q/\gamma + l_q])}(\chi) \left(\frac{t}{\zeta}\right)^k. \end{aligned}$$

It follows $[[k^*/\gamma]_0 + 1 - J_q^-] \leq 0$ from the assumption (A_1) . If $e_q < e^*$ and $|h'| = h - (e^* - e_q)$, then we have

$$\begin{aligned} \left[\frac{-e_q}{\gamma} + l_q \right] &= \left[\frac{-e_q}{\gamma} - \frac{e^* - e_q}{\gamma_0} + k^* + m^* - J_q^- \right] \\ &= \left[\frac{k^*}{\delta} + \frac{e^* - e_q}{\gamma} - \frac{e^* - e_q}{\gamma_0} + m^* - J_q^- \right] \\ (5.2) \quad &\leq \left[\frac{k^*}{\delta} + m^* - J_q^- \right] = k^* + \left[\frac{k^*}{\gamma} \right] + m^* - 1 + \left[\left[\frac{k^*}{\gamma} \right]_0 + 1 - J_q^- \right] \\ &\leq \left[\frac{k^*}{\delta} \right] + m^* - 1, \end{aligned}$$

and if $e_q = e^*$, by the same manner we have

$$(5.3) \quad \left[\frac{-e_q}{\gamma} + l_q \right] = \left[\frac{k^*}{\gamma} + k^* + m^* - 1 \right] = \left[\frac{k^*}{\delta} \right] + m^* - 1.$$

Thus by Proposition 4.3, we have

$$\begin{aligned} W_h^0(t, x) \\ \ll \sum_{\substack{e_q < e^* \\ |h'| = h - (e^* - e_q)}} C^{\sigma_q} \zeta^{e_q} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} (\tau^{\alpha_1 - m^*} U_{h_i}^0) \sum_{k \geq 0} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta] + m^*)}(\chi) \left(\frac{t}{\zeta}\right)^k \\ + \sum_{\substack{e_q = e^* \\ |h'| = h - 1}} C^{\sigma_q} \zeta^{e_q} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} (\tau^{\alpha_1 - m^*} U_{h_i}^0) \sum_{k \geq 0} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta] + m^*)}(\chi) \left(\frac{t}{\zeta}\right)^k. \end{aligned}$$

Thus by Lemma 5.1, we get the estimate (5.1) for any $h \geq 0$, where

$$\begin{aligned}
 U_h^0 := c(\tau) & \sum_{\substack{e_q < e^* \\ |h'| = h - (e^* - e_q)}} C^{\sigma_q} \zeta^{e_q - e^*} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^0 \\
 (5.4) \quad & + c(\tau) \sum_{e_q = e^*, |h'| = h-1} C^{\sigma_q} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^0.
 \end{aligned}$$

Next let us give an estimate of the coefficient U_h^0 . Let us consider the following functional equation for $Z(t)$:

$$\begin{aligned}
 Z(t) = c(\tau) & \sum_{e_q < e^*} C^{\sigma_q} \zeta^{e_q - e^*} t^{e^* - e_q} B_q \prod_{j+|\alpha| \leq m} \{\tau^{\alpha_1 - m^*} Z(t)\}^{q_{j,\alpha}} \\
 (5.5) \quad & + c(\tau) \sum_{e_q = e^*} C^{\sigma_q} t \prod_{j+|\alpha| \leq m} \{\tau^{\alpha_1 - m^*} Z(t)\}^{q_{j,\alpha}} + \zeta^{-e^*} c(\tau) F.
 \end{aligned}$$

By the implicit function theorem, we have the holomorphic solution near $t = 0$. Moreover substituting $Z(t) = \sum_{h \geq 0} Z_h^0 t^h$ into (5.5), for any $T > 0$, we can take a sufficiently large $\tau > 0$ and a sufficiently small $R > 0$ such that $U_h^0 \leq Z_h^0 T^h$. Hence we have the desired result. $\square$

We construct a true solution to (E^0) . Set $\tilde{u}^0(t, x) := \sum_{h \geq 0} u_h^0(t, x)$. Then we have:

$$\tilde{u}^0(t, x) \ll \tau^{-m^*} \sum_{h \geq 0} U_h^0 \sum_{k \geq k^*} \frac{1}{k!} \Psi_R^{([k/\delta])}(\chi) \left(\frac{t}{\zeta}\right)^k.$$

If we set $\tilde{U}_0^0 := \sum_{h \geq 0} U_h^0$, then by Proposition 5.2, we get

$$\tilde{u}^0(t, x) \ll \tau^{-m^*} \tilde{U}_0^0 \sum_{k \geq k^*} \frac{1}{k!} \Psi_R^{([k/\delta])}(\chi) \left(\frac{t}{\zeta}\right)^k.$$

For $|\theta| < \pi/(2\gamma)$, as in Lemma 3.3 we can prove the existence of a holomorphic function $u_S^0(t, x)$ such that

$$u_S^0(t, x) \sim_\gamma \tilde{u}^0(t, x) \quad \text{in } S_\theta(T).$$

So we construct a true solution $u^0(t, x)$ to (E^0) such that $u^0(t, x) = u_S^0(t, x) + v^0(t, x)$, where $v^0(t, x)$ is an unknown function; we can adapt Theorem 2.3 to the equation for $v^0(t, x)$, and we can solve (E^0) with $v^0(t, x) \in \text{Asy}_{\{\gamma\}}^0(S_\theta(T) \times D_R)$.

We can solve each equation (E^g) for $g \geq 0$ by repeating the procedure of constructing a true solution $u^0(t, x)$:

First we construct a formal solution $u^g(t, x) = \sum_{h \geq -g} u_h^g(t, x)$ for $g \geq 0$ as follows:

$$\begin{aligned} Su_{-g}^g(t, x) &= - \sum_{e_q > e^*} \sum_{\substack{|g'| = g - (e_q - e^*) \\ |h'| = -g + (e_q - e^*)}} a_q(t, x) \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^{g_i}(t, x), \\ Su_h^g(t, x) &= - \sum_{e_q > e^*} \sum_{\substack{|g'| = g - (e_q - e^*) \\ |h'| = h + (e_q - e^*)}} a_q(t, x) \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^{g_i}(t, x) \\ &\quad - \sum_{e_q < e^*} \sum_{\substack{|g'| = g \\ |h'| = h - (e_q^* - e_q)}} a_q(t, x) \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^{g_i}(t, x) \\ &\quad - \sum_{e_q = e^*} \sum_{\substack{|g'| = g \\ |h'| = h - 1}} a_q(t, x) \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^{g_i}(t, x). \end{aligned}$$

Proposition 5.3. *Under the conditions $(C'_2)_s$ and (A_1) , for any $g \in \mathbb{N}$ and $h \geq g$, there exists a $U_h^g > 0$ such that*

$$(5.6) \quad u_h^g(t, x) \ll \tau^{-m^*} U_h^g \sum_{k \geq k^* - (h)_-} \frac{1}{k!} \Psi_R^{([k/\delta + (h)_- / \gamma - (h)_- / \gamma^*])}(\chi) \left(\frac{t}{\zeta} \right)^k,$$

and that $\sum_{g \geq 0} \sum_{h \geq -g} U_h^g$ converges for any sufficiently large $\tau > 0$ and sufficiently small ζ , $R > 0$. Here $\gamma^* = \gamma_s^*$, and $(a)_- := 0$ if $a \geq 0$ and $(a)_- := a$ if $a < 0$ for $a \in \mathbb{R}$.

Proof. In the case where $g = 0$, we showed the estimate (5.6) by Lemma 5.2. Let us prove that $u_h^g(t, x)$ satisfies (5.6) by induction on $g > 0$. We assume for $g_i < g$

$$\begin{aligned} &\left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^{g_i}(t, x) \\ &\ll \tau^{\alpha_1 - m^*} \zeta^{-j} U_{h_i}^{g_i} \sum_{k \geq (k^* - (h_i)_- - j)_+} \frac{1}{k!} \Psi_R^{([k + (k + j + (h_i)_-) / \gamma - (h_i)_- / \gamma^* + j + |\alpha|])}(\chi) \left(\frac{t}{\zeta} \right)^k. \end{aligned}$$

Set $|(h')_-| = \sum_{j+|\alpha| \leq m} \sum_{i=1}^{q_{j,\alpha}} (h_i)_-$. By Proposition 4.3, we have

$$\begin{aligned} &t^{\sigma_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^{g_i}(t, x) \\ &\ll C^{\sigma_q} \zeta^{e_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^{g_i} \sum_{k \geq k_h} \frac{1}{k!} \Psi_R^{([k + (k + |(h')_-| - e_q) / \gamma - |(h')_-| / \gamma^* + l_q])}(\chi) \left(\frac{t}{\zeta} \right)^k. \end{aligned}$$

First let us estimate $u_{-g}^g(t, x)$. Under the conditions that $e_q > e^*$, $|g'| = g - (e_q - e^*)$ and $|h'| = -g + (e_q - e^*)$, we have

$$(5.7) \quad k_h \geq k^*|q| - |(h')_-| - \sum_{j+|\alpha| \leq m} jq_{j,\alpha} + \sigma_q \geq k^*|q| - |h'| + e_q \geq g,$$

and

$$(5.8) \quad \left[\frac{|(h')_-| - e_q}{\gamma} - \frac{|(h')_-|}{\gamma^*} + l_q \right] \leq \left[\frac{|h'| - e_q}{\gamma} - \frac{|h'|}{\gamma^*} + l_q \right] \\ = \left[\frac{k^*}{\delta} + \frac{-g}{\gamma} - \frac{-g}{\gamma^*} + m^* - J_q \right] \leq \left[\frac{k^*}{\delta} + \frac{-g}{\gamma} - \frac{-g}{\gamma^*} \right] + m^*.$$

Thus by Proposition 4.3 we have

$$t^{\sigma_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^{g_i}(t, x) \\ \ll C^{\sigma_q} \zeta^{e_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^{g_i} \sum_{k \geq g} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta + (-g)/\gamma - (-g)/\gamma^*] + m^*)}(\chi) \left(\frac{t}{\zeta} \right)^k.$$

Therefore by Lemma 5.1, the estimate (5.6) holds for $u_{-g}^g(t, x)$, where

$$U_{-g}^g := c(\tau) \sum_{\substack{e_q > e^* \\ |g'| = g - (e_q - e^*) \\ |h'| = -g + (e_q - e^*)}} \sum_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^{g_i}.$$

Let us assume that a function $u_{h_i}^g(t, x)$ satisfies the estimate (5.6) for $h_i < h$. Here we give an estimate for the following three cases:

- (i) $e_q > e^*$, $|g'| = g - (e_q - e^*)$ and $|h'| = h + (e_q - e^*)$;
- (ii) $e_q < e^*$, $|g'| = g$ and $|h'| = h - (e^* - e_q)$;
- (iii) $e_q = e^*$, $|g'| = g$ and $|h'| = h - 1$.

(A) Assume $h \leq 0$. Then as for (5.7), we have $k_h \geq -h$ for the all cases. First consider the case (i). Then as for the estimate (5.8) we have

$$\left[\frac{|(h')_-| - e_q}{\gamma} - \frac{|(h')_-|}{\gamma^*} + l_q \right] \leq \left[\frac{k^*}{\delta} + \frac{h}{\gamma} - \frac{h}{\gamma^*} \right] + m^*.$$

Therefore by Proposition 4.3, we have

$$t^{\sigma_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^{g_i}(t, x) \\ \ll C^{\sigma_q} \zeta^{e_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^{g_i} \sum_{k \geq -h} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta + h/\gamma - h/\gamma^*] + m^*)}(\chi) \left(\frac{t}{\zeta} \right)^k.$$

Next consider the cases (ii) and (iii). Then as for the estimates (5.2) and (5.3) we have

$$\left[\frac{|(h')_-| - e_q}{\gamma} - \frac{|(h')_-|}{\gamma^*} + l_q \right] \leq \left[\frac{k^*}{\delta} + \frac{h}{\gamma} - \frac{h}{\gamma^*} \right] + m^* - 1,$$

Therefore by the same proposition, we have

$$(5.9) \quad t^{\sigma_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^{g_i}(t, x) \\ \ll C^{\sigma_q} \zeta^{e_q} R \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^{g_i} \\ \times \sum_{k \geq -(h)_-} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta + (h)_-/\gamma - (h)_-/\gamma^*] + m^*)}(\chi) \left(\frac{t}{\zeta} \right)^k.$$

(B) If $h > 0$, then we have $k_h \geq 0$ for the all cases.

In the case (i), we have

$$\left[\frac{|(h')_-| - e_q}{\gamma} - \frac{|(h')_-|}{\gamma^*} + l_q \right] \leq \left[\frac{-e_q}{\gamma} + l_q \right] \leq \left[\frac{k^*}{\delta} + m^* - J_q \right] \leq \left[\frac{k^*}{\delta} \right] + m^*.$$

Therefore we have

$$t^{\sigma_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \left(\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha u_{h_i}^{g_i}(t, x) \\ \ll C^{\sigma_q} \zeta^{e_q} \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^{g_i} \sum_{k \geq 0} \frac{1}{k!} \Psi_R^{([(k+k^*)/\delta] + m^*)}(\chi) \left(\frac{t}{\zeta} \right)^k.$$

In the cases (ii) and (iii), as for the estimates (5.2) and (5.3) we have

$$\left[\frac{|(h')_-| - e_q}{\gamma} - \frac{|(h')_-|}{\gamma^*} + l_q \right] \leq \left[\frac{-e_q}{\gamma} + l_q \right] \leq \left[\frac{k^*}{\delta} \right] + m^* - 1.$$

Therefore we have the estimate (5.9). Hence by Lemma 5.1, we have the estimate (5.6), where

$$U_h^g = c(\tau) \sum_{\substack{e_q > e^* \\ |h'| = h + (e_q - e^*)}} \sum_{\substack{|g'| = g - (e_q - e^*)}} C^{\sigma_q} \zeta^{e_q - e^*} B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^{g_i} \\ + c(\tau) \sum_{e_q < e^*} \sum_{\substack{|g'| = g \\ |h'| = h - (e^* - e_q)}} C^{\sigma_q} \zeta^{e_q - e^*} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^{g_i} \\ + c(\tau) \sum_{e_q = e^*} \sum_{\substack{|g'| = g \\ |h'| = h - 1}} C^{\sigma_q} R B_q \prod_{j+|\alpha| \leq m} \prod_{i=1}^{q_{j,\alpha}} \tau^{\alpha_1 - m^*} U_{h_i}^{g_i}.$$

Next let us give an estimate of the coefficient U_h^g . Let us consider the following functional equation:

$$(5.10) \quad \begin{aligned} Z(t, s) = & c(\tau) \sum_{e_q > e^*} C^{\sigma_q} s^{e_q - e^*} B_q \prod_{j+|\alpha| \leq m} \{\tau^{\alpha_1 - m^*} Z(t, s)\}^{q_{j, \alpha}} \\ & + c(\tau) \sum_{e_q < e^*} C^{\sigma_q} \zeta^{e_q - e^*} t^{e^* - e_q} B_q \prod_{j+|\alpha| \leq m} \{\tau^{\alpha_1 - m^*} Z(t, s)\}^{q_{j, \alpha}} \\ & + c(\tau) \sum_{e_q = e^*} C^{\sigma_q} t B_q \prod_{j+|\alpha| \leq m} \{\tau^{\alpha_1 - m^*} Z(t, s)\}^{q_{j, \alpha}} + \zeta^{-e^*} c(\tau) F. \end{aligned}$$

As in the case of (5.5), we have the holomorphic solution near $(t, s) = (0, 0)$. Moreover substituting $Z(t, s) = \sum_{g \geq 0} \sum_{h \geq -g} Z_h^g t^h (ts)^g$ into (5.10), for any $T > 0$ and $S > 0$ we can take a sufficiently large $\tau > 0$ and sufficiently small $R > 0$ and $\zeta > 0$ such that $U_h^g \leq Z_h^g T^h (TS)^g$. Hence we have the desired result. $\square$

Next for $g > 0$ we construct a true solution to (E^g) .

Set $\tilde{u}^g(t, x) := \sum_{h \geq -g} u_h^g(t, x) = \sum_{h=-g}^{-1} u_h^g(t, x) + \tilde{u}_0^g(t, x)$. Then we have

$$\tilde{u}_0^g(t, x) \ll \sum_{h \geq 0} \tau^{-m^*} U_h^g \sum_{k \geq k^*} \frac{1}{k!} \Psi_R^{([k/\delta])}(\chi) \left(\frac{t}{\zeta}\right)^k.$$

By Proposition 5.3, $\sum_{h \geq 0} U_h^g$ converges. Further, if we set $\tilde{U}_0^g := \sum_{h \geq 0} U_h^g$, then $\sum_{g \geq 0} \tilde{U}_0^g$ also converges by Proposition 5.3. For $h < 0$ set $\tilde{u}_h^g(t, x) := u_h^g(t, x)$ and $\tilde{U}_h^g := U_h^g$. Then we have

$$\tilde{u}^g(t, x) = \sum_{h=-g}^0 \tilde{u}_h^g(t, x) \ll \sum_{h=-g}^0 \tau^{-m^*} \tilde{U}_h^g \sum_{k \geq k^* - h} \frac{1}{k!} \Psi_R^{([k/\delta + h/\gamma - h/\gamma^*])}(\chi) \left(\frac{t}{\zeta}\right)^k.$$

Therefore for each $g > 0$, we can construct a true solution of (E^g) inductively as in the case $g = 0$. Hence we have the following result for $g \geq 0$:

Proposition 5.4. *Under the assumptions $(C'_2)_s$ and (A_1) , there exists a true solution $u^g(t, x)$ to (E^g) such that for $|\theta| \leq \pi/(2\gamma)$*

$$u^g(t, x) \sim_{\gamma} \tilde{u}^g(t, x) = \sum_{-g \leq h \leq 0} \tilde{u}_h^g(t, x) = \sum_{-g \leq h \leq 0} \sum_{k \geq k^* - h} \tilde{u}_{h,k}^g(x) t^k \quad \text{in } S_{\theta}(T).$$

Here $\tilde{u}_{h,k}^g(x) \ll \tau^{-m^*} \tilde{U}_h^g \frac{1}{k! \zeta^k} \Psi_R^{([k/\delta + h/\gamma - h/\gamma^*])}(\chi)$ and $\sum_{g \geq 0} \sum_{-g \leq h \leq 0} \tilde{U}_h^g$ converges for a sufficiently large $\tau > 0$ and sufficiently small $R > 0$ and $\zeta > 0$.

Proof of Theorem 3.2. Let us construct $u_{(h)}(t, x)$. We define

$$\begin{aligned}\widehat{u}_h^g(x, \xi) &:= \sum_{k \geq k^* - h} \frac{\widetilde{u}_{h,k}^g(x)}{\Gamma(k/\gamma + 1)} \xi^{k/\gamma}, \\ u_{h,S'}^g(t, x) &:= t^{-\gamma} \int_0^{\widehat{\xi}} \exp(-\xi t^{-\gamma}) \widehat{u}_h^g(x, \xi) d\xi.\end{aligned}$$

First let us estimate $\widehat{u}_h^g(x, \xi)$. By Proposition 5.4, there exist positive constants A and B such that

$$|\widehat{u}_h^g(x, \xi)| \leq \tau^{-m^*} \widetilde{U}_h^g A^g B^h \frac{\Gamma(-h/\gamma^* + 1)}{\Gamma(-h/\gamma + 1)} |\xi|^{(k^* - h)/\gamma} \quad \text{for } g \geq -h \geq 0.$$

Therefore we have

$$|u_{h,S'}^g(t, x)| \leq \tau^{-m^*} \widetilde{U}_h^g A^g B^h \Gamma\left(-\frac{h}{\gamma^*} + 1\right) \Gamma\left(\frac{k^*}{\gamma} + 1\right) |t|^{k^* - h} \quad \text{for } g \geq -h \geq 0.$$

Setting $u_{(h)}(t, x) := \sum_{g \geq -h} u_{h,S'}^g(t, x)$, we have

$$|u_{(h)}(t, x)| \leq \tau^{-m^*} \sum_{g \geq -h} \widetilde{U}_h^g A^g B^h \Gamma\left(-\frac{h}{\gamma^*} + 1\right) \Gamma\left(\frac{k^*}{\gamma} + 1\right) |t|^{k^* - h} \quad \text{for } h \leq 0.$$

For a sufficiently large $\tau > 0$ and sufficiently small $R > 0$ and $\zeta > 0$, we can show that $\sum_{g \geq -h} \widetilde{U}_h^g A^g$ converges. Set $\widetilde{U}_{(h)} := \sum_{g \geq -h} \widetilde{U}_h^g A^g$. Then $\sum_{h \leq 0} \widetilde{U}_{(h)} t^{-h}$ also converges. This completes a proof of Theorem 3.2. $\square$

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