数理解析研究所講究録
1144 巻 2000 年 137-146

EXTENSIONS OF PARANORMAL OPERATORS AND THEIR PROPERTIES

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ABSTRACT. This report is based on the following papers:

[Y1] T.Yamazaki and M.Yanagida, *A characterization of log-hyponormal operators via p-paranormality*, Sci. Math. **3** (2000), 19–21.

[Y2] T.Yamazaki and M.Yanagida, *A further generalization of paranormal operators*, Sci. Math. **3** (2000), 23–31.

An operator T is said to be paranormal if $\|T^2x\| \geq \|Tx\|^2$ holds for every unit vector x. Several extensions of paranormal operators have been considered until now, for example, absolute-k-paranormal and p-paranormal operators introduced in [6] and [3], respectively. As a further generalization of paranormal operators, we shall introduce absolute-(p,r)-paranormal operators for $p > 0$ and $r > 0$ such that $\||T|^p|T^*|^r x\|^r \geq \||T^*|^r x\|^{p+r}$ for every unit vector x. We shall show several properties on absolute-(p,r)-paranormal operators as generalizations of the results on absolute-k-paranormal and p-paranormal operators. We shall also show a characterization of log-hyponormal operators via absolute-(p,r)-paranormality, that is, an invertible operator T satisfies $\log T^*T \geq \log TT^*$ if and only if T is absolute-(p,r)-paranormal for all $p > 0$ and $r > 0$.

1. Introduction

In this report, an operator means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

An operator T is said to be p-hyponormal for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$, and T is said to be log-hyponormal if T is invertible and $\log T^*T \geq \log TT^*$. p-Hyponormal and log-hyponormal operators were defined as extensions of hyponormal ones, i.e., $T^*T \geq TT^*$. It is easily seen that every p-hyponormal operator is q-hyponormal for $p \geq q > 0$ by the celebrated Löwner-Heinz theorem "*$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$*," and every invertible p-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function.

An operator T is said to be paranormal if

$$\|T^2x\| \geq \|Tx\|^2 \quad \text{for every unit vector } x. \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary 47B20, 47A30, 47A63.

Key words and phrases. Paranormal operator, absolute-k-paranormal operator, p-paranormal operator, class A operator, class A(k) operator, class A(p,r) operator, Hölder-McCarthy inequality.

Paranormal operators have been studied by many researchers, for example, [1][5] and [7]. Particularly, Ando [1] showed that every log-hyponormal operator is paranormal. Afterward, in [6], we gave another simple proof of this result by introducing class A as a new class of operators given by an operator inequality. An operator T belongs to class A if

$$|T^2| \geq |T|^2,$$

where $|T| = (T^*T)^{\frac{1}{2}}$, and we showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal.

In [6], we introduced class $\mathrm{A}(k)$ and absolute-k-paranormal operators for $k > 0$ as generalizations of class A and paranormal operators, respectively. An operator T belongs to class $\mathrm{A}(k)$ if

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2,$$

and T is said to be absolute-k-paranormal if

$$\| |T|^k Tx \| \geq \|Tx\|^{k+1} \quad \text{for every unit vector } x. \tag{1.2}$$

It is clear that class $\mathrm{A}(1)$ equals class A and absolute-1-paranormality equals paranormality since $\| |S|y \| = \|Sy\|$ for any $S \in B(H)$ and $y \in H$. An operator T is said to be normaloid if $\|T\| = r(T)$. We showed inclusion relations among these classes in [6]. Class A and class $\mathrm{A}(k)$ operators have been studied in [8][9] and [11].

On the other hand, Fujii, Izumino and Nakamoto [3] introduced p-paranormal operators for $p > 0$ as another generalization of paranormal operators. An operator T is said to be p-paranormal if

$$\| |T|^p U |T|^p x \| \geq \| |T|^p x \|^2 \quad \text{for every unit vector } x, \tag{1.3}$$

where the polar decomposition of T is $T = U|T|$. It is clear that 1-paranormality equals paranormality. p-Paranormality is based on the following fact [2]: $T = U|T|$ is p-hyponormal if and only if $S = U|T|^p$ is hyponormal for $p > 0$. Actually, it was shown in [3] that $T = U|T|$ is p-paranormal if and only if $S = U|T|^p$ is paranormal for $p > 0$.

Fujii, Jung, S.H.Lee, M.Y.Lee and Nakamoto [4] introduced class $\mathrm{A}(p, r)$ as a further generalization of class $\mathrm{A}(k)$. An operator T belongs to class $\mathrm{A}(p, r)$ for $p > 0$ and $r > 0$ if

$$(|T^*|^r |T|^{2p} |T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}, \tag{1.4}$$

and class $\mathrm{AI}(p, r)$ is the class of all invertible operators which belong to class $\mathrm{A}(p, r)$. It was pointed out in [11] that class $\mathrm{A}(k, 1)$ equals class $\mathrm{A}(k)$ for each $k > 0$.

In this report, as a parallel concept to class $\mathrm{A}(p, r)$, we shall introduce absolute-(p, r)-paranormality which is a further generalization of both absolute-k-paranormality and p-paranormality. Then we shall generalize the results on absolute-k-paranormal and p-paranormal operators for absolute-(p, r)-paranormal operators. We shall also show a characterization of log-hyponormal operators via absolute-(p, r)-paranormality and p-paranormality.

2. Definition and properties of absolute-(p,r)-paranormal operators

We introduce the following class of operators.

Definition ([Y2]). For positive real numbers $p>0$ and $r>0$, an operator T is absolute-(p,r)-paranormal if

$$\||T|^p|T^*|^r x\|^r \geq \||T^*|^r x\|^{p+r} \quad \text{for every unit vector } x, \tag{2.1}$$

or equivalently,

$$\||T|^p|T^*|^r x\|^r \|x\|^p \geq \||T^*|^r x\|^{p+r} \quad \text{for all } x \in H. \tag{2.2}$$

We remark that the definition of absolute-(p,r)-paranormal operators (2.1) and (2.2) are expressed in terms of only T and T^*, without U which appears in the polar decomposition of $T=U|T|$.

To consider the relations to absolute-k-paranormality and p-paranormality, we show another expression of absolute-(p,r)-paranormality as follows.

Proposition 1 ([Y2]). *For each $p>0$ and $r>0$, T is absolute-(p,r)-paranormal if and only if*

$$\||T|^p U|T|^r x\|^r \geq \||T|^r x\|^{p+r} \quad \textit{for every unit vector } x, \tag{2.3}$$

where the polar decomposition of T is $T=U|T|$.

The following result is easily obtained as a corollary of Proposition 1.

Corollary 2 ([Y2]).

(i) *For each $k>0$, T is absolute-k-paranormal iff T is absolute-$(k,1)$-paranormal.*
(ii) *For each $p>0$, T is p-paranormal iff T is absolute-(p,p)-paranormal.*
(iii) *T is paranormal iff T is absolute-$(1,1)$-paranormal.*

It turns out by Corollary 2 that absolute-(p,r)-paranormality is a further generalization of paranormality than both absolute-k-paranormality and p-paranormality.

Proof of Proposition 1. It is well known that $|T^*|^r = U|T|^r U^*$ for $r>0$, so that (2.2) is equivalent to the following (2.4):

$$\||T|^p U|T|^r U^* x\|^r \|x\|^p \geq \|U|T|^r U^* x\|^{p+r} \quad \text{for all } x \in H. \tag{2.4}$$

It is also well known that $N(S^r)=N(S)$ for any $S\geq 0$ and $r>0$. By using this fact, we have $R(|T|^r) \subseteq \overline{R(|T|^r)} = N(|T|^r)^\perp = N(|T|)^\perp = N(U)^\perp$, so that $\|U|T|^r U^* x\| = \||T|^r U^* x\|$ for all $x\in H$. Hence (2.4) is equivalent to the following (2.5):

$$\||T|^p U|T|^r U^* x\|^r \|x\|^p \geq \||T|^r U^* x\|^{p+r} \quad \text{for all } x \in H. \tag{2.5}$$

Put $x=Uy$ in (2.5), then we have the following (2.6) since $|T|^r U^* U = |T|^r$:

$$\||T|^p U|T|^r y\|^r \|Uy\|^p \geq \||T|^r y\|^{p+r} \quad \text{for all } y \in H. \tag{2.6}$$

(2.6) yields the following (2.7) since $\|y\| \geq \|Uy\|$ for all $y \in H$:

$$\| |T|^p U |T|^r y \|^r \|y\|^p \geq \| |T|^r y \|^{p+r} \quad \text{for all } y \in H. \tag{2.7}$$

Hence (2.5) implies (2.7). Here we show that (2.7) implies (2.5) conversely. Put $y = U^*x$ in (2.7), then we have

$$\| |T|^p U |T|^r U^*x \|^r \|U^*x\|^p \geq \| |T|^r U^*x \|^{p+r} \quad \text{for all } x \in H. \tag{2.8}$$

(2.8) yields (2.5) since $\|x\| \geq \|U^*x\|$ for all $x \in H$. Hence (2.7) implies (2.5), so that (2.5) is equivalent to (2.7). Consequently, the proof of Proposition 1 is complete since (2.7) is equivalent to (2.3). □

Proof of Corollary 2. We remark that $\| |S|y \| = \|Sy\|$ holds for any $S \in B(H)$ and $y \in H$.

(i) Put $p = k > 0$ and $r = 1$ in (2.3), then we have (1.2).
(ii) Put $r = p > 0$ in (2.3), then we have (1.3).
(iii) Put $r = p = 1$ in (2.3), then we have (1.1). □

It was shown in [5] and [7] that if T is invertible and paranormal, then T^{-1} is also paranormal. Here we show the following generalization of this well-known result.

Proposition 3 ([Y2]). *The following assertions hold for each $p > 0$ and $r > 0$:*

(i) *If T is invertible and absolute-(p,r)-paranormal, then T^{-1} is absolute-(r,p)-paranormal.*
(ii) *If T is invertible and p-paranormal, then T^{-1} is also p-paranormal.*

Proposition 3 can be considered as a parallel result to the following Proposition A for class $\mathrm{AI}(p,r)$ operators.

Proposition A ([Y2]). *The following assertions hold for each $p > 0$ and $r > 0$:*

(i) *If T belongs to class $\mathrm{AI}(p,r)$, then T^{-1} belongs to class $\mathrm{AI}(r,p)$.*
(ii) *If T belongs to class $\mathrm{AI}(p,p)$, then T^{-1} also belongs to class $\mathrm{AI}(p,p)$.*
(iii) *If T is invertible and belongs to class A, then T^{-1} also belongs to class A.*

We prepare the following lemma to give a proof of Proposition 3.

Lemma 4 ([Y2]). *Let T be an invertible operator. For each $p > 0$ and $r > 0$, T is absolute-(p,r)-paranormal if and only if*

$$\| |T|^p x \|^r \| |T^{-1}|^r x \|^p \geq 1 \quad \textit{for every unit vector } x. \tag{2.9}$$

Proof. (2.2) is equivalent to the following (2.10) by putting $y = |T^*|^r x$ since $R(|T^*|^r) = H$:

$$\| |T|^p y \|^r \| |T^*|^{-r} y \|^p \geq \|y\|^{p+r} \quad \text{for all } y \in H. \tag{2.10}$$

(2.10) is equivalent to the following (2.11):

$$\| |T|^p y \|^r \| |T^*|^{-r} y \|^p \geq 1 \quad \text{for every unit vector } y. \tag{2.11}$$

(2.11) is equivalent to (2.9) since $|T^*|^{-1} = |T^{-1}|$, so that the proof is complete. □

Proof of Proposition 3.

(i) Obvious by Lemma 4.

(ii) Put $r = p > 0$ in (i), then we have (ii) by (ii) of Corollary 2. □

3. Inclusion relations among the related classes

We cite the following result which plays an important role to give proofs of the results in this section.

Theorem H-M (Hölder-McCarthy inequality [10]). *Let A be a positive operator. Then the following inequalities hold for all $x \in H$:*

(i) $(A^r x, x) \leq (Ax, x)^r \|x\|^{2(1-r)}$ *for* $0 < r \leq 1$.

(i') $\|A^r x\| \leq \|Ax\|^r \|x\|^{1-r}$ *for* $0 < r \leq 1$.

(ii) $(A^r x, x) \geq (Ax, x)^r \|x\|^{2(1-r)}$ *for* $r \geq 1$.

(ii') $\|A^r x\| \geq \|Ax\|^r \|x\|^{1-r}$ *for* $r \geq 1$.

Firstly, we show the monotonicity of the classes of absolute-(p, r)-paranormal operators for $p > 0$ and $r > 0$ as generalizations of [4, Theorem 4.1] and [6, Theorem 4].

Theorem 5 ([Y2]). *Let T be absolute-(p_0, r_0)-paranormal for $p_0 > 0$ and $r_0 > 0$. Then T is absolute-(p, r)-paranormal for any $p \geq p_0$ and $r \geq r_0$. Moreover, for each $r \geq r_0$ and unit vector x,*

$$f_r(p) = \||T|^p |T^*|^r x\|^{\frac{r}{p+r}} \tag{3.1}$$

is increasing for $p \geq p_0$.

Theorem 5 can be considered as a parallel result to the following Theorem B which states the monotonicity of class $\mathrm{AI}(p, r)$ for $p > 0$ and $r > 0$.

Theorem B ([4]). *If T belongs to class $\mathrm{AI}(p_0, r_0)$ for $p_0 > 0$ and $r_0 > 0$, then T belongs to class $\mathrm{AI}(p, r)$ for any $p \geq p_0$ and $r \geq r_0$.*

Proof of Theorem 5. Assume that T is absolute-(p_0, r_0)-paranormal for $p_0 > 0$ and $r_0 > 0$, i.e.,

$$\||T|^{p_0}|T^*|^{r_0}y\|^{r_0} \geq \||T^*|^{r_0}y\|^{p_0+r_0}\|y\|^{-p_0} \quad \text{for all } y \in H. \tag{3.2}$$

Then for each $r \geq r_0$ and unit vector x,

$$\begin{aligned}
&\||T|^{p_0}|T^*|^r x\|^{r_0} \\
=&\||T|^{p_0}|T^*|^{r_0}|T^*|^{r-r_0}x\|^{r_0} \\
\geq&\||T^*|^{r_0}|T^*|^{r-r_0}x\|^{p_0+r_0}\||T^*|^{r-r_0}x\|^{-p_0} && \text{by (3.2)} \\
=&\||T^*|^r x\|^{p_0+r_0}\||T^*|^{r-r_0}x\|^{-p_0} \\
\geq&\||T^*|^r x\|^{p_0+r_0}\||T^*|^r x\|^{\frac{r-r_0}{r}\cdot(-p_0)} && \text{by (i') of Theorem H-M for } \tfrac{r-r_0}{r} \in [0,1) \\
=&\||T^*|^r x\|^{\frac{(p_0+r)r_0}{r}},
\end{aligned}$$

so that we have

$$\||T|^{p_0}|T^*|^r x\|^{\frac{r}{p_0+r}} \geq \||T^*|^r x\|. \tag{3.3}$$

Hence for each $p \geq p_0$, $r \geq r_0$ and unit vector x,

$$\begin{aligned}
&\||T|^p|T^*|^r x\| \\
\geq &\||T|^{p_0}|T^*|^r x\|^{\frac{p}{p_0}} \||T^*|^r x\|^{1-\frac{p}{p_0}} && \text{by (ii') of Theorem H-M for } \tfrac{p}{p_0} \geq 1 \\
\geq &\||T|^{p_0}|T^*|^r x\|^{\frac{p}{p_0}} \||T|^{p_0}|T^*|^r x\|^{\frac{r}{p_0+r}\cdot\frac{p_0-p}{p_0}} && \text{by (3.3)} \\
= &\||T|^{p_0}|T^*|^r x\|^{\frac{p+r}{p_0+r}} \\
\geq &\||T^*|^r x\|^{\frac{p+r}{r}} && \text{by (3.3),}
\end{aligned}$$

so that we have

$$\||T|^p|T^*|^r x\|^{\frac{r}{p+r}} \geq \||T|^{p_0}|T^*|^r x\|^{\frac{r}{p_0+r}} \geq \||T^*|^r x\|. \tag{3.4}$$

(3.4) assures that T is absolute-(p, r)-paranormal for any $p \geq p_0$ and $r \geq r_0$, and for each $r \geq r_0$ and unit vector x, $f_r(p) = \||T|^p|T^*|^r x\|^{\frac{r}{p+r}}$ is increasing for $p \geq p_0$. □

Secondly, we show inclusion relations among the class of absolute-(p, r)-paranormal operators and the related classes.

Theorem 6 ([Y2]). *The following assertions hold for each $p > 0$ and $r > 0$:*

(i) *Every class $\mathrm{A}(p, r)$ operator is absolute-(p, r)-paranormal.*

(ii) *Every absolute-(p, r)-paranormal operator is normaloid.*

(i) of Theorem 6 is a generalization of [4, Theorem 3.5] and [6, Theorem 4], and (ii) is a generalization of [6, Theorem 5] and the following result.

Theorem C ([4]). *Every p-paranormal operator is normaloid for $p > 0$.*

Proof of Theorem 6.

Proof of (i). Assume that T belongs to class $\mathrm{A}(p, r)$ for $p > 0$ and $r > 0$, i.e.,

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}. \tag{1.4}$$

Then for every unit vector x,

$$\begin{aligned}
\||T^*|^r x\|^2 &= (|T^*|^{2r}x, x) \\
&\leq ((|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}}x, x) && \text{by (1.4)} \\
&\leq (|T^*|^r|T|^{2p}|T^*|^r x, x)^{\frac{r}{p+r}} && \text{by (i) of Theorem H-M for } \tfrac{r}{p+r} \in (0,1) \\
&= \||T|^p|T^*|^r x\|^{\frac{2r}{p+r}},
\end{aligned}$$

so that we have

$$\||T|^p|T^*|^r x\|^r \geq \||T^*|^r x\|^{p+r} \quad \text{for every unit vector } x, \tag{2.1}$$

i.e., T is absolute-(p, r)-paranormal.

Proof of (ii). Assume that T is absolute-(p,r)-paranormal. Put $q=\max\{p,r\}>0$, then T is absolute-(q,q)-paranormal by Theorem 5, i.e., T is q-paranormal by (ii) of Corollary 2. Hence T is normaloid by Theorem C. □

4. A characterization of log-hyponormal operators via absolute-(p,r)-paranormality and p-paranormality

In [4], the following result was shown which is a characterization of log-hyponormal operators in terms of class $\mathrm{AI}(p,r)$.

Theorem D ([4]). *The following assertions are mutually equivalent:*

(i) *T is log-hyponormal.*
(ii) *T belongs to class $\mathrm{AI}(p,p)$ for all $p>0$.*
(iii) *T belongs to class $\mathrm{AI}(p,r)$ for all $p>0$ and $r>0$.*

Theorem D states that the class of log-hyponormal operators can be considered as the limit of class $\mathrm{AI}(p,r)$ as $p\to+0$ and $r\to+0$ since class $\mathrm{AI}(p,r)$ is monotone increasing for $p>0$ and $r>0$ by Theorem B.

Here we shall give a characterization of log-hyponormal operators in terms of absolute-(p,r)-paranormality and p-paranormality.

Theorem 7 ([Y1][Y2]). *The following assertions are mutually equivalent:*

(i) *T is log-hyponormal.*
(ii) *T is invertible and p-paranormal for all $p>0$.*
(iii) *T is invertible and absolute-(p,r)-paranormal for all $p>0$ and $r>0$.*

Theorem 7 states that the class of log-hyponormal operators can be considered as the limit of the class of invertible and absolute-(p,r)-paranormal operators as $p\to+0$ and $r\to+0$ since the class of absolute-(p,r)-paranormal operators is monotone increasing for $p>0$ and $r>0$ by Theorem 5. It is interesting to remark that class $\mathrm{AI}(p,r)$ and the class of invertible and absolute-(p,r)-paranormal operators can be considered to be parallel to each other, but their limits as $p\to+0$ and $r\to+0$ coincide. In fact, Theorem 7 gives a more precise sufficient condition for that an operator T is log-hyponormal than Theorem D by (i) of Theorem 6.

In order to give a proof of Theorem 7, we prepare the following result.

Proposition 8 ([Y2]). *The following assertions hold for each $p>0$ and $r>0$:*

(i) *T is absolute-(p,r)-paranormal if and only if*

$$r|T^*|^r|T|^{2p}|T^*|^r-(p+r)\lambda^p|T^*|^{2r}+p\lambda^{p+r}I\geq 0\quad \text{for all } \lambda>0. \tag{4.1}$$

(ii) *T is p-paranormal if and only if*

$$|T^*|^p|T|^{2p}|T^*|^p-2\lambda|T^*|^{2p}+\lambda^2 I\geq 0\quad \text{for all } \lambda>0. \tag{4.2}$$

Ando [1] gave a characterization of paranormal operators via an operator inequality as follows: T is paranormal if and only if

$$T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$$

for all $\lambda > 0$. A generalization of this result for absolute-k-paranormal operators was shown in [6, Theorem 6], and Proposition 8 is a further generalization for absolute-(p,r)-paranormal operators.

We use the following well-known fact in the proof of Proposition 8.

Lemma E. *For positive real numbers $a > 0$ and $b > 0$,*

$$\lambda a + \mu b \geq a^\lambda b^\mu$$

holds for $\lambda > 0$ and $\mu > 0$ such that $\lambda + \mu = 1$.

Proof of Proposition 8.
Proof of (i). (2.2) is equivalent to the following (4.3):

$$(4.3) \qquad (|T^*|^r|T|^{2p}|T^*|^r x, x)^{\frac{r}{p+r}}(x,x)^{\frac{p}{p+r}} \geq (|T^*|^{2r}x, x) \quad \text{for all } x \in H.$$

By Lemma E,

$$\begin{aligned}(|T^*|^r|T|^{2p}|T^*|^r x, x)^{\frac{r}{p+r}}(x,x)^{\frac{p}{p+r}} &= \{\lambda^{-p}(|T^*|^r|T|^{2p}|T^*|^r x, x)\}^{\frac{r}{p+r}}\{\lambda^r(x,x)\}^{\frac{p}{p+r}} \\ &\leq \frac{r}{p+r}\cdot\lambda^{-p}(|T^*|^r|T|^{2p}|T^*|^r x, x) + \frac{p}{p+r}\cdot\lambda^r(x,x)\end{aligned}$$

holds for all $x \in H$ and $\lambda > 0$, so that (4.3) implies the following (4.4):

$$(4.4) \qquad \frac{r}{p+r}\cdot\lambda^{-p}(|T^*|^r|T|^{2p}|T^*|^r x, x) + \frac{p}{p+r}\cdot\lambda^r(x,x) \geq (|T^*|^{2r}x, x)$$
$$\text{for all } x \in H \text{ and } \lambda > 0.$$

Conversely, (4.3) follows from (4.4) by putting $\lambda = \left\{\frac{(|T^*|^r|T|^{2p}|T^*|^r x,x)}{(x,x)}\right\}^{\frac{1}{p+r}} > 0$ in case $(|T^*|^r|T|^{2p}|T^*|^r x, x) \neq 0$, and letting $\lambda \to +0$ in case $(|T^*|^r|T|^{2p}|T^*|^r x, x) = 0$. Hence (4.3) is equivalent to (4.4). Consequently, the proof of Proposition 8 is complete since (4.4) is equivalent to (4.1).
Proof of (ii). Put $r = p > 0$ and replace λ^p with λ in (i), then we have (ii) by (ii) of Corollary 2. □

Proof of Theorem 7. It is pointed out in [4] that every log-hyponormal operator belongs to class AI(p,r) for all $p > 0$ and $r > 0$, so that (i) $\Longrightarrow$ (iii) holds by (i) of Theorem 6. And (iii) $\Longrightarrow$ (ii) can be proved by putting $r = p > 0$ by (ii) of Corollary 2. Hence we have only to prove (ii) $\Longrightarrow$ (i).

Assume that T is p-paranormal for all $p > 0$. By (ii) of Proposition 8, (4.2) holds particularly for $\lambda = 1$, that is,

$$(4.5) \qquad |T^*|^p|T|^{2p}|T^*|^p - 2|T^*|^{2p} + I \geq 0 \quad \text{for all } p > 0.$$

Since T is invertible, (4.5) can be rewritten as the following (4.6):

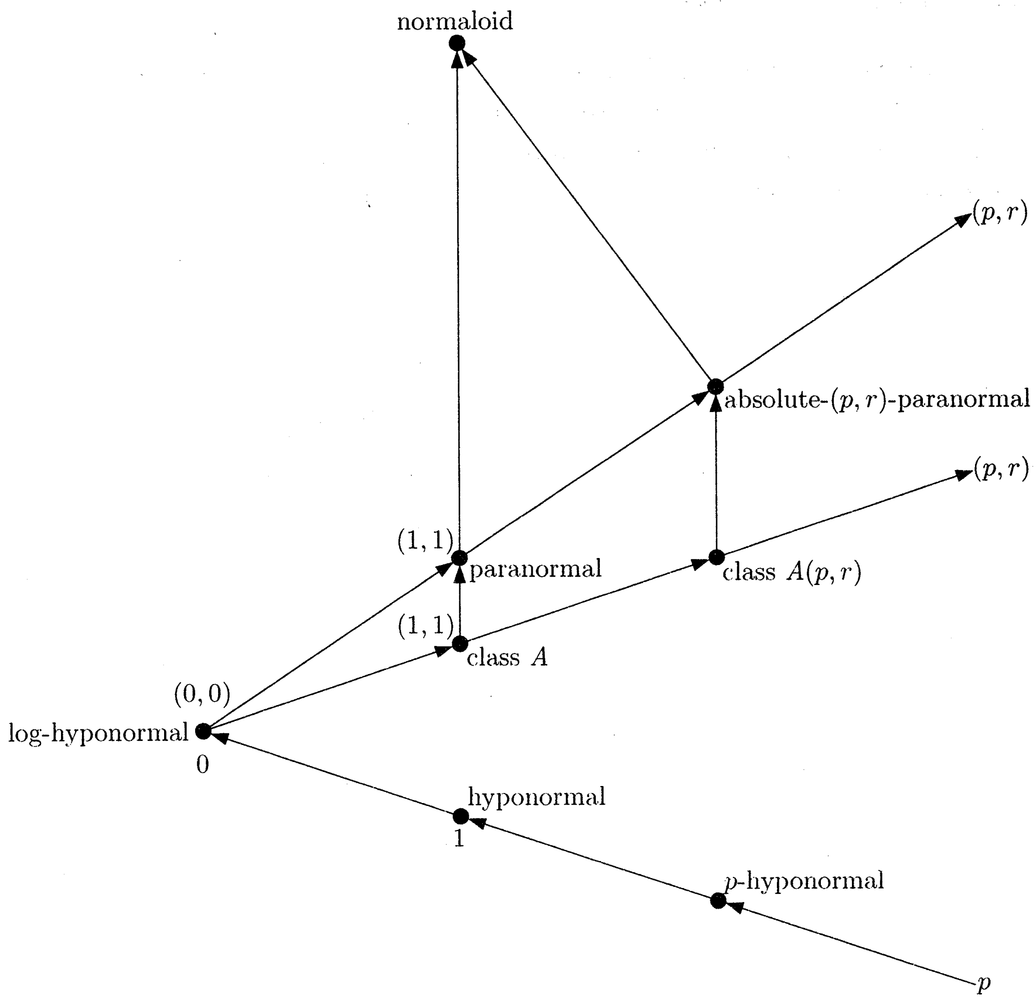
$$\frac{|T|^{2p}-I}{p} \geq \frac{|T^*|^{-2p}-I}{-p} \quad \text{for all } p>0. \tag{4.6}$$

By letting $p \to +0$ in (4.6), we have

$$\log|T|^2 \geq \log|T^*|^2,$$

i.e., T is log-hyponormal. □

The following diagram represents the inclusion relations among the classes discussed in this report.



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