

The Simple Graphs Associated with Rings and Semigroups

Mitsuo Kanemitsu¹

Let R be a commutative ring, and let $Z(R)$ denote its set of zero-divisors. We associate a simple graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisor of R . Two distinct vertices x and y are adjacent if $xy = 0$. This graphs are called the *zero-divisor graphs* of rings R .

We also associate a simple graph $\Delta(\mathbf{Z}_n)$ to $\mathbf{Z}_n$ with vertices $\mathbf{Z}_n$ and for distinct elements $x, y \in \mathbf{Z}_n$, the vertices x and y are adjacent if and only if $y = x^2$ ($x \neq y$). This graphs are called the *parabola graphs*.

For a commutative multiplicative semigroup S with 0 ($0x = 0$ for all $x \in S$), we can defined the zero-divisor graph $\Gamma(S)$ as above ([DMS]).

We denote an edge such that a and b are adjacent by $a - b$. We also denote a path by $a - b - c - d$ etc. Also, let $\chi(G)$ denote the chromatic number of the graph G and let $\chi'(G)$ denote the edge chromatic number of the graph G .

The notion of a zero-divisor graph was first introduced by I. Beck in [B1] and further investigated in [A1], though their vertices set included the zero element. Let G be a graph.

The *diameter* of G is

$$\text{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are distinct vertices of } G\},$$

where $d(x, y)$ denotes the length of the shortest path from x to y . The *grith* of G , denoted by $g(G)$, is defined as the length of the shortest cycle in G .

A complete subgraph of G is called *clique*. $\omega(G)$, the *clique number* of G , is the greatest integer $r \geq 1$ such that $K^r \subset G$. Also, $c(G)$, the *circumference* of G , is the length of the longest cycle in G . Let $n = p_1^{2n_1} \cdots p_k^{2n_k} q_1^{2m_1+1} \cdots q_r^{2m_r+1}$ for distinct primes p_i, q_j and integers $n_i, m_j \geq 0$. Then

$$\omega(\Gamma(\mathbf{Z}_n)) = p_1^{n_1} \cdots p_k^{n_k} q_1^{m_k} \cdots q_r^{m_r} + r - 1$$

¹ This is a part of an abstract and details will be published elsewhere.

by [B, Proposition 2.3].

For a graph G and an integer $n \geq 1$, we define $\lambda(G, n)$ to be the number of complete subgraphs (cliques) of G of order n . Note that $\lambda(G, 1)$ is the number of vertices of G , $\lambda(G, 2)$ is the number of edges of G , $\lambda(G, 3)$ is the number of triangles in G and $\lambda(G, n) = 0$ for all $n \geq \omega(G) + 1$ ([AFL]).

A graph G is *planar* if it can be drawn in such a way that no two edges intersect.

We examine the zero-divisor graph $\Gamma(Z_n)$, of the ring Z_n where the residue class ring modulo n where n is a positive integer. Many parts of this note are contained in [DS] and [AF].

We also investigate the parabola graphs.

For any commutative semigroup S , let $C(S)$ = the *core* of $\Gamma(S)$ be the union of the cycles in $\Gamma(S)$. A vertex x of $\Gamma(S)$ is called an *end point* in case there is at most one edge in $\Gamma(S)$ with vertex x .

§1. Some examples of zero-divisor graphs and parabola graphs.

Example 1. Let S be the commutative nilsemigroup

$$S = \langle a, b \mid a^3 = a^2b = ab^2 = b^3 = 0 \rangle = \{a, b, a^2, ab, b^2, 0\}.$$

For this zero-divisor graph $\Gamma(S) = (V(\Gamma(S)), E(\Gamma(S)))$, we have that

$$V(\Gamma(S)) = \{a, b, a^2, ab, b^2\}$$

is the set of all vertices and $E(\Gamma(S)) = \{a-a^2, a-b^2, a-ab, b^2-ab, b^2-a^2, b^2-b, \}$. This graph is connected, $d(a, b^2) = 1$, $d(a, b) = 2$ and $a - a^2 - b^2 - a$ is a cycle of length 3. And so $\text{diam}(\Gamma(S)) = 2$ and $g(\Gamma(S)) = 3$. There is not an end point in $\Gamma(S)$ and $C(S) = \Gamma(S)$. Also, we have that $\omega(\Gamma(S)) = 4$. Also we have that $\lambda(S, 1) = 5$, $\lambda(S, 2) = 9$ and $\lambda(S, 2) = 6$.

Example 2. Let S be the commutative nilsemigroup

$$S = \langle a, b \mid a^3 = a^2b = ab^2 = b^3 = 0, a^2 = b^2 \rangle = \{a, b, a^2, ab, 0\}.$$

$\Gamma(S)$ is connected, $d(a, a^2) = 1, d(a, b) = 2$ and $a - a^2 - ab - a$ is a cycle of length 3. And so $\text{diam}(\Gamma(S)) = 2$ and $g(\Gamma(S)) = 3$. There is not an end point in $\Gamma(S)$ and $C(S) = \Gamma(S)$. Also, we have that $\omega(\Gamma(S)) = 3$ and $c(\Gamma(S)) = 3$. Also we have that $\lambda(S, 1) = 4, \lambda(S, 2) = 5$ and $\lambda(S, 2) = 2$.

This graph $\Gamma(S)$ is planar.

Example 3. $V(\Gamma(\mathbf{Z}_{12}))$ of the zero-divisor graph $\Gamma(\mathbf{Z}_{12})$ is the set

$$\{2, 3, 4, 6, 8, 9, 10\} \text{ and}$$

$$E(\Gamma(\mathbf{Z}_{12})) = \{2 - 6, 3 - 4, 3 - 8, 4 - 6, 4 - 9, 6 - 8, 6 - 10, 8 - 9\}.$$

The elements 4, 9 are idempotent elements and 6 is a nilpotent element. Also, $\{0, 6\}$ is an ideal of $\mathbf{Z}_{12}$, $\omega(\Gamma(\mathbf{Z}_{12})) = 4$. Also we have that $\lambda(\mathbf{Z}_{12}, 1) = 7, \lambda(\mathbf{Z}_{12}, 2) = 8$ and $\lambda(\mathbf{Z}_{12}, 3) = 0$. Also $\text{diam}(\Gamma(\mathbf{Z}_{12})) = 3, g(\Gamma(\mathbf{Z}_{12})) = 4$ and $c(\Gamma(\mathbf{Z}_{12})) = 4$. We have that $\chi(\Gamma(\mathbf{Z}_{12})) = 2$ and $\chi'(\Gamma(\mathbf{Z}_{12})) = 4$. $\Gamma(\mathbf{Z}_{12})$ is planar.

Example 4. $V(\Gamma(\mathbf{Z}_{15}))$ of the zero-divisor graph $\Gamma(\mathbf{Z}_{15})$ is the set

$$\{3, 5, 6, 9, 10, 12\} \text{ and}$$

$$E(\Gamma(\mathbf{Z}_{15})) = \{3 - 5, 3 - 10, 5 - 6, 5 - 9, 5 - 12, 6 - 10, 9 - 10, 10 - 12\}.$$

The elements 6, 10 are idempotent elements. The ring $\mathbf{Z}_{15}$ has no ideals contained only two elements. We have that $\omega(\mathbf{Z}_{15}) = 4, c(\mathbf{Z}_{15}) = 4$ and also we have that $\lambda(\mathbf{Z}_{15}, 1) = 6, \lambda(\mathbf{Z}_{15}, 2) = 8$ and $\lambda(\mathbf{Z}_{15}, 2) = 0$. Also $\text{diam}(\Gamma(\mathbf{Z}_{15})) = 2$ and $g(\Gamma(\mathbf{Z}_{15})) = 4$. We have that $\chi(\Gamma(\mathbf{Z}_{15})) = 2$ and $\chi'(\Gamma(\mathbf{Z}_{15})) = 4$. $\Gamma(\mathbf{Z}_{15})$ is planar.

Example 5. $V(\Gamma(\mathbf{Z}_{16}))$ of the zero-divisor graph $\Gamma(\mathbf{Z}_{16})$ is the set

$$\{2, 4, 6, 8, 10, 12, 14\} \text{ and}$$

$$E(\Gamma(\mathbf{Z}_{16})) = \{2 - 8, 4 - 8, 4 - 12, 6 - 8, 8 - 10, 8 - 12, 8 - 14\}.$$

The element 8 is an idempotent element and 4, 12 are nilpotent elements. The set $\{0, 8\}$ is an ideal of $\mathbf{Z}_{16}$. We have that $\omega(\mathbf{Z}_{16}) = 3$. Also we have that $\lambda(\mathbf{Z}_{16}, 1) = 7, \lambda(\mathbf{Z}_{16}, 2) = 7$ and $\lambda(\mathbf{Z}_{16}, 3) = 1$. Also $\text{diam}(\Gamma(\mathbf{Z}_{16})) =$

$2, g(\Gamma(\mathbf{Z}_{16})) = 3$ and $c(\Gamma(\mathbf{Z}_{16})) = 3$. We have that $\chi(\Gamma(\mathbf{Z}_{16})) = 3$ and $\chi'(\Gamma(\mathbf{Z}_{16})) = 6$. $\Gamma(\mathbf{Z}_{16})$ is planar.

Example 6. Let $\Gamma(\mathbf{Z}_4[X]/(X^2)) = \Gamma(\mathbf{Z}_4[x])$ ($x^2 = 0$) be a zero-divisor graph associated with the ring $\mathbf{Z}_4[X]/(X^2)$. Set $R_4[x] = \mathbf{Z}_4[x]$. Then $V(\Gamma(R_4[x])) = \{2, x, 2+x, 2x, 2+2x, 3x, 2+3x\}$ and $E(R_4[x]) = \{2-2x, 2-2+2x, x-2x, x-3x, 2+x-2x, 2x-2+2x, 2+3x-2x\}$.

We have that $\omega(R_4[x]) = 3, c(R_4[x]) = 3$ and also we have that $\lambda(R_4[x], 1) = 7, \lambda(R_4[x], 2) = 8, \lambda(R_4[x], 3) = 2$ and $\lambda(R_4[x], 4) = 0$. Also $\text{diam}(\Gamma(R_4[x])) = 2$ and $g(\Gamma(R_4[x])) = 3$. We have that $\chi(\Gamma(R_4[x])) = 3$ and $\chi'(\Gamma(R_4[x])) = 6$.

$\Gamma(R_4[x])$ is planar.

Lemma 1. $\Gamma(\mathbf{Z}_n)$ is connected and $\text{diam}(\mathbf{Z}_n) \leq 3$.

Lemma 2. If $\Gamma(\mathbf{Z}_n)$ contains a cycle, then $g(\mathbf{Z}_n) \leq 4$.

Conjecture. The number of end points of $\Gamma(\mathbf{Z}_n)$ is an even number.

This conjecture is valid for $n \leq 30$.

Lemma 3. If $n \geq 9$ and $\Gamma(\mathbf{Z}_n)$ is a zero-divisor graph of $\mathbf{Z}_n$, then there exists an element x such that $\{0, x\}$ is an ideal of $\mathbf{Z}_n$, $\text{Ann}(x)$ is a maximal ideal of $\mathbf{Z}_n$ and $\mathbf{Z}_n/\text{Ann}(x) \cong \mathbf{Z}_2$.

If p is a prime number ($\neq 2$), then $\{0, p\}$ is an ideal of $\mathbf{Z}_{2p}$. So $\Gamma(\mathbf{Z}_{2p})$ is a star graph.

Lemma 4. If any vertex in $\Gamma(\mathbf{Z}_n)$ is either a vertex of the core $C(\Gamma(\mathbf{Z}_n))$ or else is an end point of $\Gamma(\mathbf{Z}_n)$.

Lemma 5 ([B] and [AN]). Let p, q and r be all distinct prime numbers. The following statements are satisfied.

- (1) $\omega(\Gamma(\mathbf{Z}_n)) = 1$ if and only if $n = 4$.
- (2) $\omega(\Gamma(\mathbf{Z}_n)) = 2$ if and only if $n = 8, 9, pq, 4p$ ($p \neq 2$).
- (3) $\omega(\Gamma(\mathbf{Z}_n)) = 3$ if and only if $n = prq, 4pq$ ($p \neq 2, q \neq 2$), $8p$ ($p \neq 2$), $9p$ ($p \neq 3$), $16, 27$.

§2. The parabola graph $y = x^2$

We associate a simple graph $\Delta(\mathbf{Z}_n)$ to $\mathbf{Z}_n$ with vertices $\mathbf{Z}_n$ and for distinct elements $x, y \in \mathbf{Z}_n$, the vertices x and y are adjacent if and only if $y = x^2$ ($x \neq y$).

Example 1. The vertices set $V(\Delta(\mathbf{Z}_7))$ of the simple graph $\Delta(\mathbf{Z}_7)$ is the set $\mathbf{Z}_7$ and $E(\Delta(\mathbf{Z}_7)) = \{1 - 6, 2 - 3, 2 - 4, 4 - 5\}$. $\lambda(\Delta(\mathbf{Z}_7), 1) = 7, \lambda(\Delta(\mathbf{Z}_7), 2) = 4$ and $\lambda(\Delta(\mathbf{Z}_7), 3) = 0$. We have that $\text{diam}(\Delta(\mathbf{Z}_7)) = 3, \omega(\Delta(\mathbf{Z}_7)) = 2, c(\Delta(\mathbf{Z}_7)) = g(\Delta(\mathbf{Z}_7)) = 0$.

Example 2. Let $\Delta(\mathbf{Z}_{11}) = (V(\Delta(\mathbf{Z}_{11})), E(\Delta(\mathbf{Z}_{11})))$ be a simple graph associated with $\mathbf{Z}_{11}$. We have that $V(\Delta(\mathbf{Z}_{11})) = \mathbf{Z}_{11}$ and $E(\Delta(\mathbf{Z}_{11})) = \{2 - 10, 3 - 5, 3 - 6, 3 - 9, 4 - 5, 4 - 9, 5 - 7, 8 - 9\}$. $\lambda(\Delta(\mathbf{Z}_{11}), 1) = 11, \lambda(\Delta(\mathbf{Z}_{11}), 2) = 8$ and $\lambda(\Delta(\mathbf{Z}_{11}), 3) = 0, \lambda(\Delta(\mathbf{Z}_{11}), 4) = 1$. We have that $\text{diam}(\Delta(\mathbf{Z}_{11})) = 4, \omega(\Delta(\mathbf{Z}_{11})) = 2, c(\Delta(\mathbf{Z}_{11})) = 4, g(\Delta(\mathbf{Z}_{11})) = 4$. This graph is not a forest.

Theorem. (a) Let $\Delta(\mathbf{Z}_7)$ be a parabola graph and let A_7 be an adjacent matrix of a parabola graph $\Delta(\mathbf{Z}_7)$. Also, $f_n(X) = f_{A^n}(X)$ be a minimal polynomial of A_7 . Then the following statements hold.

- (1) If n is an even natural number, then $f_n(X)$ has a divisor $X^2 - L_n X + 1$.
- (2) If n is an odd natural number, then $f_n(X)$ has a divisor $X^2 - L_n X - 1$, where L_n is a Lucas number.

(b) (1) $\Delta(\mathbf{Z}_p)$ has no triangles, that is, $\lambda(\Delta(\mathbf{Z}_p), 3) = 0$ for a prime number p .

(2) $\lambda(\Delta(\mathbf{Z}_n)) = 0$ for $1 \leq n \leq 10$ and $12 \leq n \leq 20$.

(3) For parabola graphs $\Delta(\mathbf{Z}_n)$ ($2 \leq n \leq 20$), their graphs are forest except $n = 11$.

References

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Department of Mathematics
Aichi University of Education
Kariya-shi, Aichi, 448-8542, JAPAN
e-mail: mkanemit@aecc.aichi-edu.ac.jp