

Singular set of stable mappings from 4-manifolds into 3-manifolds

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1. Introduction

The present note contains the investigation of the topology of a singular set of stable mappings. It is a continuation of the author's note [3], where analogous theorems on the topology of singular sets of certain mapping are formulated. Our main concern is the stable mapping from 4-manifolds into 3-dim euclidean space, $f: M \rightarrow \mathbb{R}^3$. Since f is stable, possible singularity types are the followings ,

- 1) $(u, x, y, z) \rightarrow (u, x, y^2 \pm z^2)$, fold point (A_1 -type)
- 2) $(u, x, y, z) \rightarrow (u, x, y^3 + uy \pm z^2)$, cusp point (A_2 -type)
- 3) $(u, x, y, z) \rightarrow (u, x, y^4 + uy^2 + xy \pm z^2)$, swallow tail (A_3 -type)

In this case it is easy to see that the singular set, $S(f)$, of the mapping f is 2-dim submanifolds of M^4 . Then it arises a question to what extent the structure of $S(f)$ is determined by the mapping and topology of M^4 . We are concerned with the location of $S(f)$ in M^4 .

2. Congruence formula

To extract embedding phenomena of $S(f)$ in M^4 , we will study the structure of the normal bundle of $S(f)$ in M . To be precise, we will formulate the relation between self-intersection number of $S(f)$ and signature of M . We set several assumptions for simplicity from now on: M is oriented and first integral homology group of M vanishes. This means that torsion of second (co)homology is free, for $H^2(M; \mathbb{Z}) = \text{Hom}(H_2(M; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_1(M; \mathbb{Z}), \mathbb{Z})$, which follows from universal coefficient theorem.

For a stable mapping $f: M^4 \rightarrow \mathbb{R}^3$, we can prove the following formula,

$$\sigma(M) \equiv -S(f) \cdot S(f) \pmod{4} \quad (1)$$

Here $\sigma(M)$ is defined as the signature of the cup product bilinear

form on the 2-nd cohomology group and the dot stands for the self intersection number of $S(f)$ in M^4 . Before proving formula (1), we need a key result. For our mapping f it holds,

$$\chi(M) \equiv \chi(S(f)) \pmod{2} \quad (2)$$

where χ denotes the Euler characteristic. The requisite technique of the proof is to consider the orthogonal projection from $\mathbb{R}^3$ to an one dimensional linear subspace L of $\mathbb{R}^3$ (See [1]). We can regard L as a line through an origin of $\mathbb{R}^3$ i.e. an element of $\mathbb{RP}^2$, 2-dim projective space. Let $A_k(f)$ stand for an A_k -type singular set of f ($1 \leq k \leq 3$). It follows from the genericity of the above projection $\pi: \mathbb{R}^3 \rightarrow L$ that for almost every line of $\mathbb{RP}^2$ the composed mapping $\pi \circ f$ and $\pi \circ f|_{A_k(f)}$ ($1 \leq k \leq 3$) are Morse functions. Let $\#C(g)$ be the number of critical points of a Morse function $g: M \rightarrow \mathbb{R}$. Since the closure of $A_2(f)$, $\overline{A_2(f)}$, is a disjoint union of a circle, $\#C(\pi \circ f|_{\overline{A_2(f)}})$ is even. Moreover, it is easy to see

$$\begin{aligned} \#C(\pi \circ f) &\equiv \#C(\pi \circ f|_{A_1(f)}) \pmod{2}, \\ \#C(\pi \circ f|_{S(f)}) &= \#C(\pi \circ f|_{A_1(f)}) + \#C(\pi \circ f|_{\overline{A_2(f)}}) \\ &\equiv \#C(\pi \circ f|_{A_1(f)}) \pmod{2}. \end{aligned}$$

From the Morse theory the Euler characteristic of a compact manifold is mod 2 congruent with the number of critical points of the Morse function over the manifold. Thus formula (2) follows.

Next, the following is well known as the generalized Whitney congruence (See [2]). Let F^2 be a characteristic surface of M^4 . If the first integral homology group of M vanishes, it holds

$$\sigma(M) \equiv F \cdot F + 2\chi(F) \pmod{4}. \quad (3)$$

F is characteristic if the mod 2 cycle $[F]_2$ is dual to the 2-nd Stiefel-Whitney class $w_2(M)$. In our situations the following fact is well known (See [4]). $S(f)$ is a mod 2 cycle and its Poincaré dual coincides with $w_2(M)$. Thus formula (1) automatically follows from (2) and (3). Formula (1) merely suggests certain kind of restriction between the topology of a domain and singular set and does not completely elucidate the location of $S(f)$.

3. Orientability of singular sets

We should need to consider the orientability of $S(f)$. For our

mapping f we have an immediate result concerning the orientability of $S(f)$.

(i) If the signature of M^4 is odd, then $S(f)$ must be unorientable. More precisely, $S(f)$ contains unorientable surfaces with odd genus, which follows from classification of closed surfaces and formula (2).

The case where $\sigma(M)$ is even is more complicated and subtle. We will continue to study the orientability of $S(f)$, assuming $\sigma(M)$ is even. Let φ be a smooth embedding of $S(f)$ into M , $\varphi: S(f) \rightarrow M$. If $\varphi^*: H(M; \mathbb{Z}/2) \rightarrow H(S(f); \mathbb{Z}/2)$ is the homomorphism, induced by φ , we assume φ^* is an injection for a while, then we obtain the following bundle isomorphism

$$\tau(S(f)) \oplus \nu_\varphi \cong \varphi^* \tau(M),$$

where ν_φ denotes the normal bundle of $S(f)$ in M . Then it holds,

$$\begin{aligned} w(\varphi^* \tau(M)) &= (1 + w_1(S(f)) + w_2(S(f))) (1 + w_1(\nu_\varphi) + w_2(\nu_\varphi)) \\ &= 1 + (w_1(S(f)) + w_1(\nu_\varphi)) + (w_2(S(f)) + w_2(\nu_\varphi) + w_1(S(f))w_1(\nu_\varphi)) \\ &\quad + (w_2(S(f))w_1(\nu_\varphi) + w_2(\nu_\varphi)w_1(S(f)) + w_2(S(f))w_2(\nu_\varphi)). \end{aligned} \quad (4)$$

If $S(f)$ is orientable, then $w_1(S(f))$ and $w_2(S(f))$ vanishes. Thus we have,

$$\begin{aligned} \varphi^* w_2(M^4) &= w_2(\varphi^* \tau(M)) \\ &= w_2(S(f)) + w_2(\nu_\varphi) + w_1(S(f))w_1(\nu_\varphi) \\ &= w_2(\nu_\varphi). \end{aligned} \quad (5)$$

On the other hand, from formula (1) the self-intersection number of $S(f)$ is even since $\sigma(M)$ is even. Hence,

$$\langle w_2(\nu_\varphi), [S(f)]_2 \rangle \equiv S(f) \cdot S(f) \equiv 0 \pmod{2}.$$

If the mod 2 cycle $[S(f)]_2$ is a zero element, then $w_2(M)$ vanishes from the fact which we state above. Otherwise, $w_2(\nu_\varphi)$ vanishes. Also in this case by equation (5) $w_2(M)$ vanishes from the injectivity of φ^* . Thus we have proven that M is spin. From the classical Rochlin's theorem, the signature of M must be divisible by 16. Therefore, assume $\sigma(M)$ is even and not divisible by 16, if φ^* is an injection then $S(f)$ is unorientable.

4. Branched covering space

In this section we will study the general case in which the target space is an arbitrary oriented 3-manifold. For any stable

mapping $f: M^4 \rightarrow N^3$, the mod 2 cycle $[S(f)]_2$ is also dual to the 2-nd Stiefel-Whitney class of M since any orientable 3-manifold is parallelizable and vanish the contributions of characteristic classes of $f^* T(N)$ in the Thom polynomial. Accordingly, it is expected that if M is a spin manifold, namely, $w_2(M)=0$, the embedding phenomena of $S(f)$ should be more accurately evaluated. We will explain our trial to calculate the self-intersection number of $S(f)$ in M hereafter. Since M is spin, $[S(f)]_2$ is a zero element of $H_2(M; \mathbb{Z}/2)$. Suppose $S(f)$ is orientable. Non-orientable case will be treated afterwards. Let $\varphi: S(f) \rightarrow M$ be a smooth embedding with $\varphi^*[S(f)] = u \in H^2(M; \mathbb{Z})$. Then we can define the divisibility of $S(f)$,

$$\text{div}(S(f)) = \max\{n \in \mathbb{Z}; u = nv \text{ for some } v \in H_2(M; \mathbb{Z})\}.$$

We have an immediate result from this definition, considering the following homomorphism

$$H_2(M; \mathbb{Z}) \longrightarrow H_2(M; \mathbb{Z}/2) \xrightarrow{\cong} H^2(M; \mathbb{Z}/2),$$

where the first arrow is mod 2Z coefficient homomorphism and second one is Poincaré duality isomorphism. We claim $\text{div}(S(f))$ is even, and we set $\text{div}(S(f)) = 2m$. This follows from the fact that $[S(f)]_2$ is a zero element since M is spin. Then the integral homology class $[S(f)]$ is dual to $2mx$ for some x of $H^2(M; \mathbb{Z})$ by Poincaré duality. Let E be the complex line bundle over M such that $c_1(E) = x$, where $c_1(E)$ denotes the first Chern class of E . Then $c_1(E^{2m}) = c_1(E \otimes \dots \otimes E) = 2mx$, where E^{2m} is the $2m$ -fold tensor product bundle of E . Choose a cross-section $s: M \rightarrow E^{2m}$ of the bundle E^{2m} which is transversal to the zero section M and equal to zero exactly on $S(f)$ embedded in M . Such a cross section always exists by Thom's transversality theorem. We define $p_{2m}: E \rightarrow E^{2m}$ by $p_{2m}(v) = v \otimes \dots \otimes v$. p_{2m} is a branched covering of order $2m$ branched along the zero section, M . Take $\tilde{M} = p_{2m}^{-1} s(M)$. The covering map $\mathbb{P}: \tilde{M} \rightarrow M$ is the composition of $p_{2m}|_M$ with the bundle projection $E^{2m} \rightarrow M$. $\tilde{M}$ is a $2m$ -fold branched covering branched along the zeroes $S(f)$ of s . It is a diffeomorphism on $S(f)$ and an usual covering on the complement of $S(f)$ in $\tilde{M}$ and M .

5. Algebraic invariant and its application

Let H be a finite dimensional real vector space and t a linear

transformation of H such that $t^n = \text{id}$. Let f be a quadratic form over H , invariant under t . Define the polynomials by the formula,

$$P_k(z) = \begin{cases} z-1 & \text{if } k=0 \\ (z-\zeta^k)(z-\zeta^{-k}) & \text{if } 1 \leq k \leq n/2 \\ z+1 & \text{if } k=n/2 \end{cases} \quad (6)$$

, where $\zeta = \exp(2\pi i/n)$.

Then it obviously holds,

$$z^n - 1 = \prod_{k=0}^{[n/2]} P_k(z). \quad (7)$$

This expansion corresponds to a decomposition of H in the direct sum of subspaces $\text{Ker} P_k(z)$ ($0 \leq k \leq [n/2]$). Then we define t -signature of a quadratic form f ,

$$\sigma(f, t) = \sum_{k=0}^{[n/2]} a(k) \cos 2k\pi/n, \quad (8)$$

where $a(k)$ denotes the signature of f over $\text{Ker} P_k(t)$. If $t = \text{id}$, then $\sigma(f, t)$ denotes the ordinary signature $\sigma(f)$ of a quadratic form f . Moreover, we define the numbers $\hat{a}(0), \dots, \hat{a}(n-1)$ by

$$\hat{a}(k) = \begin{cases} a(k) & \text{if } k=0, n/2 \\ a(k)/2 & \text{if } 1 \leq k \leq n/2 \\ a(n-k)/2 & \text{if } n/2 < k \leq n-1 \end{cases} \quad (9)$$

It is easy to see, $\sigma(f, t) = \sum_{k=0}^{n-1} \hat{a}(k) \zeta^k$. (10)

Subspaces $\text{Ker} P_k(t)$ ($0 \leq k \leq [n/2]$) are pairwise orthogonal with respect to a quadratic form f . Thus we have

$$\sigma(f, t^s) = \sum_{k=0}^{n-1} \hat{a}(k) \zeta^{ks} = \sum_{k=0}^{[n/2]} a(k) \cos 2ks\pi/n \quad (s=0, \dots, n-1) \quad (11)$$

and in particular, since $\sigma(f) = \sigma(f, \text{id})$

$$\sigma(f) = \sum_{k=0}^{n-1} \hat{a}(k) = \sum_{k=0}^{[n/2]} a(k). \quad (12)$$

Since

$$\sum_{k=0}^{n-1} \zeta^{ks} \zeta^{-ks_1} = \begin{cases} n & \text{if } s=s_1 \\ 0 & \text{if } s \neq s_1 \end{cases}$$

we have

$$\begin{aligned}\hat{a}(k) &= 1/n \sum_{s=0}^{n-1} \sigma(f, t^s) \zeta^{-ks} \quad (k=0, \dots, n-1) \\ &= 1/n \{ \sigma(f) + \sum_{s=1}^{n-1} \sigma(f, t^s) \zeta^{-ks} \}\end{aligned}\quad (13)$$

and

$$a(0) = 1/n \sum_{s=0}^{n-1} \sigma(f, t^s) = 1/n \{ \sigma(f) + \sum_{s=1}^{n-1} \sigma(f, t^s) \} \quad (14)$$

Combining (13) and (14),

$$\begin{aligned}\hat{a}(k) &= a(0) - 1/n \sum_{s=1}^{n-1} \sigma(f, t^s) (1 - \zeta^{-ks}) \\ &= a(0) - 2/n \sum_{s=1}^{n-1} \sigma(f, t^s) \sin^2 ks\pi/n.\end{aligned}\quad (15)$$

In the previous section we have constructed a $2m$ -fold branched covering $\mathbb{M}: \tilde{M} \rightarrow M$ branched along $S(f)$. Consider the homomorphism $\mathbb{M}^*: H^2(M; \mathbb{R}) \rightarrow H^2(\tilde{M}; \mathbb{R})$, induced by $\mathbb{M}$. $\mathbb{M}^*$ isomorphically maps $H^2(M; \mathbb{R})$ on the set of $H^2(\tilde{M}; \mathbb{R})^{\mathbb{Z}_{2m}}$, which are fixed with respect to $\mathbb{Z}_{2m}$ action. Consequently, $H^2(M; \mathbb{R}) \cong \text{Ker } P_0(t)$. Thus we have $a(0) = \sigma(M)$. In our situation there is an adaption of Atiyah-Singer G -signature theorem, [5].

$$\sigma(f, t^s) = e[F] / n \sin^2 \frac{s}{n} \pi \quad (s=1, \dots, n-1), \quad (16)$$

where F is the fixed point set of the diffeomorphism $h: M \rightarrow M$ of period n and $e[F]$ denotes the self-intersection number of F in M . Accordingly, we have, applying (16) to (15) in our statement

$$\hat{a}(k) = \sigma(M) - 2/n^2 S(f) \cdot S(f) \sum_{s=1}^{n-1} \sin^2 \frac{ks}{n} \pi / \sin^2 \frac{s}{n} \pi \quad (17)$$

We can easily show, considering second difference equation

$$\sum_{s=1}^{n-1} \sin^2 \frac{ks}{n} \pi / \sin^2 \frac{s}{n} \pi = k(n-k) \quad (18)$$

Thus we have

$$\hat{a}(k) = \sigma(M) - 2k(n-k)S(f) \cdot S(f)/n^2. \quad (19)$$

In our case $H = H^2(\tilde{M}; \mathbb{R})$ and $\sigma(f) = \sigma(\tilde{M})$. Hence we obtain from (12) and (19),

$$\begin{aligned}\sigma(\tilde{M}) &= \sum_{k=0}^{n-1} \hat{a}(k) \\ &= n \sigma(M) - 2S(f) \cdot S(f) / n^2 \sum_{k=0}^{n-1} k(n-k)\end{aligned}$$

$$= n \sigma(M) - (n^2 - 1)S(f) \cdot S(f) / 3n \quad (20)$$

This is Hirzebruch's formula, [6].

Since $n=2m$,

$$6m \sigma(\tilde{M}) = 12m^2 \sigma(M) - (4m^2 - 1)S(f) \cdot S(f). \quad (21)$$

M is spin, the signature of M is divisible by 16. If we can prove that $\tilde{M}$ is also spin, i.e. $\mathbb{P}^*(w_2(M)) = w_2(\tilde{M})$, we can calculate the self-intersection number of $S(f)$. Since $\text{div}(S(f))$ is even, we have

$$\begin{aligned} S(f) \cdot S(f) &\equiv 0 \pmod{4} && \text{if } m=1, \\ S(f) \cdot S(f) &\equiv 0 \pmod{16} && \text{if } m=2, \\ S(f) \cdot S(f) &\equiv 0 \pmod{36} && \text{if } m=3 \text{ and so on.} \end{aligned}$$

Applying equation (21) to this result,

$$\begin{aligned} S(f) \cdot S(f) &\equiv 0 \pmod{32} \\ S(f) \cdot S(f) &\equiv 0 \pmod{64} \\ S(f) \cdot S(f) &\equiv 0 \pmod{288} \text{ and so on.} \end{aligned}$$

If $S(f)$ is non-orientable, we can construct a double branched covering branched along $S(f)$. Thus this case is contained in $m=1$.

References

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