

COVERING PROPERTIES CHARACTERIZED BY ORTHOCOMPACTNESS AND SUBNORMALITY OF PRODUCTS

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All spaces are assumed to be T_1 , but compact spaces and paracompact spaces are assumed to be Hausdorff.

A space X is assumed to be Tychonoff when we consider the product $X \times \gamma X$, where γX denotes a compactification of X . An infinite cardinal κ is assumed to be no less than $L(X)$ when we consider the product $X \times 2^\kappa$ or the product $X \times (\kappa + 1)$, where $L(X)$ denotes the Lindelöf number of the space X .

The main purpose of this note is to give some partial answers to Problems A and C stated in Section 1.

1. CHARACTERIZATIONS OF COVERING PROPERTIES BY PRODUCTS

Let us begin with a classical result of Dowker [D].

Theorem 1.1 [D]. *For a normal space X , the following are equivalent.*

- (a) X is countably paracompact.
- (b) $X \times (\omega + 1)$ is normal.
- (c) $X \times [0, 1]$ is normal.

Theorem 1.1 is the first result which indicated an important implication between covering properties and products. Moreover, this led up to a beautiful characterization of paracompactness in terms of products.

Theorem 1.2 [T,M]. *For a Hausdorff space X , the following are equivalent.*

- (a) X is paracompact.
- (b) $X \times \gamma X$ is normal.
- (c) $X \times 2^\kappa$ is normal.
- (d) $X \times (\kappa + 1)$ is normal.

Remark. The equivalence (a) and (d) in Theorem 1.2 was proved by Kunen. It is found in [P, Corollary 3.7].

An open cover $\mathcal{V}$ of a space X is *interior-preserving* if $\bigcap \mathcal{V}'$ is open in X for each $\mathcal{V}' \subset \mathcal{V}$. A space X is *orthocompact* if every open cover of X has an interior-preserving open refinement.

Subsequently, as a nice analogue of Theorem 1.2, a characterization of metacompactness was obtained as follows.

Theorem 1.3 [Ju1,S]. *For a space X , the following are equivalent.*

- (a) X is metacompact.
- (b) $X \times \gamma X$ is orthocompact.
- (c) $X \times 2^\kappa$ is orthocompact.

This means that there are some closed relations between normality and orthocompactness of products (see [S,KY]). Moreover, as an analogue of Theorem 1.3, we proved a characterization of submetacompactness as follows.

Theorem 1.4 [Y1]. *For a space X , the following are equivalent.*

- (a) X is submetacompact.
- (b) $X \times \gamma X$ is suborthocompact.
- (c) $X \times 2^\kappa$ is suborthocompact.

Seeing Theorems 1.2 and 1.3, it is natural to raise the following problem.

Problem A [Y2]. If $X \times (\kappa + 1)$ is orthocompact, is X metacompact ?

Moreover, it is natural to ask whether there is an analogical characterization of subparacompactness in terms of products.

Recall that a space X is *subnormal* [C, Kr] (*normal*) if for any disjoint closed sets A and B in X , there are disjoint G_δ -sets (open sets) G and H such that $A \subset G$ and $B \subset H$. Note that a space X is subnormal (normal) if and only if every binary open cover of X has a countable (finite) closed refinement.

Problem B [Ju3]. If $X \times \gamma X$ is subnormal, is X subparacompact ?

Problem C [Y2]. If $X \times 2^\kappa$ is subnormal, is X subparacompact ?

Remark. As is shown later, it suffices for these three problems to prove that X is submetacompact. In fact, this follows from Lemma 2.9 and Theorem 3.3 (or Corollary 3.5) below.

2. METACOMPACTNESS AND SUBMETACOMPACTNESS OF β -SPACES

In this section, we give an affirmative answer to our Problem A under the assumption of X being a β -space.

A space X is called a β -space if there is a function $g: X \times \omega \rightarrow \text{Top}(X)$, satisfying

- (i) $x \in \bigcap_{n \in \omega} g(x, n)$,
- (ii) if $x \in g(x_n, n)$ for each $n \in \omega$, then $\{x_n\}$ has a cluster point in X .

Since the class of β -spaces contains the classes of Σ -spaces and semi-stratifiable spaces, it is very broad as a class of generalized metric spaces.

A well-ordered sequence $\{y_\alpha: \alpha \in \kappa\}$ of length κ in a space Y is a *free sequence* if $\text{Cl}\{y_\beta: \beta < \alpha\} \cap \text{Cl}\{y_\gamma: \alpha \leq \gamma < \kappa\} = \emptyset$ for each $\alpha \in \kappa$.

Theorem 2.1. *Let X be a β -space and C a compact space with a free sequence of length $\geq L(X)$. Then X is metacompact if and only if $X \times C$ is orthocompact.*

Since $\kappa + 1$ has a free sequence of length κ , Theorem 2.1 yields a partial answer to Problem A.

Corollary 2.2. *A β -space X is metacompact if and only if $X \times (\kappa + 1)$ is orthocompact.*

Moreover, Arhangel'skii's theorem in [A] and Theorem 2.1 yield

Corollary 2.3. *Let X be a β -space and C a compact space with tightness $> L(X)$. Then X is metacompact if and only if $X \times C$ is orthocompact.*

Now, we will give only a course of the proof of Theorem 2.1. On the way, we will obtain a characterization of submetacompactness of β -spaces.

A well-ordered open cover $\{U_\alpha: \alpha \in \kappa\}$ of a space X is *well-monotone* if $\beta < \alpha$ implies $U_\beta \subset U_\alpha$.

Lemma 2.4. *Let X be a space and C a compact space with a free sequence of length $\geq L(X)$. If $X \times C$ is orthocompact, then every well-monotone open cover of X has a closure-preserving closed refinement.*

By this, it seems to be effective to consider well-monotone open covers and their closure-preserving closed refinements. So we think of the following Junnila's theorem.

Theorem 2.5 [Ju1, Ju2]. *The following are equivalent for a space X .*

- (a) X is metacompact (submetacompact).
- (b) Every well-monotone open cover of X has a point-finite open refinement (θ -sequence of open refinements).
- (c) Every interior-preserving directed open cover of X has a (σ) -closure-preserving closed refinement.

Seeing Lemma 2.4 and Theorem 2.5, we raise the following problem.

Problem D. *If every well-monotone open cover of a space X has a σ -closure-preserving closed refinement, when is X submetacompact?*

Lemma 2.6 [Ji]. Let X be a β -space and $\mathcal{U}$ a well-monotone open cover of X . If $\mathcal{H}$ is an open refinement of $\mathcal{U}$, then there is a sequence $\{\mathcal{G}_{\mathcal{H},s}: s \in \omega^{<\omega}\}$ of partial refinements by open sets in X , satisfying

- (1) $\mathcal{G}_{\mathcal{H},s} \subset \mathcal{G}_{\mathcal{H},s'}$ for $s \subset s'$,
- (2) if $x \in X$ with $\text{ord}(x, \mathcal{H}) \leq n$, then $x \in \bigcup \mathcal{G}_{\mathcal{H},s}$ for each $s \in \omega^{n+1}$,
- (3) for each $x \in X$, there is some $\sigma \in \omega^\omega$ such that $\text{ord}(x, \mathcal{G}_{\mathcal{H},(\sigma \upharpoonright n)}) < \omega$ for each $n \in \omega$.

Making use of this, we prove the following lemma. A basic idea for the proof is also due to Jiang [Ji].

Lemma 2.7 (main). Let X be a β -space and $\mathcal{U}$ a well-monotone open cover of X . If $\mathcal{U}$ has a closure-preserving closed refinement, then it has a θ -sequence of open refinements.

By Lemma 2.7, we can easily obtain an answer to our Problem D.

Theorem 2.8. A β -space X is submetacompact if and only if every well-monotone open cover of X has a σ -closure-preserving closed refinement.

Now, let us return the proof of Theorem 2.1.

Let X be a space and $\mathcal{F}$ a collection of subsets of X . A collection $\{G(F): F \in \mathcal{F}\}$ of subsets in X is an *open expansion* (a G_δ -*expansion*) if $G(F)$ is an open set (a G_δ -set) in X such that $F \subset G(F)$ for each $F \in \mathcal{F}$.

A space X is *almost expandable* [SK] if every locally finite collection of closed sets in X has a point-finite open expansion.

A well-ordered sequence $\{y_\alpha: \alpha \in \kappa\}$ of length κ in a space Y is *right separated* if $y_\alpha \notin \text{Cl}\{y_\delta: \delta > \alpha\}$ for each $\alpha \in \kappa$. Note that each free sequence is right separated.

Lemma 2.9. Let X be a space and C a compact space with a right separated sequence of length $\geq L(X)$. If $X \times C$ is orthocompact, then X is almost expandable.

Since submetacompact, almost expandable spaces are metacompact (see [SK]), Theorem 2.1 follows from Lemmas 2.4 and 2.9, and Theorem 2.8. $\square$

As a similar problem to Problem D, we raise

Problem D'. If every well-monotone open cover of an orthocompact space X has a closure-preserving closed refinement, is X metacompact?

If problem D' would be affirmatively solved, it follows from Lemma 2.4 that Problem A would be affirmative.

Concerning Problem D', we get an additional result.

Lemma 2.10 [HV, Theorem 3.1]. *For a (an orthocompact) space X , the following are equivalent.*

- (a) *For every well-monotone open cover $\{U_\alpha: \alpha \in \kappa\}$ of X , there is a well-monotone closed cover $\{F_\alpha: \alpha \in \kappa\}$ of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \kappa$.*
- (b) *Every well-monotone open cover of X has a cushioned (closure-preserving) closed refinement.*
- (c) *Every infinite open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ with $\text{ord}(x, \mathcal{V}) < |\mathcal{U}|$ for each $x \in X$.*

Let $(\lambda + 1)_\lambda$ denote the space $\lambda + 1$ with the topology such that the point λ has a neighborhood base in the usual order topology and that all other points are isolated.

Using Lemma 2.10, we obtain

Theorem 2.11. *For an orthocompact space X , every well-monotone open cover of X has a closure-preserving closed refinement if and only if $X \times (\lambda + 1)_\lambda$ is orthocompact for each $\lambda (\leq L(X))$.*

We close this section with the following two unsolved problems, which seem to be related to Problems D and D'.

Problem E [Ka, Y1]. *If every directed open cover of a (suborthocompact) space X has a σ -cushioned closed refinement, is X submetacompact ?*

Problem E' [Ka, Ju3]. *If every directed open cover of a space X has a cushioned closed refinement, is X metacompact ?*

Problem E' was affirmatively solved under the assumption of X being suborthocompact (see [Y1]).

3. COUNTABLE SUBPARACOMPACTNESS

In this section, we give some partial answers to our Problem C.

A space X is *countably subparacompact* [Kr] if every countable open cover of X has a countable closed refinement. Note that countably subparacompact spaces are, equivalently, countably metacompact and subnormal (see [Kr, Theorem 2.5]).

Recently, a list of analogues of Theorem 1.1 was given in [GT, p.118]. Here we can add another analogue, answering to Problem C in the case of $\kappa = \omega$.

Theorem 3.1. *For a space X , the following are equivalent.*

- (a) *X is countably subparacompact.*
- (b) *$X \times 2^\omega$ is subnormal.*
- (c) *$X \times [0, 1]$ is subnormal.*

Remark 1. The equivalence of (a) and (c) in Theorem 3.1 was stated in [GT, p.127] without proof. However, at the 10th Summer Conference on General Topology and Application (Amsterdam, August 1994), Good and Tree kindly informed the author that this equivalence had *not* been proved yet, because they misunderstood the proof.

Theorem 3.1 immediately yields a generalization of Theorem 1.1.

Corollary 3.2. *For a normal space X , the following are equivalent.*

- (a) X is countably paracompact.
- (b) $X \times (\omega + 1)$ is normal.
- (c) $X \times [0, 1]$ is subnormal.

Remark 2. It should be noticed that Theorem 3.1 and Corollary 3.2 are essentially different from all the analogues in the list of [GT, p.118]. Because we can replace $[0, 1]$ with $\omega + 1$ in all of them, but we cannot do in Theorem 3.1 and Corollary 3.2. In fact, consider a Dowker space Y , whose existence is assured by Rudin [R1]. Since the product of a subnormal space and a countable space is subnormal, $Y \times (\omega + 1)$ is subnormal. On the other hand, Y is normal, but not countably metacompact.

A space X is *collectionwise δ -normal* [Ju3] if every discrete collection of closed sets in X has a disjoint G_δ -expansion.

Theorem 3.3 [R2]. *Let X be a space and C a compact space with weight $\geq L(X)$. If $X \times C$ is subnormal, then X is collectionwise δ -normal.*

A space X is *collectionwise subnormal* [C, Kr] if for each discrete collection $\mathcal{F}$ of closed sets in X , there is a sequence $\{\mathcal{U}_n\}$ of open expansions of $\mathcal{F}$ such that for each $x \in X$, there is some $n \in \omega$ such that at most one member of $\mathcal{U}_n$ contains x . Note “subparacompact $\Rightarrow$ collectionwise subnormal $\Rightarrow$ collectionwise δ -normal”.

Now, we get another partial answer to Problem C.

Theorem 3.4. *If $X \times 2^\kappa$ is subnormal, then X is collectionwise subnormal.*

Since collectionwise δ -normal and submetacompact spaces are subparacompact [Ju3], Theorems 1.4 and 3.3 yields a partial answer to Problems B and C.

Corollary 3.5. *For a space X , the following are equivalent.*

- (a) X is subparacompact.
- (b) $X \times \gamma X$ is subnormal and suborthocompact.
- (c) $X \times 2^\kappa$ is subnormal and suborthocompact.

4. LINDELÖF SPACES

Recall that a space X is ω_1 -compact if every closed discrete subset in X is at most countable. Note that Lindelöf spaces are ω_1 -compact.

Lemma 4.1. *Let C be a countably compact space and X a subspace of C . If the subspace $(X \times C) \cup (C \times X)$ of the square C^2 is subnormal, then X is ω_1 -compact.*

Using this, we can obtain an analogous characterization of Lindelöf spaces to Tamano's theorem for paracompactness (see Theorem 1.2).

Theorem 4.2. *For a Tychonoff space X , the following are equivalent.*

- (a) X is Lindelöf.
- (b) The subspace $(X \times \gamma X) \cup (\gamma X \times X)$ of the square $(\gamma X)^2$ is normal.
- (c) X is submetacompact and the subspace $(X \times \gamma X) \cup (\gamma X \times X)$ of the square $(\gamma X)^2$ is subnormal.

In Theorem 4.2, we can find a kind of similarity to the form of Corollary 3.2.

REFERENCES

- [A] A. V. Arhangel'skiĭ, *On bicompacta hereditarily satisfying Suslin condition. Tightness and free sequence* *Soviet Math. Dokl.* **12** (1971), 1253–1256.
- [C] J. Chaber, *On subparacompactness and related properties*, *General Topology and Appl.* **10** (1979), 13–17.
- [D] C. H. Dowker, *On countably paracompact spaces*, *Canad. J. Math.* **3** (1951), 219–224.
- [GT] C. Good and I. J. Tree, *On δ -normality*, *Topology and Appl.* **56** (1994), 117–127.
- [HV] R.E.Hodel and J.E.Vaughan, *A note on $[a, b]$ -compactness*, *General Topology and Appl.* **4** (1974), 179–189.
- [Ji] S. Jiang, *Every strict p -space is θ -refinable*, *Topology Proc.* **11** (1986), 309–316.
- [Ju1] H. J. K. Junnila, *Metacompactness, paracompactness and interior-preserving open covers*, *Trans. Amer. Math. Soc.* **249** (1979), 373–385.
- [Ju2] ———, *On submetacompactness*, *Topology Proc.* **3** (1978), 375–405.
- [Ju3] ———, *Three covering properties*, *Surveys in General Topology* (G.M.Reed, ed.), Academic Press, New York, 1980, pp. 195–245.
- [Ka] Y. Katuta, *Expandability and its generalizations*, *Fund. Math.* **87** (1975), 231–250.
- [Kr] T. R. Kramer, *A note on countably subparacompact spaces*, *Pacific J. Math.* **46** (1973), 209–213.
- [KY] N. Kemoto and Y. Yajima, *Orthocompactness and normality of products with a cardinal factor*, *Topology and Appl.* **49** (1993), 141–148.
- [M] K. Morita, *A note on paracompactness*, *Proc. Japan Acad.* **37** (1961), 1–3.
- [P] T. C. Przymusiński, *Products of normal spaces*, *Handbook of Set-theoretic Topology* (K. Kunen and J.E. Vaughan, eds), North-Holland, Amsterdam, 1984, pp. 781–826.
- [R1] M. E. Rudin, *A normal space X for which $X \times I$ is not normal*, *Fund. Math.* **73** (1971), 179–186.
- [R2] ———, *The normality of products with one compact factor*, *General Topology and Appl.* **5** (1975), 45–59.
- [S] B.M.Scott, *Toward a product theory of orthocompactness*, *Studies in Topology* (N.M.Stavrakas and K.R.Allen, ed.), Academic Press, New York, 1975, pp. 517–537.
- [SK] J. C. Smith and L. L. Krajewski, *Expandability and collectionwise normality*, *Trans. Amer. Math. Soc.* **160** (1971), 437–451.
- [T] H. Tamano, *On compactifications*, *J. Math. Kyoto Univ.* **1** (1962), 161–193.
- [Y1] Y. Yajima, *A characterizations of submetacompactness in terms of products*, *Proc. Amer. Math. Soc.* **112** (1991), 291–296.
- [Y2] ———, *Characterizations of four covering properties by products*, *Topology and Appl.* **45** (1992), 119–130.