

On the minimal 2-blocking sets in $\text{PG}(5, 2)$ *

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1 Introduction

We denote by $\text{PG}(r, q)$ the projective geometry of dimension r over the field of q elements $\mathbb{F}_q$. A j -space is a projective subspace of dimension j in $\text{PG}(r, q)$. In this paper, Π_k stands for a k -space in $\text{PG}(r, q)$. We set $\Pi_k = \emptyset$ for $k < 0$. The 0-spaces, 1-spaces, 2-spaces, 3-spaces and $(r - 1)$ -spaces are called *points*, *lines*, *planes*, *solids* and *hyperplanes*, respectively. A set of points in $\text{PG}(r, q)$ meeting every $(r - k)$ -space is called a *k-blocking set* or a blocking set with respect to $(r - k)$ -spaces [3]. A 1-blocking set is simply called a *blocking set*. A k -space in $\text{PG}(r, q)$ is the smallest k -blocking set [5] and a k -blocking set containing a k -space in $\text{PG}(r, q)$ is called *trivial*. A k -blocking set $\mathcal{B}$ is *minimal* if $\mathcal{B} \setminus \{P\}$ is no longer a k -blocking set for any point P of $\mathcal{B}$.

For an integer $r \geq 3$ and a prime power $q \geq 3$, a smallest non-trivial 1-blocking set $\mathcal{B}_0$ in a plane δ in $\text{PG}(r, q)$ is also a smallest non-trivial 1-blocking set in $\text{PG}(r, q)$. The speciality for the binary case is that a non-trivial 1-blocking set in $\text{PG}(2, 2)$ does not exist.

Denote by $\text{Cone}(\Pi_k, \mathcal{B})$ (or simply $\Pi_k \mathcal{B}$) a cone with vertex a k -space Π_k and base $\mathcal{B}$ in an s -space Δ skew to Π_k . Note that the cone is just $\mathcal{B}$ if Π_k is empty.

Govaerts and Storme proved the following.

- Theorem 1.1** ([6]). (a) *Any smallest non-trivial 1-blocking set in $\text{PG}(r, 2)$, $r \geq 3$, is an elliptic quadric in a solid in $\text{PG}(r, 2)$.*
- (b) *Every non-trivial minimal 2-blocking set in $\text{PG}(3, 2)$ is the complement of an elliptic quadric.*
- (c) *Any smallest non-trivial k -blocking set in $\text{PG}(r, 2)$, $r \geq 3$, with $2 \leq k \leq r - 1$ is $\text{Cone}(\Pi_{k-3}, \mathcal{T})$ where $\mathcal{T}$ is the set of 10 points consisting of the complement of an elliptic quadric in a solid Δ .*

*This paper is a preliminary version and the final version will be submitted to elsewhere.

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An elliptic quadric in $\text{PG}(3, 2)$ is a set of five points no four of which are coplanar, that is the only non-trivial minimal 1-blocking set in $\text{PG}(3, 2)$ up to projective equivalence. A natural question is to classify all non-trivial minimal k -blocking sets in $\text{PG}(r, 2)$ up to projective equivalence for $1 \leq k \leq r - 1$.

In this paper, the point P in $\text{PG}(r, 2)$ with coordinate vector $(p_0, p_1, \dots, p_r)$ is denoted by $(p_0, p_1, \dots, p_r)$ or simply $p_0 p_1 \dots p_r$, and the number of 1's in $\{p_0, p_1, \dots, p_r\}$ is called the *weight* of P . The hyperplane defined by the equation $a_0 x_0 + a_1 x_1 + \dots + a_r x_r = 0$ is denoted by $[a_0 a_1 \dots a_r]$. For two distinct points $P(p_0, p_1, \dots, p_r)$ and $Q(q_0, q_1, \dots, q_r)$ in $\text{PG}(r, 2)$, we denote the point $(p_0 + q_0, p_1 + q_1, \dots, p_r + q_r)$ by $P + Q$.

Let $\mathbf{e}_i = 0 \dots 010 \dots 0$ be the point of $\text{PG}(r, 2)$ the only i -th entry of which is 1. We denote by $\mathbf{1}$ the point $11 \dots 1$ and let $\mathcal{I}_r := \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{r+1}, \mathbf{1}\}$ in $\text{PG}(r, 2)$ with odd $r \geq 3$. Note that $\mathcal{I}_3 = \{1000, 0100, 0010, 0001, 1111\}$ is an elliptic quadric in $\text{PG}(3, 2)$. It is easy to see that $\mathcal{I}_r$ is a non-trivial 1-blocking set in $\text{PG}(r, 2)$ since r is odd. Since $\mathcal{I}_r$ meets the hyperplane $[\mathbf{e}_j + \mathbf{1}]$ in the point $\mathbf{e}_j$ and meets the hyperplane $[11 \dots 1]$ in the point $\mathbf{1}$, $\mathcal{I}_r$ is minimal. Thus $\mathcal{I}_r$ is a non-trivial minimal 1-blocking set in $\text{PG}(r, 2)$ for odd $r \geq 3$.

Let $P_1, P_2, \dots, P_{r+1}$ be $r + 1$ points of $\text{PG}(r, 2)$ in general position. We call the $(r + 2)$ -set $\{P_1, P_2, \dots, P_{r+1}, \sum_{i=1}^{r+1} P_i\}$ a *skeleton* in $\text{PG}(r, 2)$, which is also called a ‘frame’ [1]. Obviously, a skeleton in $\text{PG}(r, 2)$ is projectively equivalent to $\mathcal{I}_r$. Bono et al.[4] proved the following.

Theorem 1.2. *Let S be a non-trivial minimal 1-blocking set in $\text{PG}(r, 2)$, $r \geq 3$. Then, S is projectively equivalent to $\mathcal{I}_s$ in some s -space of $\text{PG}(r, 2)$ with odd $s \geq 3$.*

Corollary 1.3. *There are exactly $\lfloor (r - 1)/2 \rfloor$ non-trivial minimal 1-blocking sets up to projective equivalence in $\text{PG}(r, 2)$, $r \geq 3$.*

As a consequence of Theorem 1.2, there is only one non-trivial minimal 1-blocking set up to projective equivalence in $\text{PG}(4, 2)$, which is a skeleton in a solid. Bono et al.[4] also classified non-trivial minimal 2-blocking sets in $\text{PG}(4, 2)$ up to projective equivalence.

For t spaces $\chi_1, \dots, \chi_t$, we denote by $\langle \chi_1, \dots, \chi_t \rangle$ the smallest space containing $\chi_1, \dots, \chi_t$. From Theorem 1.1, we get (a) of the following theorem.

Theorem 1.4 ([4]). (a) *Let S_{10} be the set of 10 points in a solid Δ in $\text{PG}(4, 2)$ which is the complement of a skeleton in Δ . Then, S_{10} is the smallest non-trivial 2-blocking set in $\text{PG}(4, 2)$.*

(b) *Let $S_{11} = \text{Cone}(P, K)$ with a point P and a skeleton K in a solid Δ not containing P . Then, S_{11} is a non-trivial minimal 2-blocking set with size 11 in $\text{PG}(4, 2)$.*

(c) *Take two planes δ_1, δ_2 meeting in a point P in $\text{PG}(4, 2)$ and a point $Q_i \in \delta_i \setminus \{P\}$ for $i = 1, 2$. Let $S_{12} = (\delta_1 \setminus \{Q_1\}) \cup (\delta_2 \setminus \{Q_2\}) \cup \{Q_1 + Q_2\}$. Then, S_{12} is a non-trivial minimal 2-blocking set with size 12 in $\text{PG}(4, 2)$.*

- (d) Take three points Q_1, Q_2, Q_3 not on a line and a line l which is skew to the plane $\langle Q_1, Q_2, Q_3 \rangle$ in $PG(4, 2)$. Let $\delta_i = \langle Q_i, l \rangle$ for $i = 1, 2, 3$ and let $P = Q_1 + Q_2 + Q_3$. Then, $S_{13} = \{P\} \cup \bigcup_{i=1}^3 (\delta_i \setminus \{Q_i\})$ is a non-trivial minimal 2-blocking set with size 13 in $PG(4, 2)$.
- (e) Take a skeleton $\{Q_1, Q_2, Q_3, Q_4, P = \sum_{i=1}^4 Q_i\}$ in a solid Δ and a point R_1 out of Δ . Let $l_1, \dots, l_4$ be the lines defined by $l_1 = \{P, R_1, R'_1 = P + R_1\}$ and

$$l_j = \{P, R_j = R_{j-1} + Q_{j-1}, R'_j = R'_{j-1} + Q_{j-1}\}, \quad j = 2, 3, 4.$$

Then, $S'_{13} = \bigcup_{i=1}^4 (l_i \cup \{P + Q_i\})$ is a non-trivial minimal 2-blocking set with size 13 in $PG(4, 2)$.

- (f) A parabolic quadric $\mathcal{P}_4$ is a non-trivial minimal 2-blocking set with size 15 in $PG(4, 2)$.

Theorem 1.5 ([4]). *There are exactly six non-trivial minimal 2-blocking sets in $PG(4, 2)$ up to projective equivalence, which are described in Theorem 1.4.*

As a continuation of [4], we tried to find all non-trivial minimal 2-blocking sets in $PG(5, 2)$, and we obtained the following.

Theorem 1.6. *There are at least 84 non-trivial minimal 2-blocking sets in $PG(5, 2)$ up to projective equivalence, whose sizes n satisfy $13 \leq n \leq 20$.*

2 How to search 2-blocking sets in $PG(5, 2)$

Let B be a non-trivial minimal 2-blocking set in $\Sigma := PG(5, 2)$ with size n . For any point P of B , there exists a solid π such that $\pi \cap B = \{P\}$ since B is minimal. such a solid π is called a *tangent* of B at P . Let M_B be a $6 \times n$ matrix whose columns consist of the n points of B . We denote by $\text{rank } B$ the rank of the matrix M_B . We may assume that $\text{rank } B = 6$ because of the following.

Lemma 2.1. *Let S be a non-trivial minimal k -blocking set in a t -space of $PG(r, q)$ with $k < t < r$. Then, S is also a non-trivial k -blocking set in $PG(r, q)$.*

Imamura [8] found that there is no non-trivial minimal 2-blocking set in $\Sigma = PG(5, 2)$ for $n \leq 12$ by an exhaustive computer search. It can be proved that $n \leq 21$, see Miura [9]. Hence, we may assume

$$13 \leq n \leq 21. \quad (2.1)$$

A hyperplane meeting B in exactly t points is called a *t-hyperplane*. A *t*-solid, a *t*-plane and so on are defined similarly. For a space Π in Σ , let

$$t_i(\Pi) = \max\{|S \cap B| \mid S \text{ is an } i\text{-space in } \Pi\}.$$

We simply denote by t_i for $t_i(\Sigma)$. Let Δ be a t_3 -solid in Σ . Then, we have $t_3 \leq 12$ since B is non-trivial. For any point P of $B \cap \Delta$, a tangent of B at P meets Δ in a

1-line or a 1-plane. Hence, each point of $B \cap \Delta$ is on a 1-line, giving $t_3 \leq 10$. Suppose $t = 10$. If the complement of B in Δ contains a line, then one can take a point of B which is not on a 1-line in Δ , a contradiction. Otherwise, B is the complement of a skeleton in Δ , and $\text{rank } B = 4$, a contradiction again. Thus, $t_3 \leq 9$. The following can be proved from the known results of binary linear codes [7].

Lemma 2.2. (a) $t_3 \geq 6$.

(b) $t_4 \geq 10$ if $n = 14$ or 15 .

(c) $7 \leq t_3 \leq 9$ if $n \geq 16$.

We give the geometric description of the possible solid Δ with $t_3 = |\Delta \cap B|$.

Lemma 2.3. If Δ is a 9-solid, then Δ satisfies one of the following conditions:

(a) $\Delta \setminus B = \ell_1 \cup \ell_2$, where ℓ_1 and ℓ_2 are skew lines;

(b) $\Delta \cap B = (\delta_1 \cup \delta_2) \cap B$, where δ_1 and δ_2 are 6-planes through a 3-line.

Lemma 2.4. If Δ is a 8-solid, then Δ satisfies one of the following conditions:

(a) $\Delta \setminus B = \ell_1 \cup \ell_2 \cup \{P\}$, where ℓ_1 and ℓ_2 are skew lines and $P = P_1 + P_2$ for some $P_1 \in \ell_1$ and $P_2 \in \ell_2$;

(b) $\Delta \cap B = (\delta_1 \cup \delta_2) \cap B$, where δ_1, δ_2 are a 6-plane and a 5-plane through a 3-line.

Lemma 2.5. If Δ is a 7-solid, then Δ satisfies one of the following conditions:

(a) $\Delta \cap B = (\delta_6 \cap B) \cup \{P\}$, where δ_6 is a 6-plane not containing a point P ;

(b) $\Delta \setminus B = \delta_0 \cup \{P\}$, where δ_0 is a 0-plane not containing a point P ;

(c) $\Delta \cap B = (\delta_5 \cap B) \cup \{P_1, P_2\}$, where δ_5 is a 5-plane meeting the line $\langle P_1, P_2 \rangle$ in a 0-point;

(d) $\Delta \cap B$ consists of three non-coplanar lines through a fixed point;

(e) $\Delta \cap B = \ell_1 \cup \ell_2 \cup \{P\}$, where ℓ_1 and ℓ_2 are skew lines and $P = P_1 + P_2$ for some $P_1 \in \ell_1$ and $P_2 \in \ell_2$.

Lemma 2.6. If Δ is a 6-solid, then Δ satisfies one of the following conditions:

(a) $\Delta \cap B = (\delta_5 \cap B) \cup \{P\}$, where δ_5 is a 5-plane not containing a point P ;

(b) $\Delta \setminus B = \delta_0 \cup \{P_1, P_2\}$, where δ_0 is a 0-plane not containing the two points P_1, P_2 ;

(c) $\Delta \cap B = K \cup \{P\}$, where K is a skeleton and P is a point of $\Delta \setminus K$;

(d) $\Delta \cap B = \ell_1 \cup \ell_2$, where ℓ_1 and ℓ_2 are skew lines.

Let H_1, H_2, H_3 be the hyperplanes through Δ . Without loss of generality, we may assume that $H_1 = [000010]$, $H_2 = [000001]$, $H_3 = [000011]$ and that

$$n_1 \geq n_2 \geq n_3,$$

where $n_i = |(H_i \setminus \Delta) \cap B|$ for $i = 1, 2, 3$. So, $n = t_3 + n_1 + n_2 + n_3$.

For example, assume $n = 13$ with $t_3 = 6$. Then, the possible (n_1, n_2, n_3) 's are

$$(3, 3, 1), (3, 2, 2), (4, 3, 0), (4, 2, 1), (5, 2, 0), (5, 1, 1), (6, 1, 0).$$

Since there are six possible 6-solids by Lemma 2.6, we checked 28 cases by a computer, giving no result. Next, assume $n = 13$ with $t_3 = 7$. Then, the possible (n_1, n_2, n_3) 's are

$$(2, 2, 2), (3, 3, 0), (3, 2, 1), (4, 1, 1), (4, 2, 0),$$

giving three non-trivial minimal 2-blocking sets. In this way, we found 84 non-trivial minimal 2-blocking sets in $\text{PG}(5, 2)$ up to projective equivalence by an exhaustive computer search as Table 1.

Table 1: Possible $n = |B|$, (t_4, t_3, t_2) and the number of possible B

n	(t_4, t_3, t_2)	#	n	(t_4, t_3, t_2)	#	n	(t_4, t_3, t_2)	#
13	(9,7,6)	3	16	(10,7,5)	1	18	(12,8,5)	6
14	(10,8,6)	2		(11,8,5)	1		(12,8,6)	2
15	(10,7,5)	1		(12,8,5)	1		(12,9,6)	1
	(11,7,5)	1		(12,9,6)	30		(14,9,5)	2
	(10,8,6)	1	17	(12,8,5)	3		(14,9,6)	1
	(11,8,6)	6		(13,8,5)	2	19	(13,9,5)	3
	(11,9,6)	12		(13,9,5)	4	20	(12,8,5)	1

3 A generalization

From Table 1, there are three non-trivial minimal 2-blocking sets of size 13 in $\text{PG}(5, 2)$ up to projective equivalence. We give how to construct them in a geometric way.

We construct a minimal k -block B in $\Sigma = \text{PG}(r, 2)$ from two k -blocks B_1 and B_2 . For a point P_1 of B_1 and a set $T \subset B_2$, we denote by $(B_1; P_1) + (B_2; T)$ the set

$$(B_1 \setminus \{P_1\}) \cup (B_2 \setminus T) \cup \{P_1 + R \mid R \in T\}.$$

Lemma 3.1. *Let B_1, B_2 be k -spaces in $\Sigma = \text{PG}(r, 2)$ with $B_1 \cap B_2 = \emptyset$, $r = 2k + 1$. Let P_1 be a point of B_1 and T be a subset of B_2 with $T \neq B_2$. Then, $B = (B_1; P_1) + (B_2; T)$ is a non-trivial minimal k -block in Σ .*

Example 3.1. Let B_1, B_2 be skew planes $\Sigma = \text{PG}(5, 2)$. Take a point $P_1 \in B_1$ and a set T in B_2 . Taking T as a point, a line and six points of B_2 , we get three different non-trivial minimal 2-blocking sets of size 13 in $\text{PG}(5, 2)$.

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