

ON A NEW CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$.

We consider some subclasses of the class A . Let S denote the subclass of A whose functions are univalent in U . A function $f(z)$ belonging to the class A is said to be starlike of order α ($0 \leq \alpha < 1$) if it satisfies the inequality

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > \alpha \quad (z \in U)$$

for $0 \leq \alpha < 1$. We denote by $S^*(\alpha)$ the subclass of A , consisting of all starlike functions of order α in U . On the other hand, a function belonging to the class A is said to be convex of order α ($0 \leq \alpha < 1$) if it satisfies the inequality

$$\operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha \quad (z \in U)$$

for $0 \leq \alpha < 1$. We denote by $K(\alpha)$ the subclass of A consisting of such functions. It is well known that $K(\alpha) \subset S^*(\alpha) \subset S$. These classes were introduced by Robertson [13] in 1936, and studied subsequently by Schild [15], MacGregor [5], Pinchuk [12] and Jack [3].

Let T denote the subclass of A of the form

$$(1.1) \quad f(z) = z - \sum_{k=2}^{\infty} a_k z^k$$

where a_k are non-negative real numbers for all k . In 1975, Silverman [18] introduced the classes $T^*(\alpha) = T \cap S^*(\alpha)$ and $C(\alpha) = T \cap K(\alpha)$ for some $0 \leq \alpha < 1$, and proved the following lemmas.

Lemma A. A function $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ is in $T^*(\alpha)$

if and only if $\sum_{k=2}^{\infty} (k - \alpha) a_k \leq 1 - \alpha$.

Lemma B. A function $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ is in $C(\alpha)$

if and only if $\sum_{k=2}^{\infty} k (k - \alpha) a_k \leq 1 - \alpha$.

Several other subclasses of T were studied by Sarangi and Uralegaddi [14], Owa [6,7,8,9,10,11], Gupta and Jain [1,2] and Jain and Ahuja [4].

In 1986, Sekine and Owa [17] introduced new subclasses $T^*(\alpha, p_k)$ and $C(\alpha, p_k)$ of $T^*(\alpha)$ and $C(\alpha)$, respectively. They defined the subclass of $T^*(\alpha)$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^n \frac{1-\alpha}{k-\alpha} p_k z^k - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0),$$

where $0 \leq p_k \leq 1$ and $0 \leq \sum_{k=2}^n p_k \leq 1$, and denoted it by $T^*(\alpha, P_K)$.

They also defined the subclass of $C(\alpha)$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^n \frac{p_k(1-\alpha)}{k(k-\alpha)} z^k - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0),$$

where $0 \leq p_k \leq 1$ and $0 \leq \sum_{k=2}^n p_k \leq 1$, and denoted it by $C(\alpha, P_K)$.

In 1981, the classes $T^*(\alpha, p_2)$ and $C(\alpha, p_2)$ for $k=2$ were introduced by Silverman and Silvia [19].

In 1987, Sekine [16] introduced a new generalized subclass of T as follows. Let $\{B_k\}$ denote a sequence of positive real numbers, i.e.

$$(1.2) \quad B_k > 0 \quad (k = 2, 3, \dots).$$

Let $T(\{B_k\})$ denote the subclass of T satisfying the coefficient relation

$$(1.3) \quad \sum_{k=2}^{\infty} B_k a_k \leq 1.$$

All functions belonging to the class $T(\{B_k\})$ satisfy the coefficient relation

$$(1.4) \quad 0 \leq a_k \leq \frac{1}{B_k} \quad (k \geq 2).$$

The classes $T^*(\alpha)$ and $C(\alpha)$ become to special cases of Sekine's new class. Sekine [16] showed many relations among the new class and

various subclasses of T .

I'd like to introduce a new subclass of $T(\{B_k\})$ by using the inequality (1.4). For a finite sequence $\{p_k\}_{k=2}^n$ of real numbers satisfying the condition

$$(1.5) \quad 0 \leq p_k \leq 1 \quad (k = 2, 3, \dots, n), \quad 0 \leq \sum_{k=2}^n p_k \leq 1,$$

we define by $T(\{B_k\}, \{p_k\}_2^n)$ the subclass of $T(\{B_k\})$ consisting of functions $f(z)$ of the form :

$$f(z) = z - \sum_{k=2}^n \frac{p_k}{B_k} z^k - \sum_{k=n+1}^{\infty} a_k z^k.$$

2. Fundamental results

THEOREM 1. *Let a function f be in the class $T(\{B_k\})$. Then $f \in T(\{B_k\}, \{p_k\}_2^n)$ if and only if*

$$(2.1) \quad \sum_{k=n+1}^{\infty} B_k a_k \leq 1 - \sum_{k=2}^n p_k.$$

The result (2.1) is sharp.

Proof. Since $f \in T(\{B_k\})$, the function f has the form (1.1) and the relation (1.3) holds for a_k and B_k . We put

$$a_k = \frac{p_k}{B_k} \quad (k = 2, \dots, n), \text{ then}$$

$$f \in T(\{B_k\}, \{p_k\}_2^n) \quad \text{if and only if} \quad \sum_{k=2}^n B_k \times \frac{p_k}{B_k} + \sum_{k=n+1}^{\infty} B_k a_k \leq 1$$

This shows the result (2.1). The function $f(z)$ of the form

$$f(z) = z - \sum_{k=2}^n \frac{p_k}{B_k} z^k - \frac{1 - \sum_{k=2}^n p_k}{B_{N+1}} z^N$$

for $N \geq n + 1$ shows that the result (2.1) is sharp.

The following corollary is a kind of coefficient estimates for f .

COROLLARY 1. *Let a function f be in the class $T(\{B_k\}, \{p_k\}_2^n)$. Then,*

$$(2.2) \quad 0 \leq a_k \leq \frac{1 - \sum_{j=2}^n p_j}{B_k} \quad (k \geq n + 1).$$

The result (2.2) is sharp.

The following theorem shows an inclusion relation.

THEOREM 2. *Let sequences $\{B_k\}_2^\infty$ and $\{p_k\}_2^n$ satisfy (1.2) and (1.5), respectively. Then we have*

$$T(\{B_k\}, \{p_k\}_2^n) \subset T(\{B_k d_k\}, \{p_k d_k\}_2^m)$$

for positive integers m and n and a sequence $\{d_k\}_2^n$ such that $2 \leq m \leq n$ and $0 < d_k \leq 1$.

We can obtain the proof of Theorem 2 by using the following two lemmas.

LEMMA 1. *Under the same hypotheses as in Theorem 2, we have*

$$T(\{B_k\}, \{p_k\}_2^n) \subset T(\{B_k\}, \{p_k\}_2^m)$$

for positive integers m and n such that $2 \leq m \leq n$.

LEMMA 2. *Under the same hypotheses as in Theorem 2, we have*

$$T(\{B_k\}, \{p_k\}_2^n) \subset T(\{B_k d_k\}, \{p_k d_k\}_2^n)$$

for a sequence $\{d_k\}_2^n$ such that $0 < d_k \leq 1$.

Proof. Let f denote an element of $T(\{B_k\}, \{p_k\}_2^n)$. Then we obtain the form

$$f(z) = z - \sum_{k=2}^n \frac{p_k d_k}{B_k d_k} z^k - \sum_{k=n+1}^{\infty} a_k z^k$$

and the relation (2.1). The hypotheses $0 < d_k \leq 1$, $B_k > 0$ and $a_k \geq 0$ show

$$(2.3) \quad 0 \leq p_k d_k \leq 1 \quad (k = 2, 3, \dots, n), \quad 0 \leq \sum_{k=2}^n p_k d_k \leq 1$$

and

$$0 < \sum_{k=n+1}^{\infty} B_k d_k a_k \leq \sum_{k=n+1}^{\infty} B_k a_k \leq 1 - \sum_{k=2}^n p_k \leq 1 - \sum_{k=2}^n p_k d_k,$$

which prove $f \in T(\{B_k d_k\}, \{p_k d_k\}_2^n)$.

Theorem 2' is an analogous result as Theorem 2.

THEOREM 2'. Let sequences $\{B_k\}_2^{\infty}$, $\{p_k\}_2^n$ and $\{d_k\}_2^n$ satisfy (1.2), (2.3) and $d_k \geq 1$. Then we have

$$T(\{B_k d_k\}, \{p_k d_k\}_2^n) \subset T(\{B_k\}, \{p_k\}_2^m)$$

for positive integers m and n such that $2 \leq m \leq n$.

3. Convexity of the class $T(\{B_k\}, \{p_k\}_2^n)$

THEOREM 3. Let a sequence $\{n_j\}_1^m$ consist of integers larger

than 1 and n denote the minimum of the numbers $n_1, \dots, n_m$. Let each function

$$(3.1) \quad f_j(z) = z - \sum_{k=2}^{n_j} \frac{p_k^{(j)}}{B_k} z^k - \sum_{k=n_j+1}^{\infty} a_k^{(j)} z^k \quad (a_k^{(j)} \geq 0)$$

be in each class $T(\{B_k\}, \{p_k^{(j)}\}_2^{n_j})$ for each $j = 1, \dots, m$. Then the function $F(z)$ defined by

$$(3.2) \quad F(z) = \sum_{j=1}^m \lambda_j f_j(z),$$

where $\lambda_j \geq 0$, $\sum_{j=1}^m \lambda_j = 1$, is in the class $T(\{B_k\}, \{\sum_{j=1}^m \lambda_j p_k^{(j)}\}_2^n)$.

Proof. By (3.1) and Theorem 1, we have inequalities

$$(3.3) \quad \sum_{k=n_j+1}^{\infty} B_k a_k^{(j)} \leq 1 - \sum_{k=2}^{n_j} p_k^{(j)} \quad (j = 1, \dots, m)$$

An easy calculation shows from (3.1) and (3.2) that

$$F(z) = z - \sum_{k=2}^n \frac{\sum_{j=1}^m \lambda_j p_k^{(j)}}{B_k} z^k - \sum_{k=n+1}^{\infty} \left(\sum_{j=1}^m \lambda_j a_k^{(j)} \right) z^k,$$

where $a_k^{(j)} = \frac{p_k^{(j)}}{B_k}$ for $k = n+1, n+2, \dots, n_j$.

By (3.3) and the definition of $T(\{B_k\}, \{p_k^{(j)}\}_2^{n_j})$, we observe that

$$0 \leq \sum_{j=1}^m \lambda_j a_k^{(j)}$$

$$0 \leq \sum_{j=1}^m \lambda_j p_k^{(j)} \leq \sum_{j=1}^m \lambda_j = 1,$$

$$0 \leq \sum_{k=2}^n \left(\sum_{j=1}^m \lambda_j p_k^{(j)} \right) \leq \sum_{j=1}^m \left(\lambda_j \sum_{k=2}^{n_j} p_k^{(j)} \right) \leq \sum_{j=1}^m \lambda_j = 1,$$

and

$$\begin{aligned} \sum_{k=n+1}^{\infty} \left(B_k \sum_{j=1}^m \lambda_j a_k^{(j)} \right) &= \sum_{j=1}^m \lambda_j \left(\sum_{k=n+1}^{\infty} B_k a_k^{(j)} \right) \\ &= \sum_{j=1}^m \lambda_j \left(\sum_{k=n+1}^{n_j} p_k^{(j)} + \sum_{k=n_j+1}^{\infty} B_k a_k^{(j)} \right) \\ &\leq \sum_{j=1}^m \lambda_j \left(\sum_{k=n+1}^{n_j} p_k^{(j)} + 1 - \sum_{k=2}^{n_j} p_k^{(j)} \right) \\ &= 1 - \sum_{k=2}^n \left(\sum_{j=1}^m \lambda_j p_k^{(j)} \right), \end{aligned}$$

which prove $F \in T(\{B_k\}, \{\sum_{j=1}^m \lambda_j p_k^{(j)}\}_2^n)$ with the aid of Theorem 1.

Immediately, the following corollaries are obtained by Theorem 3.

COROLLARY 2. Let functions f and g be in the class $T(\{B_k\}, \{p_k\}_2^n)$ and $T(\{B_k\}, \{p_k\}_2^{n'})$, respectively. Then we have

$$\lambda f + \lambda' g \in T(\{B_k\}, \{\lambda p_k + \lambda' p_k\}_2^n)$$

where $0 \leq \lambda \leq 1$, $0 \leq \lambda' \leq 1$, $\lambda + \lambda' = 1$ and $n \leq n'$.

The next corollary shows convexity of the class $T(\{B_k\}, \{p_k\}_2^n)$.

COROLLARY 3. If f and g are functions in the class $T(\{B_k\}, \{p_k\}_2^n)$ and λ is a real number such that $0 \leq \lambda \leq 1$, then the function $\lambda f + (1 - \lambda)g$ is also in the class $T(\{B_k\}, \{p_k\}_2^n)$.

We like to obtain a generalization of Corollary 2.

THEOREM 4. Let f and g be functions in the class

$T(\{B_k'\}, \{p_k'\}_2^{n'})$ and $T(\{B_k'''\}, \{p_k'''\}_2^{n''})$, respectively. Then the function $\lambda' f + \lambda'' g$, where $0 \leq \lambda' \leq 1$, $0 \leq \lambda'' \leq 1$ and $\lambda' + \lambda'' = 1$, is in the class

$$T\left(\left\{\frac{B_k' B_k'''}{B_k}\right\}, \left\{\frac{B_k' p_k' \lambda' + B_k''' p_k''' \lambda''}{B_k}\right\}_2^n\right),$$

where $B_k = \max\{B_k', B_k'''\}$ and $n = \min\{n', n''\}$.

Proof. We may consider the case of $n' = n'' = n$, by virtue of Lemma 1. We can put, with the definitions of f and g and aid of Theorem 1,

$$f(z) = z - \sum_{k=2}^n \frac{p_k'}{B_k'} z^k - \sum_{k=n+1}^{\infty} a_k' z^k$$

and

$$g(z) = z - \sum_{k=2}^n \frac{p_k'''}{B_k'''} z^k - \sum_{k=n+1}^{\infty} a_k''' z^k,$$

where

$$(3.4) \quad \sum_{k=n+1}^{\infty} B_k' a_k' \leq 1 - \sum_{k=2}^n p_k', \quad \sum_{k=n+1}^{\infty} B_k''' a_k''' \leq 1 - \sum_{k=2}^n p_k'''.$$

Then we have

$$\begin{aligned} & \lambda' f(z) + \lambda'' g(z) \\ &= z - \sum_{k=2}^n \left(\frac{p_k'}{B_k'} \lambda' + \frac{p_k'''}{B_k'''} \lambda'' \right) z^k - \sum_{k=n+1}^{\infty} \left(\lambda' a_k' + \lambda'' a_k''' \right) z^k \\ &= z - \sum_{k=2}^n \frac{q_k}{c_k} z^k - \sum_{k=n+1}^{\infty} b_k z^k, \end{aligned}$$

where $c_k = \frac{B_k' B_k'''}{B_k}$, $b_k = \lambda' a_k' + \lambda'' a_k'''$ and $q_k = \frac{B_k' p_k' \lambda' + B_k''' p_k''' \lambda''}{B_k}$.

Since, by (3.4) and a simple calculation,

$$0 \leq q_k \leq p_k \lambda' + p_k \lambda'' \leq \lambda' + \lambda'' = 1,$$

$$0 \leq \sum_{k=2}^n q_k \leq \lambda' \sum_{k=2}^n p_k + \lambda'' \sum_{k=2}^n p_k \leq \lambda' + \lambda'' = 1$$

and

$$\begin{aligned} \sum_{k=n+1}^{\infty} c_k b_k &\leq \sum_{k=n+1}^{\infty} \left(B_k \lambda' a_k + B_k \lambda'' a_k \right) \\ &\leq \lambda' \left(1 - \sum_{k=2}^n p_k \right) + \lambda'' \left(1 - \sum_{k=2}^n p_k \right) \\ &= 1 - \sum_{k=2}^n \left(\lambda' p_k + \lambda'' p_k \right) \leq 1 - \sum_{k=2}^n q_k, \end{aligned}$$

we obtain that

$$\lambda' f + \lambda'' g \in T(\{c_k\}, \{q_k\}_2^n)$$

with virtue of Theorem 1.

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