

Berger-Shaw's theorem for p -hyponormal operators

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For a n -multicyclic p -hyponormal operator T , we shall show that $|T|^{2p} - |T^*|^{2p}$ belongs to the Schatten $\frac{1}{p}$ -class $\mathcal{C}_{\frac{1}{p}}$ and that $\text{tr}(|T|^{2p} - |T^*|^{2p})^{\frac{1}{p}} \leq \frac{n}{\pi} \text{Area}(\sigma(T))$.

1. Introduction For a bounded linear operator T on Hilbert space $\mathcal{H}$, $\mathcal{R}(\sigma(T))$ denotes the set of all rational functions analytic on $\sigma(T)$, where $\sigma(T)$ is the spectrum of T . The operator T is said to be n -multicyclic if there are n vectors $x_1, \dots, x_n \in \mathcal{H}$, called generating vectors, such that $\vee\{g(T)x_i; i = 1, \dots, n, g \in \mathcal{R}(\sigma(T))\} = \mathcal{H}$. For a p such as $0 < p \leq 1$, T is said to be p -hyponormal, if $(T^*T)^p \geq (TT^*)^p$. In particular, 1-hyponormal is called hyponormal and $\frac{1}{2}$ -hyponormal is called semihyponormal. Xia([7]) gave an example which is not hyponormal but semihyponormal. Thus, the class of p -hyponormal operators properly contains 1-hyponormal operators. Putnam([6]) obtained the norm estimation for the self-commutator of a hyponormal operator, so called Putnam's inequality. This inequality is extended for a p -hyponormal operator by Xia([8]) and Cho-Itoh([3]). Berger-Shaw([2]) showed the trace norm estimation for the self-commutator of n -multicyclic hyponormal operator, so called Berger-Shaw's inequality. In this paper we shall extend this inequality to the case of a n -multicyclic p -hyponormal operator.

2. Preliminary lemmas

For p -hyponormal operator T with its polar decomposition $T = U|T|$, the operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is said to be the Aluthge transform.

It is known, by Aluthge([1]), that $\widetilde{T}$ is hyponormal if $\frac{1}{2} \leq p \leq 1$ and $(p + \frac{1}{2})$ -hyponormal if $0 < p \leq \frac{1}{2}$ and if U is unitary. In this paper, we deal with the operator $\widehat{T} = |T|^{\frac{1}{2m}-p} U |T|^{1-\frac{1}{2m}+p}$ for $\frac{1}{2m+1} \leq p \leq \frac{1}{2m}$, where m is non-negative integer.

Lemma 1. *If T is p -hyponormal operator for a p such as $\frac{1}{2m+1} \leq p \leq \frac{1}{2m}$, then*

$$\begin{aligned} (\widehat{T}\widehat{T}^*)^{\frac{1}{2m}} &\leq |T|^{\frac{1}{2m}-p} |T^*|^{2p} |T|^{\frac{1}{2m}-p} \\ &\leq |T|^{\frac{1}{2m-1}} \leq (\widehat{T}^*\widehat{T})^{\frac{1}{2m}}, \end{aligned}$$

and hence $\widehat{T}$ is $\frac{1}{2m}$ -hyponormal.

Proof. Since $\frac{1}{2m+1} \leq p \leq \frac{1}{2m}$, $\frac{1}{2m}-p \leq p$ and T is $(\frac{1}{2m}-p)$ -hyponormal by Heinz's inequality and hence

$$\begin{aligned} \widehat{T}^*\widehat{T} &= |T|^{1-\frac{1}{2m}+p} U^* |T|^{2(\frac{1}{2m}-p)} U |T|^{1-\frac{1}{2m}+p} \\ &\geq |T|^{1-\frac{1}{2m}+p} U^* |T^*|^{2(\frac{1}{2m}-p)} U |T|^{1-\frac{1}{2m}+p} \\ &= |T|^{1-\frac{1}{2m}+p} |T|^{2(\frac{1}{2m}-p)} |T|^{1-\frac{1}{2m}+p} = |T|^2. \end{aligned}$$

Thus, by Heinz's inequality, we have the inequality,

$$|\widehat{T}|^s \geq |T|^s \quad \forall s \in (0, 2].$$

Since, by the $(\frac{1}{2m}-p)$ -hyponormality of T ,

$$\begin{aligned} \widehat{T}\widehat{T}^* &= |T|^{\frac{1}{2m}-p} U |T|^{2-\frac{1}{2m-1}+2p} U^* |T|^{\frac{1}{2m}-p} \\ &= |T|^{\frac{1}{2m}-p} |T^*|^{2-\frac{1}{2m-1}+2p} |T|^{\frac{1}{2m}-p} \\ &= |T|^{\frac{1}{2m}-p} |T^*|^{1-\frac{1}{2m-1}+2p} |T^*|^{2(\frac{1}{2m}-p)} |T^*|^{1-\frac{1}{2m-1}+2p} |T|^{\frac{1}{2m}-p} \\ &\leq |T|^{\frac{1}{2m}-p} |T^*|^{1-\frac{1}{2m-1}+2p} |T|^{2(\frac{1}{2m}-p)} |T^*|^{1-\frac{1}{2m-1}+2p} |T|^{\frac{1}{2m}-p} \\ &= (|T|^{\frac{1}{2m}-p} |T^*|^{1-\frac{1}{2m-1}+2p} |T|^{\frac{1}{2m}-p})^2, \end{aligned}$$

We have, by Heinz's inequality and by $(\frac{1}{2^m} - p)$ -hyponormality of T ,

$$\begin{aligned}
 (\widehat{TT^*})^{\frac{1}{2}} &\leq |T|^{\frac{1}{2^m}-p} |T^*|^{1-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p} \\
 &= |T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} |T^*|^{2(\frac{1}{2^m}-p)} |T^*|^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p} \\
 &\leq |T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} |T|^{2(\frac{1}{2^m}-p)} |T^*|^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p} \\
 &= (|T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p})^2,
 \end{aligned}$$

and, by repeating the same arguments as above, we obtain

$$\begin{aligned}
 &(\widehat{TT^*})^{\frac{1}{2^2}} \\
 &\leq |T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p} \\
 &= |T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} |T^*|^{2(\frac{1}{2^m}-p)} |T^*|^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p} \\
 &\leq |T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} |T|^{2(\frac{1}{2^m}-p)} |T^*|^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p} \\
 &= (|T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p})^2,
 \end{aligned}$$

and

$$\begin{aligned}
 &(\widehat{TT^*})^{\frac{1}{2^3}} \\
 &\leq |T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2^2}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p} \\
 &= |T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2^3}-\frac{1}{2^{m-1}}+2p} |T^*|^{2(\frac{1}{2^m}-p)} |T^*|^{\frac{1}{2^3}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p} \\
 &\leq |T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2^3}-\frac{1}{2^{m-1}}+2p} |T|^{2(\frac{1}{2^m}-p)} |T^*|^{\frac{1}{2^3}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p} \\
 &= (|T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2^3}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p})^2.
 \end{aligned}$$

Eventually, we have

$$\begin{aligned}
 &(\widehat{TT^*})^{\frac{1}{2^{m-1}}} \\
 &\leq |T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2^{m-2}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p} \\
 &= |T|^{\frac{1}{2^m}-p} |T^*|^{\frac{1}{2^{m-1}}-\frac{1}{2^{m-1}}+2p} |T^*|^{2(\frac{1}{2^m}-p)} |T^*|^{\frac{1}{2^{m-1}}-\frac{1}{2^{m-1}}+2p} |T|^{\frac{1}{2^m}-p} \\
 &= |T|^{\frac{1}{2^m}-p} |T^*|^{2p} |T^*|^{2(\frac{1}{2^m}-p)} |T^*|^{2p} |T|^{\frac{1}{2^m}-p} \\
 &\leq |T|^{\frac{1}{2^m}-p} |T^*|^{2p} |T|^{2(\frac{1}{2^m}-p)} |T^*|^{2p} |T|^{\frac{1}{2^m}-p} \\
 &= (|T|^{\frac{1}{2^m}-p} |T^*|^{2p} |T|^{\frac{1}{2^m}-p})^2,
 \end{aligned}$$

and hence

$$\begin{aligned} (\widehat{T}\widehat{T}^*)^{\frac{1}{2^m}} &\leq |T|^{\frac{1}{2^m}-p} |T^*|^{2p} |T|^{\frac{1}{2^m}-p} \\ &\leq |T|^{\frac{1}{2^m}-p} |T|^{2p} |T|^{\frac{1}{2^m}-p} \\ &= |T|^{\frac{1}{2^{m-1}}} \leq (\widehat{T}^*\widehat{T})^{\frac{1}{2^m}}. \end{aligned}$$

Therefore $\widehat{T}$ is $\frac{1}{2^m}$ -hyponormal.

Lemma 2. *If T is n -multicyclic p -hyponormal, then $\widehat{T}$ is also n -multicyclic and $\sigma(\widehat{T}) = \sigma(T)$.*

Proof. For a p -hyponormal operator T , $\text{Ker} T$ is a reducing subspace of T and also $\widehat{T}$. Hence, we may assume that $\text{Ker} T = \{0\}$. Put $s = \frac{1}{2^m} - p$, where $\frac{1}{2^{m+1}} \leq p \leq \frac{1}{2^m}$.

$$\sigma(\widehat{T}) = \sigma(|T|^s U |T|^{1-s}) \subset \sigma(U |T|^{1-s} |T|^s) \cup \{0\} = \sigma(T) \cup \{0\}.$$

Similarly

$$\sigma(T) = \sigma(U |T|^{1-s} |T|^s) \subset \sigma(|T|^s U |T|^{1-s}) \cup \{0\} = \sigma(\widehat{T}) \cup \{0\}.$$

Since T is invertible if and only if $\widehat{T}$ is invertible, we have $\sigma(\widehat{T}) = \sigma(T)$ and $\mathcal{R}(\sigma(T)) = \mathcal{R}(\sigma(\widehat{T}))$. Since T is n -multicyclic,

$$\exists x_1, \dots, x_n \in \mathcal{H} \text{ s.t.}$$

$$\vee \{g(T)x_i; i = 1, \dots, n, g \in \mathcal{R}(\sigma(T))\} = \mathcal{H}.$$

Put $y_i = |T|^s x_i$, $i = 1, \dots, n$. We shall show that $\{y_i\}_{i=1}^n$ are n -multicyclic vectors for T .

$$\widehat{T}^k |T|^s = \{|T|^s U |T|^{1-s}\}^k |T|^s = |T|^s \{U |T|\}^k = |T|^s T^k.$$

If $\lambda \in \rho(T)$, then $\lambda - U(|T| + \epsilon)$ is invertible for sufficiently small $\epsilon > 0$. Therefore,

$$\begin{aligned} &(|T| + \epsilon)^s (\lambda - U(|T| + \epsilon))^{-1} \\ &= \{(\lambda - U(|T| + \epsilon))(|T| + \epsilon)^{-s}\}^{-1} \\ &= \{(|T| + \epsilon)^{-s} (\lambda - (|T| + \epsilon)^s U(|T| + \epsilon)^{1-s})\}^{-1} \\ &= (\lambda - (|T| + \epsilon)^s U(|T| + \epsilon)^{1-s})^{-1} (|T| + \epsilon)^s \end{aligned}$$

Letting $\epsilon \downarrow 0$, we have

$$(\lambda - \widehat{T})^{-1} |T|^s = |T|^s (\lambda - T)^{-1}.$$

Hence, we have

$$g(\widehat{T})|T|^s = |T|^s g(T), \quad \forall g \in \mathcal{R}(\sigma(T)),$$

and

$$g(\widehat{T})y_i = |T|^s g(T)x_i, \quad \forall g \in \mathcal{R}(\sigma(T)) \quad i = 1, \dots, n.$$

Thus,

$$\begin{aligned} & \vee \{g(\widehat{T})y_i; i = 1, \dots, n, g \in \mathcal{R}(\sigma(\widehat{T}))\} \\ &= \vee \{g(\widehat{T})y_i; i = 1, \dots, n, g \in \mathcal{R}(\sigma(T))\} \\ &= \vee \{|T|^s g(T)x_i; i = 1, \dots, n, g \in \mathcal{R}(\sigma(T))\} \\ &= [|T|^s \mathcal{H}] = \mathcal{H} \quad \text{because } \text{Ker} T = \{0\}. \end{aligned}$$

This implies that $\widehat{T}$ is n -multicyclic.

3. Main theorem

Berger-Shaw's Theorem. If T is a n -multicyclic hyponormal operator, then $[T^*, T] = T^*T - TT^*$ is in the trace class, and $\text{tr}([T^*, T]) \leq \frac{n}{\pi} \text{Area}(\sigma(T))$, where Area means the planar Lebesgue measure.

The following result is our main theorem.

Theorem. If T is a n -multicyclic p -hyponormal operator for p such as $0 < p \leq 1$, then for p such as $0 < p \leq 1$, then $|T|^{2p} - |T^*|^{2p}$ belongs to the Schatten $\frac{1}{p}$ -class $\mathcal{C}_{\frac{1}{p}}$ and

$$\text{tr}\left(|T|^{2p} - |T^*|^{2p}\right)^{\frac{1}{p}} \leq \frac{n}{\pi} \text{Area}(\sigma(T)).$$

When $p = 1$, this theorem is exactly Berger-Shaw's theorem.

The following is the key for our purpose.

Lemma 3. If T is n -multicyclic p -hyponormal, then

$$\text{tr}\left(|T|^{1-p}(|T|^{2p} - |T^*|^{2p})|T|^{1-p}\right) \leq \frac{n}{\pi} \text{Area}(\sigma(T)).$$

Proof. We shall show this lemma for p such as $\frac{1}{2^{m+1}} \leq p \leq \frac{1}{2^m}$, $m = 0, 1, 2, \dots$, by the induction in m .

If $m = 0$, then $\widehat{T}$ is a n -multicyclic hyponormal operator by Lemmas 1 and 2. Thus Berger-Shaw's theorem implies that

$$\text{tr}(\widehat{T}^*\widehat{T} - \widehat{T}\widehat{T}^*) \leq \frac{n}{\pi} \text{Area}(\sigma(\widehat{T})) = \frac{n}{\pi} \text{Area}(\sigma(T)),$$

because $\sigma(\widehat{T}) = \sigma(T)$ by Lemma 2. Since, by Lemma 1,

$$\begin{aligned} \widehat{T}^*\widehat{T} - \widehat{T}\widehat{T}^* &\geq |T|^2 - |T|^{1-p} |T^*|^{2p} |T|^{1-p} \\ &= |T|^{1-p} (|T|^{2p} - |T^*|^{2p}) |T|^{1-p}, \end{aligned}$$

we have

$$\begin{aligned} \text{tr}(|T|^{1-p} (|T|^{2p} - |T^*|^{2p}) |T|^{1-p}) &\leq \text{tr}(\widehat{T}^*\widehat{T} - \widehat{T}\widehat{T}^*) \\ &\leq \frac{n}{\pi} \text{Area}(\sigma(T)). \end{aligned}$$

Hence, the assertion holds for $m = 0$.

Next, we assume that the assertion holds for $m = k$ ($k \geq 0$). If $m = k + 1$, then $\widehat{T}$ is $\frac{1}{2^{k+1}}$ -hyponormal by Lemma 1. Hence by the assumption and by Lemmas 1 and 2, we have

$$\begin{aligned} \frac{n}{\pi} \text{Area}(\sigma(T)) &= \frac{n}{\pi} \text{Area}(\sigma(\widehat{T})) \\ &\geq \text{tr}(|\widehat{T}|^{1-\frac{1}{2^{k+1}}} (|\widehat{T}|^{2\frac{1}{2^{k+1}}} - |\widehat{T}^*|^{2\frac{1}{2^{k+1}}}) |\widehat{T}|^{1-\frac{1}{2^{k+1}}}) \\ &\geq \text{tr}(|\widehat{T}|^{1-\frac{1}{2^{k+1}}} (|T|^{2\frac{1}{2^{k+1}}} - |T|^{\frac{1}{2^{k+1}}-p} |T^*|^{2p} |T|^{\frac{1}{2^{k+1}}-p}) |\widehat{T}|^{1-\frac{1}{2^{k+1}}}) \\ &= \text{tr}((|T|^{2\frac{1}{2^{k+1}}} - |T|^{\frac{1}{2^{k+1}}-p} |T^*|^{2p} |T|^{\frac{1}{2^{k+1}}-p})^{\frac{1}{2}} |\widehat{T}|^{2(1-\frac{1}{2^{k+1}})}) \\ &\quad \times (|T|^{2\frac{1}{2^{k+1}}} - |T|^{\frac{1}{2^{k+1}}-p} |T^*|^{2p} |T|^{\frac{1}{2^{k+1}}-p})^{\frac{1}{2}}) \\ &\leq \text{tr}((|T|^{2\frac{1}{2^{k+1}}} - |T|^{\frac{1}{2^{k+1}}-p} |T^*|^{2p} |T|^{\frac{1}{2^{k+1}}-p})^{\frac{1}{2}} |T|^{2(1-\frac{1}{2^{k+1}})}) \\ &\quad \times (|T|^{2\frac{1}{2^{k+1}}} - |T|^{\frac{1}{2^{k+1}}-p} |T^*|^{2p} |T|^{\frac{1}{2^{k+1}}-p})^{\frac{1}{2}}) \\ &= \text{tr}(|T|^{1-\frac{1}{2^{k+1}}} (|T|^{2\frac{1}{2^{k+1}}} - |T|^{\frac{1}{2^{k+1}}-p} |T^*|^{2p} |T|^{\frac{1}{2^{k+1}}-p}) |T|^{1-\frac{1}{2^{k+1}}}) \\ &= \text{tr}(|T|^{1-p} (|T|^{2p} - |T^*|^{2p}) |T|^{1-p}). \end{aligned}$$

Hence, the assertion holds for $m = k + 1$. This completes the proof of Lemma 3.

Corollary 1. *If T is an invertible n -multicyclic p -hyponormal operator, then $(T^*T)^p - (TT^*)^p \in \mathcal{C}_1$ and*

$$\mathrm{tr}\left((T^*T)^p - (TT^*)^p\right) \leq \|T^{-1}\|^{2(1-p)} \frac{n}{\pi} \mathrm{Area}(\sigma(T)).$$

Proof. Since T is invertible, $T^*T \geq \|T^{-1}\|^{-2}$, and n -multicyclic p -hyponormality of T implies that

$$\begin{aligned} & \frac{n}{\pi} \mathrm{Area}(\sigma(T)) \\ & \geq \mathrm{tr}\left(|T|^{1-p} \{(T^*T)^p - (TT^*)^p\} |T|^{1-p}\right) \quad \text{by Lemma 3} \\ & = \mathrm{tr}\left(\{(T^*T)^p - (TT^*)^p\}^{\frac{1}{2}} (T^*T)^{1-p} \{(T^*T)^p - (TT^*)^p\}^{\frac{1}{2}}\right) \\ & \geq \|T^{-1}\|^{-2(1-p)} \mathrm{tr}\left((T^*T)^p - (TT^*)^p\right). \end{aligned}$$

We have $(T^*T)^p - (TT^*)^p \in \mathcal{C}_1$ and

$\mathrm{tr}\left((T^*T)^p - (TT^*)^p\right) \leq \|T^{-1}\|^{2(1-p)} \frac{n}{\pi} \mathrm{Area}(\sigma(T))$. This completes the proof of Corollary 1.

Proof of Theorem. By Lemma 3,

$$\mathrm{tr}\left(|T|^{1-p} (|T|^{2p} - |T^*|^{2p}) |T|^{1-p}\right) \leq \frac{n}{\pi} \mathrm{Area}(\sigma(T)).$$

And by the property of trace,

$$\begin{aligned} & \mathrm{tr}\left(|T|^{1-p} (|T|^{2p} - |T^*|^{2p}) |T|^{1-p}\right) \\ & = \mathrm{tr}\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}} |T|^{2-2p} (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}}\right) \\ & = \mathrm{tr}\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}} \{|T|^{2p}\}^{\frac{1-p}{p}} (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}}\right). \end{aligned}$$

If $\frac{1}{2} \leq p \leq 1$, then $0 \leq \frac{1-p}{p} \leq 1$. Thus, by Heinz's inequality, we obtain

$$\begin{aligned} & \mathrm{tr}\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}} \{|T|^{2p}\}^{\frac{1-p}{p}} (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}}\right) \\ & \geq \mathrm{tr}\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}} (|T|^{2p} - |T^*|^{2p})^{\frac{1-p}{p}} (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}}\right) \\ & = \mathrm{tr}\left((|T|^{2p} - |T^*|^{2p})^{1+\frac{1-p}{p}}\right) \\ & = \mathrm{tr}\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{p}}\right). \end{aligned}$$

Therefore, we have

$$\mathrm{tr}\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{p}}\right) \leq \frac{n}{\pi} \mathrm{Area}(\sigma(T)).$$

Thus, the assertion of Theorem holds for $p \in [\frac{1}{2}, 1]$.

If $0 < p \leq \frac{1}{2}$, then by Furuta's inequality([4]),

$$\begin{aligned} & ((|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}} \{|T|^{2p}\}^{\frac{1-p}{p}} (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}})^{2p} \\ & \geq ((|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}} (|T|^{2p} - |T^*|^{2p})^{\frac{1-p}{p}} (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}})^{2p} \\ & = (|T|^{2p} - |T^*|^{2p})^2. \end{aligned}$$

Thus $((|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}} \{|T|^{2p}\}^{\frac{1-p}{p}} (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}})^{2p}$ and $(|T|^{2p} - |T^*|^{2p})^2$ are both compact positive operators. Let $s_n[A]$ be the n -th singular number of a positive compact operator A . Then,

$$\begin{aligned} & s_n\left[\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}} \{|T|^{2p}\}^{\frac{1-p}{p}} (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}}\right)^{2p}\right] \\ & = s_n\left[\left(|T|^{2p} - |T^*|^{2p}\right)^{\frac{1}{2}} \{|T|^{2p}\}^{\frac{1-p}{p}} (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}}\right]^{2p} \\ & \geq s_n\left[\left(|T|^{2p} - |T^*|^{2p}\right)^2\right], \end{aligned}$$

and hence,

$$\begin{aligned} & s_n\left[\left(|T|^{2p} - |T^*|^{2p}\right)^{\frac{1}{2}} \{|T|^{2p}\}^{\frac{1-p}{p}} (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}}\right] \\ & \geq s_n\left[\left(|T|^{2p} - |T^*|^{2p}\right)^2\right]^{\frac{1}{2p}} \\ & = s_n\left[\left(|T|^{2p} - |T^*|^{2p}\right)^{2\frac{1}{2p}}\right] \\ & = s_n\left[\left(|T|^{2p} - |T^*|^{2p}\right)^{\frac{1}{p}}\right]. \end{aligned}$$

Hence,

$$\begin{aligned} & \mathrm{tr}\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}} \{|T|^{2p}\}^{\frac{1-p}{p}} (|T|^{2p} - |T^*|^{2p})^{\frac{1}{2}}\right) \\ & \geq \mathrm{tr}\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{p}}\right). \end{aligned}$$

Therefore, we have

$$\mathrm{tr}\left((|T|^{2p} - |T^*|^{2p})^{\frac{1}{p}}\right) \leq \frac{n}{\pi} \mathrm{Area}(\sigma(T)).$$

The assertion of Theorem also holds for $p \in (0, \frac{1}{2}]$. This completes the proof of Theorem.

For the restriction of a p -hyponormal operator to invariant subspace, we have the following.

Lemma 4. *Let $\mathcal{M}$ be an invariant subspace for a p -hyponormal operator T , and T' be the restriction of T to $\mathcal{M}$. Then*

$$\begin{aligned} \{T'T'^*\}^p &\leq P(TT^*)^p P \\ &\leq P(T^*T)^p P \leq \{T'^*T'\}^p, \end{aligned}$$

and T' is also p -hyponormal, where P denotes the projection onto $\mathcal{M}$.

Proof. Since $T' = TP$,

$$T'^*T' = PT^*TP,$$

and hence, for any $s \in (0, 1]$,

$$\{T'^*T'\}^s = \{PT^*TP\}^s \geq P(T^*T)^s P \text{ by Hansen's inequality ([5])}.$$

While,

$$T'T'^* = TPT^* = PTPT^*P,$$

we have, for any $s \in (0, 1]$,

$$\begin{aligned} \{T'T'^*\}^s &= (TPT^*)^s \\ &= P(TPT^*)^s P \\ &\leq P(TT^*)^s P \text{ by Heinz's inequality.} \end{aligned}$$

Therefore, if T is p -hyponormal for p such as $0 < p \leq 1$, then

$$\begin{aligned} \{T'T'^*\}^p &\leq P(TT^*)^p P \\ &\leq P(T^*T)^p P \\ &\leq \{T'^*T'\}^p. \end{aligned}$$

Thus, T' is also p -hyponormal.

Corollary 2. *If T is p -hyponormal operator, then*

$$\|(T^*T)^p - (TT^*)^p\| \leq \left\{ \frac{1}{\pi} \text{Area}(\sigma(T)) \right\}^p.$$

Proof. Let x be an arbitrary unit vector in $\mathcal{H}$. We define

$$\mathcal{H}_0 = \vee \{g(T)x; g \in \mathcal{R}(\sigma(T))\}.$$

Since $\mathcal{H}_0$ is an invariant subspace for T , Lemma 4 implies that $T'|_{\mathcal{H}_0}$ is a (1-multicyclic) p -hyponormal operator. If $\lambda \in \rho(T)$, then, for any $y \in \mathcal{H}_0$, $(T - \lambda)^{-1}y \in \mathcal{H}_0$. Therefore, $\lambda \in \rho(T')$. Hence, $\sigma(T') \subset \sigma(T)$. By Theorem,

$$\begin{aligned} \text{tr} \left(\{ (T'^*T')^p - (T'T'^*)^p \}^{\frac{1}{p}} \right) &\leq \frac{1}{\pi} \text{Area}(\sigma(T')) \\ &\leq \frac{1}{\pi} \text{Area}(\sigma(T)) \end{aligned}$$

and the maximal eigenvalue of positive trace class operator $\{ (T'^*T')^p - (T'T'^*)^p \}^{\frac{1}{p}}$ is equal to or less than $\frac{1}{\pi} \text{Area}(\sigma(T))$. Thus, the maximal eigenvalue of $(T'^*T')^p - (T'T'^*)^p$ is equal to or less than $\{ \frac{1}{\pi} \text{Area}(\sigma(T)) \}^p$. Therefore,

$$\| (T'^*T')^p - (T'T'^*)^p \| \leq \{ \frac{1}{\pi} \text{Area}(\sigma(T)) \}^p.$$

Let P be the projection onto $\mathcal{H}_0$. Then, by Lemma 4,

$$\begin{aligned} &\{ \frac{1}{\pi} \text{Area}(\sigma(T)) \}^p \\ &\geq \langle \{ (T'^*T')^p - (T'T'^*)^p \} x, x \rangle \\ &\geq \langle \{ P(T^*T)^p P - P(TT^*)^p P \} x, x \rangle \\ &= \langle \{ (T^*T)^p - (TT^*)^p \} x, x \rangle. \end{aligned}$$

Since $x \in \mathcal{H}$ is arbitrary unit vector,

$$\| (T^*T)^p - (TT^*)^p \| \leq \{ \frac{1}{\pi} \text{Area}(\sigma(T)) \}^p.$$

This inequality is an extension of the Putnam inequality to the case of p -hyponormal operator.

Corollary 3. *If T is an invertible p -hyponormal operator, then*

$$\| (T^*T)^p - (TT^*)^p \| \leq \| T^{-1} \|^{2(1-p)} \frac{1}{\pi} \text{Area}(\sigma(T)).$$

Proof. Put $\mathcal{H}_0$, $T' = T|_{\mathcal{H}_0}$ and P as Corollary 2, then T' is an invertible (1-multicyclic) p -hyponormal operator. Therefore by Lemma 3,

$$\begin{aligned}
& \frac{1}{\pi} \text{Area}(\sigma(T)) \geq \frac{1}{\pi} \text{Area}(\sigma(T')) \\
& \geq \text{tr} \left(|T'|^{1-p} \{ (T'^* T')^p - (T' T'^*)^p \} |T'|^{1-p} \right) \\
& = \text{tr} \left(\{ (T'^* T')^p - (T' T'^*)^p \}^{\frac{1}{2}} (T'^* T')^{1-p} \{ (T'^* T')^p - (T' T'^*)^p \}^{\frac{1}{2}} \right) \\
& = \text{tr} \left(\{ (T'^* T')^p - (T' T'^*)^p \}^{\frac{1}{2}} (P T^* T P)^{1-p} \{ (T'^* T')^p - (T' T'^*)^p \}^{\frac{1}{2}} \right) \\
& \geq \text{tr} \left(\{ (T'^* T')^p - (T' T'^*)^p \}^{\frac{1}{2}} P (T^* T)^{1-p} P \{ (T'^* T')^p - (T' T'^*)^p \}^{\frac{1}{2}} \right) \\
& \quad \quad \quad (\text{by Hansen's inequality}) \\
& = \text{tr} \left(\{ (T'^* T')^p - (T' T'^*)^p \}^{\frac{1}{2}} (T^* T)^{1-p} \{ (T'^* T')^p - (T' T'^*)^p \}^{\frac{1}{2}} \right) \\
& \geq \|T^{-1}\|^{-2(1-p)} \text{tr} \left((T'^* T')^p - (T' T'^*)^p \right).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \|T^{-1}\|^{2(1-p)} \frac{1}{\pi} \text{Area}(\sigma(T)) \\
& \geq \text{tr} \left((T'^* T')^p - (T' T'^*)^p \right) \\
& \geq \| (T'^* T')^p - (T' T'^*)^p \| \\
& \geq \langle \{ (T'^* T')^p - (T' T'^*)^p \} x, x \rangle \\
& \geq \langle \{ P (T^* T)^p P - P (T T^*)^p P \} x, x \rangle \quad \text{by Lemma 4} \\
& = \langle \{ (T^* T)^p - (T T^*)^p \} x, x \rangle.
\end{aligned}$$

Since $x \in \mathcal{H}$ is arbitrary unit vector,

$$\| (T^* T)^p - (T T^*)^p \| \leq \|T^{-1}\|^{2(1-p)} \frac{1}{\pi} \text{Area}(\sigma(T)).$$

Remark. Putnam inequality was extended to the p -hyponormal operator by Xia in the case of $\frac{1}{2} \leq p \leq 1$, and by Cho-Itoh in the case of $0 < p \leq \frac{1}{2}$. Their estimation is different from ours.

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