

Critical L^2 -decay for the dissipative nonlinear Schrödinger equation

By

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Abstract

We first consider the Cauchy problem for the nonlinear Schrödinger equation with a power type nonlinearity $\lambda|u|^{p-1}u$. It has been known that $p = 1 + 2/n$ is the critical exponent to exhibit the L^2 -decay of the Cauchy problem under the dissipative condition: $\text{Im } \lambda < 0$. In this paper, we relax an initial condition that corresponding solutions have L^2 -uniformly bound with sub-critical case: $p > 1 + 2/n$. Moreover, we show the L^2 -decay property of dissipative solutions to a problem with an exponential type nonlinearity. Even if this nonlinearity has not only the cubic powered monomial but also the higher order ones, we are able to show the L^2 -decay property of dissipative solutions.

§ 1. Introduction

We first consider the Cauchy problem of the nonlinear Schrödinger equation with a p -th power type nonlinearity

$$(1.1) \quad \begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{p-1}u, & t \in \mathbb{R}, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $p > 1$, $\lambda \in \mathbb{C}$, $u : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{C}$ is the unknown function and $u_0 : \mathbb{R}^n \rightarrow \mathbb{C}$ is initial data. We denote ∂_t and Δ by the partial derivative with respect to t and the n -dimensional Laplacian, respectively. When we consider the Cauchy problem (1.1), it

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is natural to consider in a framework of a energy space $L^2(\mathbb{R}^n)$ or $H^1(\mathbb{R}^n)$, since the problem (1.1) has conservation laws:

$$(1.2) \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad t \in \mathbb{R},$$

$$(1.3) \quad E[u(t)] \equiv \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{\lambda}{p+1} \|u(t)\|_{L^{p+1}}^{p+1} = E[u_0], \quad t \in \mathbb{R}.$$

Here, $L^2(\mathbb{R}^n)$ denotes the Banach space of squared integrable functions with the Lebesgue measure in $x \in \mathbb{R}^n$ and $H^1(\mathbb{R}^n)$ consists of functions $f \in L^2(\mathbb{R}^n)$ such that one spatial derivative of f exists and belongs to $L^2(\mathbb{R}^n)$. For $\lambda \in \mathbb{C} \setminus \{0\}$, the local well-posedness of (1.1) was established by Ginibre-Velo [10], [11], Kato [19], and Yajima [41] (see also [2], [7], [8]): If $1 < p < 1 + 4/(n-2)$, then for any $u_0 \in H^1(\mathbb{R}^n)$, there exists $T = T(\|u_0\|_{H^1}) > 0$ and a unique solution $u \in C([0, T]; H^1(\mathbb{R}^n))$ to (1.1). In the case that $\lambda \in \mathbb{C}$ and $\text{Im } \lambda \leq 0$, Tsutsumi [39] showed that the Cauchy problem (1.1) is locally well-posed in $L^2(\mathbb{R}^n)$ and using the mass conservation law (1.2), the local solution is extended globally in time when $1 < p < 1 + 4/n$. Cazenave-Weissler [8] generalized the result [39] to the Sobolev space $H^s(\mathbb{R}^n)$ for $s \geq 0$. Those results are considered along the theory of evolution equations using the semi-group theory developed by Hille-Yosida and it is common idea to introduce the notion of the mild solutions, i.e., u is called as the mild solutions to (1.1) if

$$u(t) = U(t)u_0 - i\lambda \int_0^t U(t-s)|u(s)|^{p-1}u(s) ds$$

in $C([0, T]; L^2(\mathbb{R}^n))$, where $U(t) = e^{i\frac{t}{2}\Delta}$ is the free Schrödinger evolution operator.

§ 2. L^2 -lower bound of solutions to the dissipative nonlinear Schrödinger equation

In this section, we consider a global dynamics of solutions to (1.1) with the dissipative condition, $\text{Im } \lambda < 0$. Here, we review known facts of the asymptotic behavior of solutions to (1.1) with the non-dissipative case. In this case, the exponent $p = 1 + 2/n$ is known to be the borderline for the scattering theory. Namely, when $p > 1 + 2/n$, the solution to (1.1) behaves as a free solution for the large time, while there is no scattering state for the case $p \leq 1 + 2/n$ (see for details [1], [11], [15], [36], [40]). The limiting case $p = 1 + 2/n$ exhibits that the solution to (1.1) shows so-called *nonlinear long range scattering* discovered by Ozawa [33] (cf. [9], [12], [32]).

For the case $\text{Im } \lambda < 0$, there are several works for the large time behavior of dissipative solutions (see [3], [4], [5] and [14]) and the exponent $p = 1 + 2/n$ plays a critical role not for the scattering but the large time behavior. Cazenave-Naumkin

[6] and Hoshino [17] showed that there exists a scattering state $u_+ \in L^2(\mathbb{R}^n)$ for the dissipative solution $u(t)$ to (1.1) when $p > 1 + 2/n$, i.e.,

$$(2.1) \quad \|u(t) - U(t)u_+\|_2 \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

However the mass (L^2 -norm) of solutions to (1.1) with $\text{Im } \lambda < 0$ monotonically decreases from a nonlinear dissipation, namely the dissipative solution satisfies

$$(2.2) \quad \|u(t)\|_{L^2}^2 + |\text{Im } \lambda| \int_0^t \|u(\tau)\|_{L^{p+1}}^{p+1} d\tau = \|u_0\|_{L^2}^2, \quad t \geq 0.$$

By the dissipative structure (2.2), it is not clear if the scattering state $u_+ \in L^2(\mathbb{R}^n)$ in (2.1) might not be trivial. Then we show that the mass of dissipative solutions have a lower bound as $t \rightarrow \infty$ and it ensures not only the existence of a non-zero scattering state u_+ but also showing that $p = 1 + 2/n$ is a critical exponent to exhibit the large time behavior of dissipative solutions to (1.1) in L^2 -topology.

To state main theorem, we define several functional spaces. Let $\mathcal{S}(\mathbb{R})$ be the set of rapidly decreasing functions. For any $f \in \mathcal{S}(\mathbb{R})$, we define the Fourier transform and the Fourier inverse transform, respectively.

$$(2.3) \quad \begin{aligned} \mathcal{F}[f](\xi) &= \widehat{f}(\xi) \equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \\ \mathcal{F}^{-1}[f](x) &\equiv \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi. \end{aligned}$$

For $s > 0$ and $r > 0$,

$$H_r^s(\mathbb{R}^n) \equiv \left\{ f \in H^s(\mathbb{R}^n); \|f\|_{H_r^s} \equiv \|\langle x \rangle^r \langle \nabla \rangle^s f\|_{L^2} < \infty \right\},$$

where $\langle x \rangle \equiv (1 + |x|^2)^{1/2}$ and $\langle \nabla \rangle^s f \equiv \mathcal{F}^{-1}[\langle \xi \rangle^s \widehat{f}]$ and we denote $H_r^0(\mathbb{R}^n) = L_r^2(\mathbb{R}^n)$. Here, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are defined by (2.3).

We state that any small solutions to (1.1) with the dissipative nonlinearity have the positive L^2 -lower bound when $p > 1 + 2/n$.

Theorem 2.1. *Let $1 \leq n \leq 3$, $1 + 2/n < p < p(n)$, where $p(n) = \infty$ if $n = 1, 2$ and $p(n) = 1 + 4/(n - 2)$ if $n = 3$, and let $n/2 < s < p$, $\text{Im } \lambda < 0$. there exists small $\delta_0 > 0$ such that for any nontrivial $u_0 \in H^s(\mathbb{R}^n) \cap L_s^2(\mathbb{R}^n)$ with $\|u_0\|_{H^s} + \||x|^s u_0\|_{L^2} < \delta_0$, the Cauchy problem (1.1) has a global solution $u \in C([0, \infty); H^s(\mathbb{R}^n))$ such that $J^s(t)u \in C([0, \infty); L^2(\mathbb{R}^n))$ and it satisfies*

$$(2.4) \quad \sup_{t \geq 0} \left(\|u(t)\|_{H^s} + \|J^s(t)u(t)\|_{L^2} \right) \leq C(\|u_0\|_{H^s} + \||x|^s u_0\|_{L^2}).$$

Moreover, the solution $u(t)$ satisfies a following lower bound

$$(2.5) \quad \|u(t)\|_{L^2} \geq C, \quad t \geq 0,$$

where $C > 0$ is independent of t .

In the previous work [35], we needed to control the nonlinear dissipation by the size of initial data to obtain a L^2 -lower bound of solutions to (1.1). Indeed the solution is estimated by

$$(2.6) \quad \|u(t)\|_{L^2}^2 \geq \|u_0\|_{L^2}^2 - C(\|u_0\|_{H^1} + \| |x|^s u_0 \|_{L^2})^{p+1}$$

and we restricted the initial condition such that $u_0 = \delta v_0$ for small $\delta > 0$ to obtain the positivity in the right side of (2.6). However we remove this initial restriction in Theorem 2.1 and we show the solution has the uniform lower bound for general small initial data. Theorem 2.1 gives the lower estimate of solutions to (1.1) when $p > 1 + 2/n$, while lower estimates was shown by [22], [26] and [29] under the critical case $p = 1 + 2/n$.

We define an energy of solutions to (3.2) as follows.

$$(2.7) \quad E[u(t)] \equiv \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u(t, x)|^2 dx + \frac{\operatorname{Re} \lambda}{p+1} \int_{\mathbb{R}^n} |u(t, x)|^{p+1} dx, \quad t \geq 0.$$

Large dissipative solutions to (1.1) also have the positive L^2 -lower bound by adding a special oscillation to the initial datum.

Theorem 2.2 ([35]). *Let $n \geq 1$, $1 + 2/n < p < 1 + 4/(n-2)$, $\operatorname{Re} \lambda > 0$ and $\operatorname{Im} \lambda < 0$. For any nontrivial $v_0 \in H^1(\mathbb{R}^n) \cap L_1^2(\mathbb{R}^n)$ and*

$$(2.8) \quad b_0 > \begin{cases} \frac{4|\operatorname{Re} \lambda|(p+1)E[v_0]}{n|\operatorname{Im} \lambda|(p-1-2/n)\|v_0\|_{L^2}^2}, & 1 + 2/n < p < 1 + 4/n, \\ \frac{2|\operatorname{Re} \lambda|(p+1)E[v_0]}{|\operatorname{Im} \lambda|\|v_0\|_{L^2}^2}, & 1 + 4/n \leq p < 1 + 4/(n-2), \end{cases}$$

the Cauchy problem (1.1) with $u_0 = (\exp i \frac{|x|^2}{4} b_0) v_0$ has a unique global solution $u \in C([0, \infty); H^1(\mathbb{R}^n))$. Moreover, the solution satisfies that for any $t \geq 0$,

$$\|u(t)\|_{L^2} \geq \sqrt{B_0},$$

where $B_0 > 0$ is given by

$$(2.9) \quad B_0 = \begin{cases} \|v_0\|_{L^2}^2 - \frac{4|\operatorname{Re} \lambda|(p+1)E[v_0]}{n|\operatorname{Im} \lambda|(p-1-2/n)b_0}, & 1 + 2/n < p < 1 + 4/n, \\ \|v_0\|_{L^2}^2 - \frac{2|\operatorname{Re} \lambda|(p+1)E[v_0]}{|\operatorname{Im} \lambda|b_0}, & 1 + 4/n \leq p < 1 + 4/(n-2). \end{cases}$$

From Theorem 2.1 and 2.2, the dissipative solution to (1.1) has the L^2 -lower bound for $p > 1 + 2/n$, while the solution to (1.1) with $p \leq 1 + 2/n$ shows the L^2 -decay

$$\|u(t)\|_{L^2} \rightarrow 0, \quad t \rightarrow \infty$$

which is obtained by Kita-Shimomura [24], [25] (cf. Shimomura [38] and Sunagawa [37], see also [23] and [30]). Thus Theorem 2.1 and 2.2 imply that $p = 1 + 2/n$ is a *critical exponent for the L^2 -decay* of the dissipative solution to (1.1).

We prepare several lemmas to prove the main theorems.

Lemma 2.3 ([12], [20], [21], [24], [25]). *Let $n \geq 1$, $s > n/2$, $p > 1 + 2/n$, and $\tilde{\gamma} > 0$ be sufficiently small. Assume that $u \in \mathcal{S}(\mathbb{R}^n)$. Then there exists $C > 0$ such that for any $t \geq 1$,*

$$(2.10) \quad \|f\|_{L^\infty} \leq t^{-\frac{n}{2}} \|\mathcal{F}[U^{-1}f]\|_{L^\infty} + C t^{-\frac{n}{2}-\tilde{\gamma}} \|J^s(t)f\|_{L^2},$$

$$(2.11) \quad \left\| \mathcal{F}[U^{-1}(|u|^{p-1}u)] - t^{-\frac{n(p-1)}{2}} |\mathcal{F}[U^{-1}u]|^{p-1} (\mathcal{F}[U^{-1}u]) \right\|_{L^\infty \cap L^2} \\ \leq C t^{-\frac{n(p-1)}{2}-\tilde{\gamma}} (\|u\|_{L^2} + \|J^s(t)u\|_{L^2})^p,$$

where $J(t) = x + it\nabla$ is a generator of the Galilei transformation and $\|\cdot\|_{L^\infty \cap L^2} \equiv \|\cdot\|_{L^\infty} + \|\cdot\|_{L^2}$.

Lemma 2.4. *Let $n \geq 1$, $s > n/2$. There exists $C > 0$ such that for any $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$(2.12) \quad \|f\|_{L^2}^2 \leq C \|\nabla|^s f\|_{L^2} \| |x|^s f \|_{L^2}.$$

Proof of Lemma 2.4. It easily follows from the Hölder inequality and the fractional Hardy inequality that

$$\|f\|_{L^2} \leq \| |x|^{-s} f \|_{L^2} \| |x|^s f \|_{L^2} \leq C \|\nabla|^s f\|_{L^2} \| |x|^s f \|_{L^2}$$

for some $C > 0$ depending on only s . □

We state the proof of Theorem 2.1.

Proof of Theorem 2.1. The upper bound of small solutions has already seen by the previous works [20], [31] and [34]. Let u be the solution to (1.1) in $C([0, \infty); L^2(\mathbb{R}^n))$ and $U(t) \equiv e^{it\Delta}$. Then, $\mathcal{F}[U^{-1}u](t, \xi)$ satisfies

$$(2.13) \quad i\partial_t \mathcal{F}[U^{-1}u](t, \xi) = \frac{\lambda}{t^{\frac{n}{2}(p-1)}} |\mathcal{F}[U^{-1}u](t, \xi)|^{p-1} \mathcal{F}[U^{-1}u](t, \xi) + R(t, \xi),$$

where $R(t, \xi) = \lambda \mathcal{F}[U^{-1}(|u|^{p-1}u)] - \lambda t^{-\frac{n}{2}(p-1)} |\mathcal{F}[U^{-1}u](t, \xi)|^{p-1} \mathcal{F}[U^{-1}u](t, \xi)$. Multiplying (4.3) by $\overline{\mathcal{F}[U^{-1}u](t, \xi)}$ and taking the imaginary part, we obtain

$$(2.14) \quad \frac{1}{2} \partial_t |\mathcal{F}[U^{-1}u](t, \xi)|^2 = -\frac{|\operatorname{Im} \lambda|}{t^{\frac{n}{2}(p-1)}} |\mathcal{F}[U^{-1}u](t, \xi)|^{p+1} + \operatorname{Im}\{R(t, \xi) \overline{\mathcal{F}[U^{-1}u](t, \xi)}\}.$$

By the decay estimate associated with the free Schrödinger evolution (2.11), $R(t, \xi)$ satisfies for small $\tilde{\gamma} > 0$,

$$(2.15) \quad \|R(t, \xi)\|_{L^\infty} \leq CM^p t^{-\frac{n}{2}(p-1)-\tilde{\gamma}}, \quad t \geq 1,$$

where $M = \|u_0\|_{H^s} + \| |x|^s u_0 \|_{L^2}$. The proof of (2.15) relies on Katayama-Li-Sunagawa [20] and Kim [21]. By using the Hölder inequality and using (2.15), we have

$$(2.16) \quad \frac{1}{2} \partial_t |\mathcal{F}[U^{-1}u](t, \xi)|^2 \leq -\frac{|\operatorname{Im} \lambda|}{t^{\frac{n}{2}(p-1)}} |\mathcal{F}[U^{-1}u](t, \xi)|^{p+1} + CM^{p+1} t^{-\frac{n}{2}(p-1)-\tilde{\gamma}}.$$

Hence we obtain by the similar argument in [26] that $\|\mathcal{F}[U^{-1}u](t)\|_{L^\infty} \leq CM$ and we have by (2.10) that for any $t \geq 0$,

$$(2.17) \quad \|u(t)\|_{L^\infty} \leq CM(1+t)^{-\frac{n}{2}} + CM(1+t)^{-\frac{n}{2}-\tilde{\gamma}}$$

where $C > 0$ is some constant. Combining (2.17) and the L^2 -dissipative identity:

$$(2.18) \quad \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 - 2|\operatorname{Im} \lambda| \int_0^t \|u(\tau)\|_{L^{p+1}}^{p+1} d\tau, \quad t \geq 0,$$

we deduce that

$$(2.19) \quad \begin{aligned} \frac{d}{dt} \|u(t)\|_{L^2}^2 &\geq -2|\operatorname{Im} \lambda| \|u(t)\|_{L^\infty}^{p-1} \|u(t)\|_{L^2}^2 \\ &\geq -C(M(1+t)^{-\frac{n}{2}} + M(1+t)^{-\frac{n}{2}-\tilde{\gamma}})^{p-1} \|u(t)\|_{L^2}^2, \end{aligned}$$

where $M \equiv \|v_0\|_{H^s} + \| |x|^s v_0 \|_{L^2}$. By solving (2.19), one can obtain the L^2 -lower bound for small solutions to (1.1) since $p > 1 + 2/n$. $\square$

§ 3. L^2 -decay of dissipative solutions

In this section, we show a relation between the L^2 -decay order and the smoothness of dissipative solutions to (1.1) under the critical setting $p = 1 + 2/n$. Hayashi-Li-Naumkin [13] obtained the L^2 -decay rate of the dissipative solution with $H^1(\mathbb{R}^n)$ -regularity. For any $t \geq 0$ and $\varepsilon > 0$,

$$(3.1) \quad \|u(t)\|_{L^2} \leq \begin{cases} C_\varepsilon (1+t)^{-(\frac{1}{p-1}-\frac{n}{2})\frac{2}{n+2}+\varepsilon}, & p < 1 + 2/n, \\ C_\varepsilon \{\log(1+t)\}^{-\frac{n}{n+2}+\varepsilon}, & p = 1 + 2/n, \end{cases}$$

where $C_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Hoshino [18] showed the L^2 -decay order of the dissipative solutions in the Sobolev space $H^r(\mathbb{R}^n)$ for $0 < r < 1$. Ogawa-Sato [31] and Sato [34] showed the L^2 -decay rate for smooth solutions in higher order Sobolev spaces. Our purpose is to consider the asymptotic behavior of solutions to the Cauchy problem for

the dissipative nonlinear Schrödinger equation combined with critical power and higher power nonlinearities. In particular we study the following Cauchy problem with an exponential type nonlinearity which contains the critical power nonlinearity:

$$(3.2) \quad \begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda(e^{|u|^2} - 1)u, & t \in \mathbb{R}, x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $\lambda \in \mathbb{C}$, $\text{Im } \lambda < 0$. We restrict to $n = 1$ to construct smooth solutions to the problem (3.2). We show that the L^2 -decay rate of solutions to (3.2) is determined by the effect of the critical cubic power nonlinearity.

Proposition 3.1. *Let $k \in \mathbb{N}$. Then, there exists a small constant $\varepsilon_0 > 0$ such that for any $u_0 \in H^{k+1}(\mathbb{R}) \cap H_1^k(\mathbb{R})$ with $\|u_0\|_{H^{k+1}} + \|u_0\|_{H_1^k} \leq \varepsilon_0$, the Cauchy problem (3.2) has a unique global solution $u(t, x)$ satisfying*

$$u \in C([0, \infty); H^{k+1}(\mathbb{R})), \quad J(t)\partial_x^k u \in C([0, \infty); L^2(\mathbb{R})).$$

Moreover, there exists $C > 0$ such that for any $t \geq 0$,

$$(3.3) \quad \|u(t)\|_{L^2} \leq C(\log t)^{-\frac{1}{2} + \frac{1}{2(2k+1)}}.$$

We easily show Proposition 3.1 by connecting with the argument in [31] and a proof of Theorem 3.2.

We remark that the estimate (3.3) is a generalization of the result (3.1) by [13] and [18] for $k \geq 1$. Namely the smoothness of the solution shows the faster decay for the critical dissipative case. Since the regularity exponent can be arbitrary large, one can expect that the decay rate of the solution to (3.2) should be $(\log t)^{-\frac{1}{2} + \varepsilon}$ with $0 < \varepsilon \ll 1$ as $k \rightarrow \infty$. Indeed, if dissipative solutions have the Gevrey regularity which is the class between C^∞ -class and an analytic regular class defined by

$$(3.4) \quad G_v^s(\mathbb{R}) \equiv \left\{ f \in L^2(\mathbb{R}); \|f\|_{G_v^s} \equiv \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k f\|_{L^2} < \infty \right\},$$

then the solution shows a slightly rapid L^2 -decay than the L^2 -decay of dissipative solutions with the H^k -regularity.

Theorem 3.2. *Let $s \geq 1$. Then, there exists a small constant $\varepsilon_0 > 0$ such that for any $u_0 \in G_v^s(\mathbb{R})$ with $\|\partial_x u_0\|_{G_v^s} + \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|x \partial_x^k u_0\|_{L^2} \leq \varepsilon_0$, the Cauchy problem (3.2) has a unique global solution u in $C([0, \infty); G_v^s(\mathbb{R}))$. Moreover, the solution satisfies*

$$\partial_x u \in C([0, \infty); G_v^s(\mathbb{R})), \quad \sum_{k=0}^{\infty} \frac{1}{(k!)^s} J(t)\partial_x^k u \in C([0, \infty); L^2(\mathbb{R}))$$

and there exists $C > 0$ such that

$$(3.5) \quad \|u(t)\|_{L^2} \leq C(\log \log t)^{\frac{s}{2}}(\log t)^{-\frac{1}{2}}$$

for any $t \geq e^e$.

It was shown by [31], [34] that dissipative solutions to (1.1) shows the L^2 -decay and its rate is the same order given by (3.3) and (3.5) when $n = 1$ and $p = 3$. In [35], It was shown that

$$\liminf_{t \rightarrow \infty} \|u(t)\|_{L^2} > 0,$$

where u is the solution to (1.1) containing the higher order monomial $|u|^{p-1}u$ with $p > 1 + 2/n$. However the problem (3.2) contains the both of cubic and higher order nonlinearities, L^2 -decay property of solutions can be observed by the critical cubic power part in the exponential nonlinearity, even if (3.2) has the other higher order nonlinearities.

The decay rate in (3.5) would not be optimal in general. Indeed, Kita-Sato [26] showed the existence of solutions to (1.1) such that $\|u(t)\|_{L^2} = O((\log t)^{-1/2})$ as $t \rightarrow \infty$ when $n = 1$ and $p = 3$, and we found that this rate is optimal (cf. Li-Nishii-Sagawa-Sunagawa [28] and Li-Sunagawa [27]).

We remark that constructing the global solution in $G_v^s(\mathbb{R})$ for $s \in (0, 1)$ is difficult because of an estimate such that for any $k_1, k_2, k_3, k \in \mathbb{N} \cup \{0\}$ with $k_1 + k_2 + k_3 = k$,

$$\frac{k!}{k_1!k_2!k_3!} \leq \left(\frac{k!}{k_1!k_2!k_3!} \right)^s,$$

which holds only if $s \geq 1$. In the analytic case for $s = 1$, Ogawa-Sato [31] proved the L^2 -decay of dissipative solutions to (1.1) with $n = 1$ and $p = 3$ by applying an operator $e^{iy\partial_x}$ which is used by Hoshino [16] to obtain the analytic smoothing effect of solutions to the nonlinear Schrödinger system.

To show the L^2 -decay rate for the dissipative solution to (3.2) in the Gevrey class, we prove the global a priori estimate of the solution in the Gevrey class defined by (3.4). Let u be the solution to (3.2) on $[0, T]$. For $0 < \gamma \leq 1/4$, we define

$$\begin{aligned} \|u\|_{X_T} \equiv \sup_{t \in [0, T]} \Big\{ & (1+t)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(t)\|_{L^\infty} \\ & + (1+t)^{-\gamma} \left(\|\partial_x u(t)\|_{G_v^s} + \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|J(t) \partial_x^k u(t)\|_{L^2} \right) \Big\}. \end{aligned}$$

For $s \geq 1$, set

$$(3.6) \quad \varepsilon_s \equiv \|\partial_x u_0\|_{G_v^s} + \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|x \partial_x^k u_0\|_{L^2}.$$

Lemma 3.3. *There exists a positive constant C , independent of $T > 0$ such that*

$$(3.7) \quad \|u\|_{X_T} \leq C\varepsilon_s,$$

where ε_s is sufficiently small.

Proof of Lemma 3.3. Let $s \geq 1$. To prove the estimate (3.7), it suffices to show that there exists a constant $C > 0$ such that

$$(3.8) \quad \|u\|_{X_T} \leq C \left(\varepsilon_s + (e^{\|u\|_{X_T}^2} - 1) \|u\|_{X_T} \right).$$

The estimate (3.8) yields the uniform Gevrey bound (3.7) if we choose ε_s sufficiently small. The existence of the global solution to the Cauchy problem (3.2) is an immediate consequence of the a priori bound (3.7) and the standard local existence theorem proved by Ginibre-Velo [10], Yajima [41], Cazenave-Weissler [7].

In what follows, we denote several positive constants by the same letter C . We find estimates for $\|\partial_x u(t)\|_{G_v^s}$, $\sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|J(t) \partial_x^k u(t)\|_{L^2}$. Applying the energy inequality, and using the unitarity of the free Schrödinger group, we obtain that

$$(3.9) \quad \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(t)\|_{\dot{H}^1} \leq \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u_0\|_{\dot{H}^1} + C \int_0^t \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \left\| \partial_x^k \left((e^{|u|^2} - 1) u \right) \right\|_{\dot{H}^1} d\tau.$$

We first estimate the cubic power nonlinearity, that is the critical power part in the exponential nonlinearity. By the Leibniz rule, it follows that

$$\begin{aligned} & \int_0^t \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \left\| \sum_{k_1+k_2+k_3=k} \frac{k!}{k_1!k_2!k_3!} (\partial_x^{k_1} u(\tau) \overline{\partial_x^{k_2} u(\tau)} \partial_x^{k_3} u(\tau)) \right\|_{\dot{H}^1} d\tau \\ & \leq C \int_0^t \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \sum_{k_1+k_2+k_3=k} \left(\frac{k!}{k_1!k_2!k_3!} \right)^s \left\| (\partial_x^{k_1} u(\tau) \overline{\partial_x^{k_2} u(\tau)} \partial_x^{k_3} u(\tau)) \right\|_{\dot{H}^1} d\tau \\ & \leq C \int_0^t \sum_{k=0}^{\infty} \sum_{k_1+k_2+k_3=k} \left(\frac{\|\partial_x^{k_1} u(\tau)\|_{\dot{H}^1}}{(k_1!)^s} \frac{\|\partial_x^{k_2} u(\tau)\|_{L^\infty}}{(k_2!)^s} \frac{\|\partial_x^{k_3} u(\tau)\|_{L^\infty}}{(k_3!)^s} \right. \\ & \quad \left. + \frac{\|\partial_x^{k_1} u(\tau)\|_{L^\infty}}{(k_1!)^s} \frac{\|\partial_x^{k_2} u(\tau)\|_{\dot{H}^1}}{(k_2!)^s} \frac{\|\partial_x^{k_3} u(\tau)\|_{L^\infty}}{(k_3!)^s} \right. \\ & \quad \left. + \frac{\|\partial_x^{k_1} u(\tau)\|_{L^\infty}}{(k_1!)^s} \frac{\|\partial_x^{k_2} u(\tau)\|_{L^\infty}}{(k_2!)^s} \frac{\|\partial_x^{k_3} u(\tau)\|_{\dot{H}^1}}{(k_3!)^s} \right) d\tau \\ & \leq C \int_0^t \left(\sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(\tau)\|_{\dot{H}^1} \right) \left(\sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(\tau)\|_{L^\infty} \right)^2 d\tau \\ & \leq C \|u\|_{X_T}^3 \int_0^t (1+\tau)^{-1+\gamma} d\tau \leq C \|u\|_{X_T}^3 (1+t)^\gamma, \end{aligned}$$

where ε_s is sufficiently small. For other powers of nonlinearities, we see that for any $\ell = 2, 3, \dots$,

$$\int_0^t \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k(|u|^{2\ell}u)\|_{\dot{H}^1} d\tau \leq C\|u\|_{X_T}^{2\ell+1} \int_0^t (1+\tau)^{-\ell+\gamma} d\tau \leq C\|u\|_{X_T}^{2\ell+1}.$$

Then it holds from (3.9) that

$$(3.10) \quad \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(t)\|_{\dot{H}^1} \leq \varepsilon_s + C(1+t)^\gamma (e^{\|u\|_{X_T}^2} - 1) \|u\|_{X_T}.$$

By the representation of $J(t)f = e^{i\frac{x^2}{2t}} it \partial_x [e^{-i\frac{x^2}{2t}} f]$, the similar argument to that in (3.9) derives

$$(3.11) \quad \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|J(t)\partial_x^k u(t)\|_{L^2} \leq \varepsilon_s + C(1+t)^\gamma (e^{\|u\|_{X_T}^2} - 1) \|u\|_{X_T}.$$

Secondly, we consider the L^∞ -bound for the solution:

$$(3.12) \quad (1+t)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(t)\|_{L^\infty} \leq C\varepsilon_s + C(e^{\|u\|_{X_T}^2} - 1) \|u\|_{X_T}$$

for any $t \geq 0$. For $t \in [0, 1]$, the standard Sobolev embedding leads to

$$(3.13) \quad (1+t)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(t)\|_{L^\infty} \leq C \sum_{k=0}^{\infty} \frac{1}{(k!)^s} (\|\partial_x^k u(t)\|_{L^2} + \|\partial_x^k u(t)\|_{\dot{H}^1}).$$

From (2.12), it follows that

$$\|\partial_x^k u(t)\|_{L^2} \leq C(\|J(t)\partial_x^k u\|_{L^2} + \|\partial_x^k u\|_{\dot{H}^1}).$$

Hence by (3.13), we have

$$(3.14) \quad (1+t)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(t)\|_{L^\infty} \leq C \sum_{k=0}^{\infty} \frac{1}{(k!)^s} (\|J(t)\partial_x^k u\|_{L^2} + \|\partial_x^k u\|_{\dot{H}^1})$$

for any $t \in [0, 1]$. From (3.10), (3.11), (3.13) and (3.14), the estimate (3.12) holds for any $t \in [0, 1]$. Hence, we prove (3.12) for $t \geq 1$. Let u be the solution to (3.2). Then it follows that for any $k \in \mathbb{N} \cup \{0\}$,

$$(3.15) \quad \begin{aligned} i\partial_t \mathcal{F}[U(-t)\partial_x^k u](t, \xi) &= \mathcal{F}\left[U(-t)\left\{\partial_x^k\left(i\partial_t u + \frac{1}{2}\partial_x^2 u\right)\right\}\right] \\ &= \lambda \mathcal{F}\left[U(-t)\partial_x^k\left((e^{|u|^2} - 1)u\right)\right] = \lambda \mathcal{F}\left[U(-t)\partial_x^k\left(\sum_{\ell=1}^{\infty} \frac{1}{\ell!}|u|^{2\ell}u\right)\right]. \end{aligned}$$

By applying the similar argument in [31], we have that for any $\ell \in \mathbb{N}$,

$$\lambda \mathcal{F} \left[U(-t) \partial_x^k \left(\frac{1}{\ell!} |u|^{2\ell} u \right) \right] = \frac{1}{\ell!} \frac{\lambda}{t^\ell} |\mathcal{F}U(-t)u|^{2\ell} (\mathcal{F}U(-t) \partial_x^k u) + R_{k,\ell},$$

where $R_{k,\ell} = R_{k,\ell}(t, \xi)$ is defined by

$$(3.16) \quad R_{k,\ell}(t, \xi) \equiv \lambda \mathcal{F} \left[U(-t) \partial_x^k \left(\frac{1}{\ell!} |u|^{2\ell} u \right) \right] - \frac{1}{\ell!} \frac{\lambda}{t^\ell} |\mathcal{F}U(-t)u|^{2\ell} (\mathcal{F}U(-t) \partial_x^k u)$$

and $R_{k,\ell}$ satisfies

$$(3.17) \quad \begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|R_{k,\ell}(t)\|_{L^\infty \cap L^2} &\leq C(1+t)^{-\ell-\tilde{\gamma}+(2\ell+1)\gamma} \frac{\|u\|_{X_T}^{2\ell+1}}{\ell!} \\ &\leq C(1+t)^{-1-(\tilde{\gamma}-3\gamma)} \frac{\|u\|_{X_T}^{2\ell+1}}{\ell!} \end{aligned}$$

for any $t \geq 1$ and $\tilde{\gamma} > 3\gamma$. Note that

$$-\ell - \tilde{\gamma} + (2\ell + 1)\gamma = -1 - (\tilde{\gamma} - 3\gamma) + (\ell - 1)(-1 + 2\gamma) \leq -1 - (\tilde{\gamma} - 3\gamma)$$

since $(\ell - 1)(-1 + 2\gamma) \leq 0$. By multiplying (3.15) by $\overline{(\mathcal{F}[U(-t) \partial_x^k u](t, \xi))}$ and taking the imaginary part and $\text{Im } \lambda < 0$, we have

$$(3.18) \quad \partial_t |\mathcal{F}[U(-t) \partial_x^k u](t, \xi)|^2 \leq 2 \sum_{\ell=1}^{\infty} |R_{k,\ell}(t, \xi)| |\overline{\mathcal{F}[U(-t) \partial_x^k u](t, \xi)}|,$$

where R_k is given by (3.16). The inequality (3.18) is equivalent to

$$(3.19) \quad \partial_t |\mathcal{F}[U(-t) \partial_x^k u](t, \xi)| \leq \sum_{\ell=1}^{\infty} |R_{k,\ell}(t, \xi)|.$$

Integrating the inequality (3.19) over $[1, t]$ and taking the summation, we see that

$$(3.20) \quad \begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(k!)^s} |\mathcal{F}[U(-t) \partial_x^k u](t, \xi)| \\ \leq \sum_{k=0}^{\infty} \frac{1}{(k!)^s} |\mathcal{F}[U(-\cdot) \partial_x^k u](1, \xi)| + \int_1^t \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \frac{1}{(k!)^s} |R_{k,\ell}(\tau, \xi)| d\tau. \end{aligned}$$

By (3.11) and an expression of $J(t) = U(t)xU(-t)$,

$$(3.21) \quad \begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(k!)^s} |\mathcal{F}[U(-\cdot) \partial_x^k u](1, \xi)| &\leq \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \int_{\mathbb{R}} |U(-\cdot) [\partial_x^k u](1, x)| dx \\ &\leq C \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \left(\int_{\mathbb{R}} x^2 |U(-\cdot) [\partial_x^k u](1, x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \left(\varepsilon_s + (e^{\|u\|_{X_T}^2} - 1) \|u\|_{X_T} \right). \end{aligned}$$

The inequalities (3.20) and (3.21) derive

$$\sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\mathcal{F}[U(-t)\partial_x^k u](t)\|_{L^\infty} \leq C \left(\varepsilon_s + (e^{\|u\|_{X_T}^2} - 1) \|u\|_{X_T} \right).$$

By (2.10) in Lemma 2.3, we have

$$(3.22) \quad (1+t)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(t)\|_{L^\infty} \leq C \left(\varepsilon_s + (e^{\|u\|_{X_T}^2} - 1) \|u\|_{X_T} \right)$$

for any $t \geq 1$. Combining the inequalities (3.10), (3.11), (3.12) and (3.22), we obtain (3.8). $\square$

We prove the uniform bound of dissipative solutions to (3.2) in the Gevrey class by using the estimate (3.8).

Lemma 3.4. *Let u be the solution to (3.2) satisfying (3.8). Then, there exists $C > 0$ such that for any $t \geq 0$,*

$$(3.23) \quad \|u(t)\|_{G_v^s} \leq C \varepsilon_s,$$

where ε_s is sufficiently small, which is given by (3.6).

Proof of Lemma 3.4. From the Plancherel theorem and unitarity of $U(t)$, we see that for any $f \in L^2(\mathbb{R})$, $\|\mathcal{F}[U(-t)f](t)\|_{L^2} = \|f\|_{L^2}$. Hence, by integrating (3.18) over $\mathbb{R}_\xi \times [1, t]$ and applying the Hölder inequality to the remainder term, it holds that for any $k \in \mathbb{N} \cup \{0\}$ and $t \geq 1$,

$$(3.24) \quad \begin{aligned} \|\partial_x^k u(t)\|_{L^2} &\leq \left(\|\partial_x^k u(1)\|_{L^2}^2 + \int_1^t \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \|R_{k,\ell}(\tau)\|_{L^2} \|\partial_x^k u(\tau)\|_{L^2} d\tau \right)^{\frac{1}{2}} \\ &\leq \|\partial_x^k u(1)\|_{L^2} + \left(\int_1^t \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \|R_{k,\ell}(\tau)\|_{L^2} \|\partial_x^k u(\tau)\|_{L^2} d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

Multiplying (3.24) by $1/(k!)^s$ and taking the summation,

$$(3.25) \quad \begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(t)\|_{L^2} &\leq \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(1)\|_{L^2} \\ &\quad + \left\{ \int_1^t \left(\sum_{k=0}^{\infty} \frac{1}{(k!)^s} \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \|R_{k,\ell}(\tau)\|_{L^2} \right) \left(\sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(\tau)\|_{L^2} \right) d\tau \right\}^{\frac{1}{2}}. \end{aligned}$$

By the unitarity of $U(t)$, the representation of $J(t)f = U(t)xU(-t)f$, the uncertainly

principle (2.12) and the uniform bound (3.7), we see that for any $\tau \in [1, t]$,

$$\begin{aligned}
 \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|\partial_x^k u(\tau)\|_{L^2} &= \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|U(-t)[\partial_x^k u](\tau)\|_{L^2} \\
 (3.26) \quad &\leq \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \left(\|J(\tau)\partial_x^k u(\tau)\|_{L^2} + \|\partial_x^k u(\tau)\|_{\dot{H}^1} \right) \\
 &\leq C(1+\tau)^\gamma \|u\|_{X_T} \leq C\varepsilon_s(1+\tau)^\gamma.
 \end{aligned}$$

By (3.17), the estimates (2.11) and (3.7) yield that for any $\tilde{\gamma} > 4\gamma$ and $\tau \in [1, t]$,

$$(3.27) \quad \sum_{k=0}^{\infty} \frac{1}{(k!)^s} \|R_{k,\ell}(\tau)\|_{L^2} \leq C\varepsilon_s \tau^{-1-(\tilde{\gamma}-3\gamma)}.$$

Combining (3.7), (3.25), (3.26) and (3.27), there exists a constant $C > 0$ such that

$$\|u(t)\|_{G_v^s} \leq C\varepsilon_s, \quad t \geq 1,$$

where ε_s is sufficiently small, which is given by (3.6). By (3.7), the estimate (3.23) holds for any $t \in [0, 1]$. $\square$

§ 4. Proof of Theorem 3.2

Proof of Theorem 3.2. For $r > 0$, $\|f\|_{L^2([-r,r])} \equiv \left(\int_{-r}^r |f(x)|^2 dx \right)^{\frac{1}{2}}$. Then it follows that for any $r > 0$,

$$(4.1) \quad \|f\|_{L^2([-r,r])} \leq (2r)^{\frac{1}{4}} \|f\|_{L^4} \quad \text{for any } f \in L^4(\mathbb{R})$$

and for any $a > 0$, there exists a constant $C > 0$ such that

$$(4.2) \quad \|f\|_{L^2([-r,r])}^2 \leq \frac{a^s}{2^{2-s}} |\operatorname{Im} \lambda| (\log \log t)^{-s} (\log t) \|f\|_{L^2([-r,r])}^4 + C (\log \log t)^s (\log t)^{-1}$$

for any $f \in L^2(\mathbb{R})$ by the Young inequality. Let u be the global solution to (3.2) given via a priori estimate (3.8). Then, $\mathcal{F}[U(-t)u](t, \xi)$ satisfies

$$(4.3) \quad i\partial_t \mathcal{F}[U(-t)u](t, \xi) = \lambda(e^{\frac{|\mathcal{F}[U(-t)u]|^2}{t}} - 1) \mathcal{F}[U(-t)u](t, \xi) + R(t, \xi),$$

where $R(t, \xi) = \lambda \mathcal{F}[U(-t)(e^{|u|^2} - 1)u] - \lambda(e^{\frac{|\mathcal{F}[U(-t)u]|^2}{t}} - 1) \mathcal{F}[U(-t)u]$. By multiplying the equation (4.3) by $\overline{\mathcal{F}[U(-t)u](t, \xi)}$ and taking imaginary part, we obtain

$$\begin{aligned}
 (4.4) \quad \frac{1}{2} \partial_t |\mathcal{F}[U(-t)u]|^2 &= -|\operatorname{Im} \lambda| (e^{\frac{|\mathcal{F}[U(-t)u]|^2}{t}} - 1) |\mathcal{F}[U(-t)u]|^2 + \operatorname{Im}\{R \overline{\mathcal{F}[U(-t)u]}\} \\
 &\leq -\frac{|\operatorname{Im} \lambda|}{t} |\mathcal{F}[U(-t)u]|^4 + \operatorname{Im}\{R \overline{\mathcal{F}[U(-t)u]}\}
 \end{aligned}$$

By (2.11) and (3.8), $R = R(t, \xi)$ satisfies for any $\tilde{\gamma} > 3\gamma$,

$$(4.5) \quad \|R(t)\|_{L^2} \leq C\varepsilon_s(e^{|\varepsilon_s|^2} - 1)t^{-1-(\tilde{\gamma}-3\gamma)}$$

for any $t \geq 1$. This remainder is controlled by the initial size with the infinite order, and this is the difference of previous works [31] and [34], where we only treated the monomial nonlinearity. Using the L^2 -a priori estimate $\|\mathcal{F}[U(-t)u](t)\|_{L^2} \leq \|u_0\|_{L^2}$ and integrating (4.4) in ξ over $\mathbb{R}$, we obtain that, for any positive $\varepsilon_s \ll 1$,

$$(4.6) \quad \frac{1}{2} \frac{d}{dt} \|\mathcal{F}[U(-t)u](t)\|_{L^2}^2 \leq -\frac{|\operatorname{Im} \lambda|}{t} \|\mathcal{F}[U(-t)u](t)\|_{L^4}^4 + C\varepsilon_s t^{-1-(\tilde{\gamma}-3\gamma)}.$$

For $0 < a < 1$, let $r(t) \equiv \left(\frac{1}{2a} \log \log t\right)^s$. From (4.1), (4.2) and (4.6), we deduce that, for any positive $\varepsilon_s \ll 1$,

$$(4.7) \quad \begin{aligned} & \frac{d}{dt} \{(\log \log t)^{-s} (\log t)^2 \|\mathcal{F}[U(-t)u](t)\|_{L^2}^2\} \\ & \leq 2(\log \log t)^{-s} (\log t) t^{-1} \left(\|\mathcal{F}[U(-t)u](t)\|_{L^2([-r, r])}^2 + \|\mathcal{F}[U(-t)u](t)\|_{L^2(\mathbb{R} \setminus [-r, r])}^2 \right) \\ & \quad + (\log \log t)^{-s} (\log t)^2 \left(-\frac{|\operatorname{Im} \lambda|}{t} \|\mathcal{F}[U(-t)u](t)\|_{L^4}^4 + C t^{-1-(\tilde{\gamma}-3\gamma)} \right) \\ & \leq 2(\log \log t)^{-s} (\log t) t^{-1} \left\{ \left(\frac{a^s}{2^{2-s}} |\operatorname{Im} \lambda| (\log \log t)^{-s} (\log t) \|\mathcal{F}[U(-t)u](t)\|_{L^2([-r, r])}^4 \right) \right. \\ & \quad \left. + C(\log \log t)^s (\log t)^{-1} \right\} + \|\mathcal{F}[U(-t)u](t)\|_{L^2(\mathbb{R} \setminus [-r, r])}^2 \\ & \quad - |\operatorname{Im} \lambda| (2r(t))^{-1} (\log \log t)^{-s} (\log t)^2 t^{-1} \|\mathcal{F}[U(-t)u](t)\|_{L^2([-r, r])}^4 + C t^{-1-\frac{\tilde{\gamma}-3\gamma}{2}} \\ & \leq C t^{-1} + C(\log \log t)^{-s} (\log t) t^{-1} \|\mathcal{F}[U(-t)u](t)\|_{L^2(\mathbb{R} \setminus [-r, r])}^2 + C t^{-1-\frac{\tilde{\gamma}-3\gamma}{2}}. \end{aligned}$$

Here, we prove the following Lemma to obtain the uniform estimate for the high frequency part of dissipative solutions.

Lemma 4.1. *Let $0 < a < 1$ and $s \geq 1$. Then, the following inequality holds:*

$$(4.8) \quad \|e^{a|\xi|^{\frac{1}{s}}} \widehat{f}\|_{L^2} \leq \left(\frac{1}{1 - a^{1-\frac{1}{s}}} \right)^{1-\frac{1}{s}} \|f\|_{L^2}^{1-\frac{1}{s}} \|f\|_{G_v^s}^{\frac{1}{s}}$$

for any $f \in G_v^s(\mathbb{R})$.

Proof of Lemma 4.1. Let $0 < a < 1$ and $s \geq 1$. By using the Taylor expansion for

the function $e^{a|\xi|^{\frac{1}{s}}}$ and Hölder's inequality, we obtain that

$$\begin{aligned} \|e^{a|\xi|^{\frac{1}{s}}} \widehat{f}\|_{L^2} &= \left\| \sum_{k=0}^{\infty} \frac{a^k}{k!} |\xi|^{\frac{k}{s}} \widehat{f} \right\|_{L^2} \\ &\leq \left\| \left(\sum_{k=0}^{\infty} a^{\frac{k}{1-\frac{1}{s}}} |\widehat{f}| \right)^{1-\frac{1}{s}} \left(\sum_{k=0}^{\infty} \frac{1}{(k!)^s} |\xi|^k |\widehat{f}| \right)^{\frac{1}{s}} \right\|_{L^2} \\ &\leq \left(\frac{1}{1-a^{1-\frac{1}{s}}} \right)^{1-\frac{1}{s}} \|f\|_{L^2}^{1-\frac{1}{s}} \|f\|_{G_v^s}^{\frac{1}{s}} \end{aligned}$$

for any $f \in G_v^s(\mathbb{R})$. □

By (4.8) and (3.23), we estimate for $0 < a < 1$,

$$\begin{aligned} (4.9) \quad \|\mathcal{F}[U(-t)u](t)\|_{L^2(\mathbb{R} \setminus [-r, r])}^2 &\leq e^{-2ar^{\frac{1}{s}}} \|e^{a|\xi|^{\frac{1}{s}}} \mathcal{F}[U(-t)u](t)\|_{L^2(\mathbb{R} \setminus [-r, r])}^2 \\ &\leq e^{-\log \log t} \|e^{a|\xi|^{\frac{1}{s}}} \mathcal{F}[U(-t)u](t)\|_{L^2}^2 \leq C(\log t)^{-1} \end{aligned}$$

for any $t \geq e^e$. Therefore the inequalities (4.7) and (4.9) yield

$$\begin{aligned} (4.10) \quad \frac{d}{dt} \{(\log \log t)^{-s} (\log t)^2 \|\mathcal{F}[U(-t)u](t)\|_{L^2}^2\} &\leq Ct^{-1} + C(\log \log t)^{-s} t^{-1} + Ct^{-1-\frac{\tilde{\gamma}-3\gamma}{2}} \\ &\leq Ct^{-1} + Ct^{-1-\frac{\tilde{\gamma}-3\gamma}{2}} \end{aligned}$$

for any $t \geq e^e$. By the dissipative effect of the solution $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$, we integrate both sides of the inequality (4.10) over $[e^e, t]$ and deduce that

$$\begin{aligned} (4.11) \quad (\log \log t)^{-s} (\log t)^2 \|\mathcal{F}[U(-t)u](t)\|_{L^2}^2 &\leq (\log \log e^e)^{-s} (\log e^e)^2 \|\mathcal{F}[U(-t)u](e^e)\|_{L^2}^2 \\ &\quad + C \log t + C \\ &\leq C(1 + \log t). \end{aligned}$$

Therefore we conclude

$$\|u(t)\|_{L^2} = \|\mathcal{F}[U(-t)u](t)\|_{L^2} \leq C(\log \log t)^{\frac{s}{2}} (\log t)^{-\frac{1}{2}}$$

for any $t \geq e^e$. □

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