

THE SECOND PLURIGENUS OF SURFACE SINGULARITIES

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INTRODUCTION

Let (X, x) be a normal surface singularity over the complex number field $\mathbb{C}$ and $f: (M, A) \rightarrow (X, x)$ the minimal good resolution of the singularity (X, x) , i.e., the smallest resolution for which an exceptional divisor A consists of non-singular curves intersecting transversally, with no three through one point. It is well known that there exists a unique minimal good resolution. Let $A = \bigcup_{i=1}^k A_i$ be the decomposition of the exceptional set A into irreducible components. The weighted dual graph of (X, x) is the graph such that each vertex of which represents a component of A weighted by the self-intersection number, while each edge connecting the vertices corresponding to A_i and A_j , $i \neq j$, corresponds to the point $A_i \cap A_j$. Giving the weighted dual graph is equivalent to giving the information of the genera of the A_i 's and the intersection matrix $(A_i \cdot A_j)$. The geometric genus of the singularity (X, x) is defined by

$$p_g(X, x) = \dim_{\mathbb{C}} H^1(M, \mathcal{O}_M).$$

The m -th L^2 -plurigenus of the singularity (X, x) is the integer $\delta_m(X, x)$ which was introduced in [Wt] and can be computed as

$$\delta_m(X, x) = \dim_{\mathbb{C}} H^0(M - A, \mathcal{O}_M(mK)) / H^0(M, \mathcal{O}_M(mK + (m-1)A)),$$

where K denotes the canonical divisor on M . Note that $p_g(X, x) = \delta_1(X, x)$. The plurigenera of a Gorenstein surface singularity are determined by the weighted dual graph and p_g (cf. [O2]). In this paper we consider relations among the invariants δ_2 , p_g , μ , τ and the modality of certain normal surface singularities, so "a singularity" always means a normal surface singularity over $\mathbb{C}$.

1. PRELIMINARIES

(1.1) Let (X, x) be a surface singularity and $f: (M, A) \rightarrow (X, x)$ the minimal good resolution of the singularity (X, x) . Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_M$ -modules and D a divisor on M . We will use the following notation: $\mathcal{F}(D) = \mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_M(D)$,

$$\begin{aligned} H^i(\mathcal{F}) &= H^i(M, \mathcal{F}), & H_A^i(\mathcal{F}) &= H_A^i(M, \mathcal{F}), \\ h^i(\mathcal{F}) &= \dim_{\mathbb{C}} H^i(\mathcal{F}), & h_A^i(\mathcal{F}) &= \dim_{\mathbb{C}} H_A^i(\mathcal{F}). \end{aligned}$$

We denote by K the canonical divisor on M .

(1.2) We take the following characterization of minimally elliptic singularities as its definition.

Theorem 1.3 (Laufer [La1, Theorem 3.10]). *A singularity (X, x) is minimally elliptic if and only if (X, x) is an elliptic Gorenstein singularity.*

Theorem 1.4 (cf. [O1, O2]). *Let (X, x) be a singularity. Then*

$$\delta_2(X, x) = h_A^1(\mathcal{O}_M(2K + A)) = h^1(\mathcal{O}_M(-K - A)).$$

If (X, x) is a Gorenstein singularity with $p_g \geq 1$, then we have

$$\delta_2(X, x) = -(K + L_1) \cdot L_1 / 2 + p_g(X, x) = -K \cdot L_1 + \chi(\mathcal{O}_A) + p_g(X, x).$$

Corollary 1.5 (cf. [O1]). *Let (X, x) be a hypersurface (resp. complete intersection) minimally elliptic singularity. Then $\delta_2(X, x) \leq 4$ (resp. ≤ 5).*

(1.6) Let $\Omega_M^1\langle A \rangle$ be the sheaf of 1-forms with logarithmic poles along A , and $\mathcal{S}$ its dual. Then there are exact sequences (cf. [Wh3]):

$$(1.6.1) \quad 0 \rightarrow \Omega_M^1 \rightarrow \Omega_M^1\langle A \rangle \rightarrow \bigoplus_{i=1}^k \mathcal{O}_{A_i} \rightarrow 0;$$

$$(1.6.2) \quad 0 \rightarrow \mathcal{S} \rightarrow \Theta_M \rightarrow \bigoplus_{i=1}^k \mathcal{O}_{A_i}(A_i) \rightarrow 0;$$

$$(1.6.3) \quad 0 \rightarrow \Theta_M(-A) \rightarrow \mathcal{S} \rightarrow \Theta_A \rightarrow 0.$$

Corollary 1.7. *Let (X, x) be a singularity. Then $\delta_2(X, x) \geq h^1(\Theta_A)$.*

Proof. For a locally free sheaf $\mathcal{F}$ of rank 2 on M , $\mathcal{F} \cong \mathcal{H}om_{\mathcal{O}_M}(\mathcal{F}, \mathcal{O}_M) \otimes_{\mathcal{O}_M} \bigwedge^2 \mathcal{F}$. Thus we get isomorphisms $\Theta_M(-A) \cong \Omega_M^1(-K - A)$ and $\mathcal{S} \cong \Omega_M^1(A)(-K - A)$. Then the exact sequences (1.6.1) and (1.6.3) give

$$(1.7.1) \quad h^1(\Theta_A) \cong h^1 \left(\bigoplus_{i=1}^k \mathcal{O}_{A_i}(-K - A) \right).$$

From the following exact sequence

$$0 \rightarrow \mathcal{O}_A \rightarrow \bigoplus_{i=1}^k \mathcal{O}_{A_i} \rightarrow \bigoplus_{i < j} \mathcal{O}_{A_i \cap A_j} \rightarrow 0,$$

we have a surjective map

$$H^1(\mathcal{O}_A(-K - A)) \rightarrow H^1 \left(\bigoplus_{i=1}^k \mathcal{O}_{A_i}(-K - A) \right).$$

By Theorem 1.4 and (1.7.1), we get

$$\delta_2(X, x) \geq h^1(\mathcal{O}_A(-K - A)) \geq h^1(\Theta_A). \quad \square$$

(1.8) Note that $h^1(\Theta_A)$ is the tangent space of locally trivial deformation of A .

2. EQUISINGULAR DEFORMATIONS

(2.1) In this section, we discuss deformations. Let (X, x) be a singularity and $f: (M, A) \rightarrow (X, x)$ the minimal good resolution of (X, x) . Let $A = \bigcup_{i=1}^k A_i$ be the decomposition into irreducible components. We denote by D_X the functor (cf. [Sc]) of deformations of a singularity (X, x) . In [Wh2], Wahl introduced the equisingular functor ES_M of deformations of (M, A) to which all A_i lift, and which blow down to deformations of (X, x) . A deformation of the singularity (X, x) is called an equisingular deformation if it is obtained from an equisingular deformation of (M, A) . It is well known that a deformation of M blows down if and only if $h^1(\mathcal{O}_M)$ does not jump (cf. [Wh2, (4.3)]). Hence equisingular deformations preserve the geometric genera and the weighted dual graphs of singularities, and so the plurigena of Gorenstein singularities (cf. Introduction). In [La2, La3, La4, La5], Laufer studied deformations of M in the analytic category. For a Gorenstein singularity (X, x) , an equisingular deformation of

(M, A) induces a topologically constant deformation of (X, x) , and the converse holds, too (see [La5, V, VI]).

By (1.6.2), We have the following exact sequence

$$0 \rightarrow H^1(\mathcal{S}) \rightarrow H^1(\Theta_M) \rightarrow H^1\left(\bigoplus_{i=1}^k \mathcal{O}_{A_i}(A_i)\right) \rightarrow 0.$$

There exists the versal deformation $\pi: \overline{M} \rightarrow (Q, o)$ of (M, A) with tangent space $T_{Q,o} \cong H^1(\Theta_M)$, and a submanifold (P, o) with tangent space $T_{P,o} \cong H^1(\mathcal{S})$ such that all of the A_i lift to above P and P is the maximal subspace of Q above which all of the A_i lift (cf. [La5, p. 26]).

Theorem 2.2 (Wahl [Wh2]). (1) ES_M has a hull (in the sense of [Sc]) and the natural map $ES_M \rightarrow D_X$ is injective.

(2) If any deformation of (M, A) to which all A_i lift blows down to a deformation of (X, x) , then $T(ES_M) = H^1(\mathcal{S})$, where $T(ES_M)$ denotes the tangent space of ES_M . If $p_g(X, x) \leq 1$, then this condition is satisfied.

(2.3) Let $B = \mathbb{C}\{z_1, \dots, z_n\}$. Let (X, x) be a q-h singularity defined by an ideal $I \subset B$. Let us recall that the tangent space T_X^1 of D_X is given by the exact sequence

$$\mathrm{Hom}_R(\Omega_B^1 \otimes R, R) \rightarrow \mathrm{Hom}_R(I/I^2, R) \rightarrow T_X^1 \rightarrow 0,$$

where $R = B/I$. Since $\mathrm{Hom}_R(I/I^2, R)$ is graded, so is T_X^1 : we write as $T_X^1 = \bigoplus_{i \in \mathbb{Z}} T_X^1(i)$.

Theorem 2.4 (Pinkham [P2, 4.6]). $T(ES_M) = \bigoplus_{i \geq 0} T_X^1(i)$.

Definition 2.5. A function $h \in \mathbb{C}\{z_0, z_1, z_2\} = \mathcal{O}_{\mathbb{C}^3,o}$ is called a quasi-homogeneous (q-h, for short) polynomial of degree d with weights $(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}^3$, if

$$t^d h(z_0, z_1, z_2) = h(t^{\alpha_0} z_0, t^{\alpha_1} z_1, t^{\alpha_2} z_2)$$

for any $t \in \mathbb{C}$. We assume that α_0, α_1 and α_2 are relatively prime.

A function $h \in \mathcal{O}_{\mathbb{C}^3,o}$ is said to be semi-quasi-homogeneous (s-q-h, for short) of degree d with weights $(\alpha_0, \alpha_1, \alpha_2)$ if it is of the form $h = h_0 + h_1$, where h_0 is a q-h polynomial of degree d with weights $(\alpha_0, \alpha_1, \alpha_2)$ which defines an isolated singularity and all of the monomials of h_1 have degree strictly greater than d or $h_1 = 0$ (cf. [AGV, 12.1]). A singularity is said to be s-q-h if it is defined by a s-q-h function.

(2.6) Assume that $h \in \mathbb{C}\{z_0, z_1, z_2\} = \mathcal{O}_{\mathbb{C}^3, o}$ define an isolated singularity (X, o) at the origin. Let J_h be an ideal of $\mathcal{O}_{\mathbb{C}^3, o}$ generated by $\partial h / \partial z_0, \partial h / \partial z_1$ and $\partial h / \partial z_2$. $Q_h = \mathcal{O}_{\mathbb{C}^3, o} / J_h$ is called Jacobian algebra. Then we have $T_X^1 \cong \mathcal{O}_{\mathbb{C}^3, o} / (h, J_h)$. It is well known that $(h, J_h) = J_h$ if and only if h is q-h (after a change of coordinates) (see [Sa]).

If h is a q-h polynomial of degree d with weights $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, then α induces a grading on $\mathcal{O}_{\mathbb{C}^3, o}$, and so on Q_h . Let $Q_h = \bigoplus_{i \geq 0} Q_h(i)$. Recall that a morphism of graded modules $\varphi \in \text{Hom}_{\mathcal{O}_X}((h)/(h^2), \mathcal{O}_X)$ has degree n if $\varphi(h)$ has degree $d+n$. Hence we have $T_X^1(i) \cong Q_h(i+d)$ (cf. (2.3)), and $T(ES_M) \cong \bigoplus_{i \geq d} Q_h(i)$. We see that a s-q-h singularity is a fibre in an equisingular deformation of a q-h singularity by Theorem 2.4 (cf. [AGV, Theorem 12.1]).

(2.7) We assume that the weighted dual graph of (X, x) is a star-shaped graph. Let us introduce some results of [TW].

We set $A = A_0 + \sum_{i=1}^{\beta} S_i$, where A_0 is the central curve, and S_i the branches. The curves of S_i are denoted by $A_{i,j}$, $1 \leq j \leq r_i$, where $A_0 \cdot A_{i,1} = A_{i,j} \cdot A_{i,j+1} = 1$ ($j = 1, \dots, r_i - 1$). Let $b_{i,j} = -A_{i,j} \cdot A_{i,j}$. For each branch S_i , positive integers e_i and d_i are defined by

$$d_i/e_i = b_{i,1} - \frac{1}{b_{i,2} - \frac{1}{\dots - \frac{1}{b_{i,r_i}}}}$$

where $e_i < d_i$, and e_i and d_i are relatively prime. Let D be a divisor on A_0 such that $\mathcal{O}_{A_0}(D)$ is the conormal sheaf of A_0 . We define a $\mathbb{Q}$ -divisor C on A_0 and a graded ring R as follows: $C = D - \sum_{i=1}^{\beta} q_i P_i$, where $q_i = e_i/d_i$ and $P_i = A_0 \cap A_{i,1}$;

$$R = \bigoplus_{n \geq 0} H^0(\mathcal{O}_{A_0}(nC)) T^n \subset \mathbb{C}(A_0)[T],$$

where $\mathbb{C}(A_0)$ is the field of rational functions of A_0 , and T an indeterminate. Then $\text{Spec}(R)$ is a q-h normal surface singularity, we denote by (Y, y) , and the weighted dual graph of (Y, y) is the same as that of (X, x) (cf. [P1]).

By contracting the branches $S_1 \cup \dots \cup S_{\beta}$, we get a normal surface M' with cyclic quotient singularities. Let $\Phi: (M', A') \rightarrow (X, x)$ be the morphism induced canonically, where A' is the image of A_0 . We define a filtration on $\mathcal{O}_X$ by $F^n = \Phi_* \mathcal{O}_{M'}(-nA')$ for $n \in \mathbb{Z}$. Note that $F^n = \mathcal{O}_X$ for $n \leq 0$. Let $\mathcal{R} = \bigoplus_{n \in \mathbb{Z}} F^n T^n$ and $G = \bigoplus_{n \geq 0} (F^n / F^{n+1}) T^n$. Then the natural map $\mathbb{C}[T^{-1}] \rightarrow \mathcal{R}$ defines a deformation of $\text{Spec}(G)$ with general fibre isomorphic to (X, x) , since $G \cong \mathcal{R}/T^{-1}\mathcal{R}$ and $\mathcal{O}_X \cong \mathcal{R}/(T^{-1} - a)\mathcal{R}$ for $a \in \mathbb{C} - \{0\}$ (cf. [TW, (5.15)]). By [TW, (6.3)], R is the normalization of G , and $R = G$ if and

only if $p_g(Y, y) = p_g(X, x)$. By [Wh4, (1.12), (3.4)], (X, x) is a fibre in an equisingular deformation of (Y, y) if $p_g(Y, y) = p_g(X, x)$.

Proposition 2.8. *Let (X, x) be a minimally elliptic singularity with a star-shaped graph. Then there exist a q-h minimally elliptic singularity (Y, y) and an equisingular deformation $\pi: \bar{Y} \rightarrow \mathbb{C}$ of (Y, y) such that $X = \pi^{-1}(a)$ for $a \in \mathbb{C} - \{0\}$.*

Proof. We use the notation in (2.7). Since the weighted dual graph of (Y, y) is the same as that of (X, x) , we see that (Y, y) is a minimally elliptic singularity. $\square$

(2.9) Under the same notation as above, if (X, x) is a hypersurface minimally elliptic singularity, then so is (Y, y) by [La1, Theorem 3.13]. By Proposition 2.8 and (2.6), a hypersurface minimally elliptic singularity with star-shaped graph is a s-q-h singularity.

3. HYPERSURFACE SINGULARITIES

(3.1) We use the same notation as in Section 2. Let (X, x) be a Gorenstein singularity with contractible X . Let Z be a cycle such that $\mathcal{O}_M(K) \cong \mathcal{O}_M(-Z)$. If (X, x) is not a rational double point, then $Z \geq A$.

Let $\mathcal{C}$ be a sheaf on M defined by an exact sequence

$$0 \rightarrow \mathcal{C} \rightarrow \mathbb{C}_M \rightarrow \mathbb{C}_A \rightarrow 0.$$

If $Z \geq A$, then the exterior differentiation gives an exact sequence (cf. [Wh3, (1.5), (1.6)])

$$(3.1.1) \quad 0 \rightarrow \mathcal{C} \rightarrow \mathcal{O}_M(-Z) \xrightarrow{d} \Omega_M^1(A)(-Z) \xrightarrow{d} \Omega_M^2(-Z + A) \rightarrow 0.$$

As X is contractible, $H^i(\mathcal{C}) = 0$ for all i . Hence $H^i(\mathcal{O}_M(-Z)) \cong H^i(d\mathcal{O}_M(-Z))$ for all i . In particular, $H^i(d\mathcal{O}_M(-Z)) \cong H^i(\mathcal{O}_M(K)) = 0$ for $i \geq 1$.

(3.2) In the rest of this section, we always assume that (X, x) is a complete intersection singularity which is not a rational double point. Let $\mu(X, x)$ and $\tau(X, x)$ denote Milnor number and Tjurina number of (X, x) , respectively. We need the following results of Greuel [Gr1, Gr2] (cf. [LS]).

Proposition 3.3. (1) $\mu(X, x) = h_{\{x\}}^1(d\Omega_X^1)$, and $\tau(X, x) = h_{\{x\}}^1(\Omega_X^1)$ [Gr2, p. 168].

(2) $H_{\{x\}}^q(\Omega_X^p) = 0$ for $p + q \leq 1$ [Gr2, Proposition 2.3].

(3) The following sequences are exact [Gr1, Satz 4.4]:

$$\begin{aligned} 0 &\rightarrow \mathbb{C}_X \rightarrow \mathcal{O}_X \rightarrow d\mathcal{O}_X \rightarrow 0; \\ 0 &\rightarrow d\mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow d\Omega_X^1 \rightarrow 0. \end{aligned}$$

(4) $H_{\{x\}}^0(d\mathcal{O}_X^1) = 0$ [Gr1, Lemma 4.5].

(3.4) From (3.1.1), we have an exact sequence

$$\begin{aligned} 0 \rightarrow H_A^1(d\mathcal{O}_M(-Z)) \rightarrow H_A^1(\Omega_M^1\langle A\rangle(K)) \rightarrow H_A^1(\mathcal{O}_M(2K+A)) \\ \rightarrow H_A^2(d\mathcal{O}_M(-Z)) \rightarrow H_A^2(\Omega_M^1\langle A\rangle(K)). \end{aligned}$$

By Theorem 1.4, we have $h_A^1(\mathcal{O}_M(2K+A)) = \delta_2(X, x)$. By the Serre duality, we have $h_A^1(\Omega_M^1\langle A\rangle(K)) = h^1(\mathcal{S})$. If we set

$$\rho = \dim_{\mathbb{C}} \ker (H_A^2(d\mathcal{O}_M(-Z)) \rightarrow H_A^2(\Omega_M^1\langle A\rangle(K))),$$

then we have

$$(3.4.1) \quad \delta_2(X, x) = h^1(\mathcal{S}) + \rho - h_A^1(d\mathcal{O}_M(-Z)).$$

We note that $h_A^1(d\mathcal{O}_M(-Z)) \leq h^1(\mathcal{S})$.

Let $U = M - A \cong X - \{x\}$.

Lemma 3.5. $h_A^1(d\mathcal{O}_M(-Z)) = h_{\{x\}}^1(d\mathcal{O}_X) + p_g(X, x) - 1$.

Proof. From the exact sequence

$$0 \rightarrow H^0(d\mathcal{O}_M(-Z)) \rightarrow H^0(d\mathcal{O}_U) \rightarrow H_A^1(d\mathcal{O}_M(-Z)) \rightarrow 0,$$

and isomorphisms

$$H^0(d\mathcal{O}_M(-Z)) \cong H^0(\mathcal{O}_M(K)) \cong H^0(f_*\mathcal{O}_M(K)),$$

we see that

$$(3.5.1) \quad H_A^1(d\mathcal{O}_M(-Z)) \cong H^0(d\mathcal{O}_U)/H^0(f_*\mathcal{O}_M(K)).$$

Using (2) and (3) of Proposition 3.3, we obtain $H_{\{x\}}^0(d\mathcal{O}_X) = 0$ and hence

$$(3.5.2) \quad H_{\{x\}}^1(d\mathcal{O}_X) \cong H^0(d\mathcal{O}_U)/H^0(d\mathcal{O}_X).$$

Let $\mathcal{M}$ be an ideal sheaf of $\mathcal{O}_X$ which defines the singular point x . Note that $d\mathcal{O}_X \cong d\mathcal{M}$. Since X is contractible, we have

$$(3.5.3) \quad H^0(\mathcal{M}) \cong H^0(d\mathcal{M}) \cong H^0(d\mathcal{O}_X).$$

As (X, x) is a Gorenstein singularity with $p_g(X, x) \geq 1$, we have $f_*\mathcal{O}_M(K) \subset \mathcal{M}$. It is well known that $p_g(X, x) = \dim_{\mathbb{C}} H^0(\mathcal{O}_X)/H^0(f_*\mathcal{O}_M(K))$ for a Gorenstein singularity (X, x) . From (3.5.1), (3.5.2) and (3.5.3), we have the following

$$\begin{aligned} h_A^1(d\mathcal{O}_M(-Z)) - h_{\{x\}}^1(d\mathcal{O}_X) &= \dim_{\mathbb{C}} H^0(d\mathcal{O}_X)/H^0(f_*\mathcal{O}_M(K)) \\ &= \dim_{\mathbb{C}} H^0(\mathcal{M})/H^0(f_*\mathcal{O}_M(K)) = p_g(X, x) - 1. \quad \square \end{aligned}$$

Lemma 3.6. $\rho = \mu(X, x) - \tau(X, x) + h_{\{x\}}^1(d\mathcal{O}_X)$.

Proof. Since $H^1(d\mathcal{O}_M(-Z)) = H^2(d\mathcal{O}_M(-Z)) = 0$, we have

$$H_A^2(d\mathcal{O}_M(-Z)) \cong H^1(d\mathcal{O}_U) \cong H_{\{x\}}^2(d\mathcal{O}_X).$$

By the vanishing theorem of Wahl [Wh1], $H^1(\Omega_M^1\langle A\rangle(K)) = 0$. Similarly, we get

$$H_A^2(\Omega_M^1\langle A\rangle(K)) \cong H_{\{x\}}^2(\Omega_X^1).$$

Then

$$\rho = \dim_{\mathbb{C}} \ker \left(H_{\{x\}}^2(d\mathcal{O}_X) \rightarrow H_{\{x\}}^2(\Omega_X^1) \right).$$

From Proposition 3.3, $H_{\{x\}}^0(d\Omega_X^1) = 0$ and we have an exact sequence

$$0 \rightarrow H_{\{x\}}^1(d\mathcal{O}_X) \rightarrow H_{\{x\}}^1(\Omega_X^1) \rightarrow H_{\{x\}}^1(d\Omega_X^1) \rightarrow H_{\{x\}}^2(d\mathcal{O}_X) \rightarrow H_{\{x\}}^2(\Omega_X^1),$$

and hence $\rho = \mu(X, x) - \tau(X, x) + h_{\{x\}}^1(d\mathcal{O}_X)$. $\square$

Theorem 3.7. $\delta_2(X, x) = h^1(\mathcal{S}) + \mu(X, x) - \tau(X, x) - p_g(X, x) + 1$.

Proof. The theorem is immediately obtained from (3.4.1), Lemma 3.5 and Lemma 3.6. $\square$

Corollary 3.8. Let $\pi: \bar{X} \rightarrow T$ be an equisingular deformation of (X, x) . We set $X_t = \pi^{-1}(t)$ for $t \in T$. Then

$$\tau(X_t) \geq \mu(X, x) - \delta_2(X, x)$$

for any $t \in T$. In particular, if $p_g(X, x) = 1$, then $\tau(X_t) \geq \mu(X, x) - 5$.

Proof. We note that X_t is a complete intersection isolated singularity for any $t \in T$ (cf. [KS]). From (3.4) and Lemma 3.5, $h^1(\mathcal{S}) \geq p_g - 1$. By Theorem 3.7, we have that $\delta_2(X_t) \geq \mu(X_t) - \tau(X_t)$. By Theorem 1.4, δ_2 is determined by p_g and the weighted dual graph of the singularity, and so is μ by [St, (2.26)]. The property of the equisingular deformations implies that $\delta_2(X_t) = \delta_2(X, x)$ and $\mu(X_t) = \mu(X, x)$. Then we get the first formula. If $p_g(X, x) = 1$, then $\delta_2(X, x) \leq 5$ by Corollary 1.5. $\square$

(3.9) For the remainder of this section, (X, o) denotes a hypersurface singularity defined by a function $h \in \mathbb{C}\{z_0, z_1, z_2\} = \mathcal{O}_{\mathbb{C}^3, o}$. It is well known that

$$\mu(X, o) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3, o} / J_h \quad \text{and} \quad \tau(X, o) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3, o} / (J_h, h),$$

and that $\mu(X, o) = \tau(X, o)$ if and only if h is q-h (after a change of coordinates).

We set $\mu = \mu(X, o)$. Let $\varphi_1, \dots, \varphi_\mu$ be functions in $\mathcal{O}_{\mathbb{C}^3, o}$ which induce $\mathbb{C}$ -basis of $\mathcal{O}_{\mathbb{C}^3, o}/J_h$. Then we define a function $H(z, t) \in \mathbb{C}\{z_0, z_1, z_2, t_1, \dots, t_\mu\} = \mathcal{O}_{\mathbb{C}^3 \times \mathbb{C}^\mu, o}$ by

$$H(z, t) = h + \sum_{i=1}^{\mu} t_i \varphi_i,$$

and we set

$$Y(X, o) = \{ (t_0) \in (\mathbb{C}^\mu, o) \mid \mu(H(z, t_0)) = \mu \},$$

where $\mu(H(z, t_0))$ denotes Milnor number of the singularity defined by $H(z, t_0)$. Then $Y(X, o)$ is an analytic subset of $(\mathbb{C}^\mu, o)$.

Definition 3.10. The modality $m(X, o)$ of the singularity (X, o) is the dimension of $Y(X, o)$ (cf. [Ga]). If (X, o) is defined by a quasi-homogeneous polynomial h of degree d , then the inner modality $m_0(X, o)$ of the singularity (X, o) is defined as the dimension of the vector space $\bigoplus_{i \geq d} Q_h(i)$ (cf. [YW]). Note that $m_0(X, o) \leq m(X, o)$ if (X, o) is a q-h singularity (see the proof of the follow).

Proposition 3.11. If $p_g(X, o) = 1$, then $\delta_2(X, o) \leq m(X, o)$.

If (X, o) is a q-h singularity, then $\delta_2(X, o) = m_0(X, o) \leq 4$.

Proof. Let $(\mathbb{C}^{\tau(X, o)}, o)$ be the versal deformation space of the singularity (X, o) and

$$p: (\mathbb{C}^{\mu(X, o)}, o) \rightarrow (\mathbb{C}^{\tau(X, o)}, o)$$

be a projection corresponding to the natural map of the tangent spaces

$$\mathcal{O}_{\mathbb{C}^3, o}/J_h \rightarrow \mathcal{O}_{\mathbb{C}^3, o}/(J_h, h).$$

There is a submanifold P of $(\mathbb{C}^{\tau(X, o)}, o)$ which represents ES_M . By the property of the equisingular deformations, $p^{-1}(P) \subset Y(X, o)$. By Theorem 2.2, we see that the dimension of $p^{-1}(P)$ is $h^1(\mathcal{S}) + \mu(X, o) - \tau(X, o)$. Hence

$$h^1(\mathcal{S}) + \mu(X, o) - \tau(X, o) \leq m(X, o).$$

From Theorem 3.7, we get $\delta_2(X, o) \leq m(X, o)$.

We assume that h is a q-h polynomial of degree d . Then Theorem 3.7 and 2.2, and (2.6) implies that $\delta_2(X, o) = h^1(\mathcal{S}) = \dim_{\mathbb{C}} \bigoplus_{i \geq d} Q_h(i) = m_0(X, o)$. By Corollary 1.5, $\delta_2(X, o) \leq 4$. $\square$

Remark 3.12. If the invariance of Milnor number implies the invariance of the topological type for two dimensional hypersurface singularities (cf. [LR]), then, in the proof above, we have $p^{-1}(P) = Y(X, o)$. In this case, $Y(X, o)$ is nonsingular, and $\delta_2(X, o) = m(X, o)$ holds.

Proposition 3.13. *Let (X, o) be a singularity defined by a s-q-h function $h \in \mathcal{O}_{\mathbb{C}^3, o}$ with weights $(1, 1, 1)$. Then $\delta_2(X, o) \geq m(X, o)$.*

Proof. We write $h = h_0 + h_1$ as in Definition 2.5. Let (X_0, o) be a singularity defined by h_0 . Then by [GK], $m_0(X_0, o) = m(X_0, o)$. Hence we have that $\delta_2(X_0, o) \geq m(X_0, o)$ by [YW]. On the other hand, (X, o) is a fibre in an equisingular deformation of (X_0, o) by (2.6). Thus $\delta_2(X, o) = \delta_2(X_0, o)$. Since the modality is upper semi-continuous by [Ga], we have $\delta_2(X, o) = \delta_2(X_0, o) \geq m(X_0, o) \geq m(X, o)$. $\square$

Proposition 3.14. *If $p_g(X, o) = 1$, $\delta_2(X, o) \leq 2$ and the weighted dual graph of (X, o) is a star-shaped graph, then $\delta_2(X, o) = m(X, o)$.*

Proof. We know that (X, o) is a s-q-h singularity by (2.9). Let us use the notation in the proof of Proposition 3.13. Then $\delta_2(X, o) = \delta_2(X_0, o) = m(X_0, o)$ by Proposition 3.11, and $p_g(X, o) = 1$ Q-h hypersurface singularities with $p_g = 1$ and $m_0 \leq 4$ are listed in [YW]. The lists of all the singularities for which $m \leq 2$ are given in [AGV, 15.1]. Then we can see that for a s-q-h function of which the q-h part has inner modality $m_0 \leq 2$, we have $m = m_0$. Thus $m(X, o) = m_0(X_0, o) = \delta_2(X_0, o) = \delta_2(X, o)$. $\square$

(3.15) We can classify the weighted dual graphs of minimally elliptic singularities with $\delta_2 \leq 2$. In the following, the symbol “ $\bigcirc$ ” corresponds to a component with self-intersection number -2 and “ $\square_i$ ” corresponds to a component A_i . We set $b_i = -A_i \cdot A_i$.

Proposition 3.16 (cf. [WO]). *Let (X, x) be a minimally elliptic singularity with $\delta_2(X, x) \leq 2$.*

(1) $\delta_2(X, x) = 1$ if and only if (X, x) is a simple elliptic, cusp singularity or a singularity with the weighted dual graph

$$D_{b_1, b_2, b_3} : \quad \square_1 - \square_0 - \square_3 \quad \begin{array}{c} \square_2 \\ | \\ \square_0 \end{array}$$

Where $b_0 = 1 < b_1 \leq b_2 \leq b_3$ and $1/b_1 + 1/b_2 + 1/b_3 < 1$.

(2) $\delta_2(X, x) = 2$ if and only if the weighted dual graph of (X, x) is one of the following.

$$\tilde{E}_6 : \quad \square_1 - \bigcirc - \bigcirc - \bigcirc - \square_3 \quad \begin{array}{c} \square_2 \\ | \\ \bigcirc \end{array} \quad 2 \leq b_1 \leq b_2 \leq b_3, 2 < b_3$$

$$\tilde{E}_7: \quad \square_1 - \bigcirc - \bigcirc - \bigcirc - \bigcirc - \bigcirc - \square_2 \quad 2 \leq b_1 \leq b_2, 2 < b_2$$

$$\tilde{E}_8: \quad \begin{array}{c} \circ \\ | \\ \circ - \circ - \circ - \circ - \circ - \circ - \square_1 \end{array} \quad 2 < b_1$$

$$\tilde{D}_4: \begin{array}{c} \square_2 \\ | \\ \square_1 - \bigcirc - \square_3 \\ | \\ \square_4 \end{array} \quad 2 \leq b_1 \leq b_2 \leq b_3 \leq b_4, 2 < b_4$$

$\tilde{D}_{i+4} (i \geq 1):$

 $2 \leq b_1 \leq b_2, 2 \leq b_3 \leq b_4, 2 < b_4$
 The number of "○" is $i + 1$.

(3) The list of the (b_i) corresponding to a hypersurface is the following.

$type$	(b_i)
D_{b_1, b_2, b_3}	$(2.3.7), (2.3.8), (2.3.9), (2.4.5), (2.4.6), (2.4.7), (2.5.5), (2.5.6)$ $(3.3.4), (3.3.5), (3.3.6), (3.4.4), (3.4.5), (4.4.4)$
$\tilde{E}_6$	$(2.2.3), (2.2.4), (2.2.5), (2.3.3), (2.3.4), (3.3.3),$
$\tilde{E}_7$	$(2.3), (2.4), (2.5), (3.3), (3.4)$
$\tilde{E}_8$	$(3), (4), (5)$
$\tilde{D}_4$	$(2.2.2.3), (2.2.2.4), (2.2.2.5), (2.2.3.3)$ $(2.2.3.4), (2.3.3.3)$
$\tilde{D}_{i+4} \ (i \geq 1)$	$(2.2.2.3), (2.2.2.4), (2.2.2.5), (2.2.3.3)$ $(2.3.2.3), (2.2.3.4), (2.3.2.4), (2.3.3.3)$

Corollary 3.17. *Let (X, o) be a hypersurface singularity. Then $\delta_2(X, o) = 1$ if and only if $m(X, o) = 1$.*

Remark 3.18. Minimally elliptic singularities with $\delta_2 \leq 2$ are Kodaira singularities (cf. [Kr]).

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