

On the Teichmüller spaces of Fuchsian groups of Schottky type and the Schwarzian derivatives of univalent functions

TOSHIYUKI SUGAWA 須川 敏幸 (京大)

Department of Mathematics, Faculty of Science, Kyoto University

§1. The main result.

Let Γ be a Fuchsian group acting on the upper half plane $\mathbb{H}$. We denote by $B_2(\Gamma)$ the Banach space of all the holomorphic function φ on $\mathbb{H}$ which satisfies the functional equation $(\varphi \circ \gamma)(\gamma')^2 = \varphi$ for all $\gamma \in \Gamma$, with finite norm $\|\varphi\| = \sup_{z \in \mathbb{H}} |\varphi(z)|(\operatorname{Im} z)^2$. We shall consider the following subsets of $B_2(\Gamma)$:

$$S(\Gamma) = \{\varphi \in B_2(\Gamma) : \exists \text{ univalent function } f \text{ on } \mathbb{H} \text{ with } S_f = \varphi\},$$

$$T(\Gamma) = \{S_f \in S(\Gamma) : f \text{ extends to a } (\Gamma\text{-compatible}) \text{ qc-map of } \widehat{\mathbb{C}},$$

where S_f denotes the Schwarzian derivative of f defined as follows: $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$.

It is known that $S(\Gamma)$ is closed and $T(\Gamma)$ is open in $B_2(\Gamma)$. $T(\Gamma)$ is called (the Bers model of) the Teichmüller space of Γ . It is an interesting problem how near $T(\Gamma)$ is to $S(\Gamma)$. For a cofinite Fuchsian group (i.e., finitely generated Fuchsian group of the first kind) Γ , the statement $\overline{T(\Gamma)} = S(\Gamma)$ is equivalent to the Bers conjecture: every B-group is obtained as a boundary group of Teichmüller space. (This conjecture is still now unsolved.)

On the other hand, for any Fuchsian group Γ of the second kind, it is known that $\overline{T(\Gamma)} \subsetneq S(\Gamma)$ (cf. [G2], [Sug]).

But a weaker statement that $T(\Gamma) = \operatorname{Int} S(\Gamma)$ is proved for some cases ([G1: $\Gamma = 1$], [Shiga: cofinite Γ]). The main result of this article is the validity of the above statement for all Fuchsian groups of Schottky type, where a Fuchsian group Γ is called Schottky type in this article, if Γ is a Schottky group simultaneously, in other words, Γ uniformizes a topologically finite Riemann surface of genus g with m holes, where $m \geq 1$. Also, the Schottky type Fuchsian group can be characterized as the finitely generated, purely hyperbolic Fuchsian group of the second kind.

MAIN THEOREM. $\operatorname{Int} S(\Gamma) = T(\Gamma)$ for any Fuchsian group Γ of Schottky type.

§2. Sketch of proof.

Let Γ be a Fuchsian group of Schottky type. Then, the quotient surface $S_0 = \mathbb{H}/\Gamma$ is a topologically finite Riemann surface of genus g with m holes and its double $S = \Omega(\Gamma)/\Gamma$ is a compact Riemann surface of genus $N = 2g + m - 1$, where $\Omega(\Gamma) \subset \widehat{\mathbb{C}}$ denotes the region of discontinuity of Γ . Let $\varphi \in \operatorname{Int} S(\Gamma)$ and $F : \mathbb{H} \rightarrow \widehat{\mathbb{C}}$ be a holomorphic map such that $S_F = \varphi$. By the Γ -automorphy of

φ , $G = F\Gamma F^{-1}$ is a subgroup of Möb which acts on $D = f(\mathbb{H})$. Since φ is an interior point of $S(\Gamma)$, it turns out that G is purely loxodromic. Since $G(\cong \Gamma)$ is a free group of finite rank, Maskit's characterization theorem tells us that G is also a Schottky group of rank $N = 2g + m - 1$. So, the quotient surface $R = \Omega(G)/G$ is a compact genus N surface. Let $p_0 : \Omega(\Gamma) \rightarrow S$ and $p : \Omega(G) \rightarrow R$ be the natural projections. Set $R_0 = p(D) = D/G$, which is isomorphic to $S_0 = \mathbb{H}/\Gamma$ by the conformal map f induced by $F : \mathbb{H} \rightarrow D$. We shall investigate the way of embedding $R_0 \hookrightarrow R$. Now, the proof of Main Theorem divides into several steps.

STEP 1. ∂R_0 consists of mutually disjoint m simple closed curves.

This step needs a localization of Gehring's method [G1]. In this step, essential is the fact that φ is an interior point of $S(\Gamma)$.

STEP 2. There exists a self-homeomorphism h of R with the following properties:

- (i) $h \circ h = \text{id}_R$,
- (ii) $h|_{\partial R_0} = \text{id}_{\partial R_0}$,
- (iii) $h(R_0) \cap R_0 = \emptyset$,
- (iv) there exists a homeomorphism $H : \Omega(G) \rightarrow \Omega(G)$ such that $p \circ H = h \circ p$ on $\Omega(G)$.

This step is covered by rather algebraic arguments. For example, the following lemma is utilized.

LEMMA (GENERAL PROPERTY OF THE NORMAL COVERINGS).

Suppose that $p : (\Omega, z_0) \rightarrow (R, a_0)$ is a normal covering between (connected) pointed manifolds. Let R_0 be a subdomain of R such that $a_0 \in R_0$ and $\iota : R_0 \rightarrow R$ denote the inclusion map. Then ι naturally induces the homomorphism $\iota_* : \pi_1(R_0, a_0) \rightarrow \pi_1(R, a_0)$. Let $\lambda : \pi(R, a_0) \rightarrow G$ be the lifting homomorphism with respect to z_0 , where G is a covering transformation group of $p : \Omega \rightarrow R$. Namely, $g = \lambda[\alpha]$ for $g \in G$ and $[\alpha] \in \pi_1(R, a_0)$ iff the final point of the lift $\tilde{\alpha}$ of α with initial point z_0 coincides with $g(z_0)$. Then, the followings hold.

- (i) Each component of $p^{-1}(R_0)$ is simply connected $\iff \lambda \circ \iota_*$ is injective.
- (ii) $p^{-1}(R_0)$ is connected $\iff \lambda \circ \iota_*$ is surjective.

In particular, if $p^{-1}(R_0)$ is a simply connected domain, then $\iota_* : \pi_1(R_0, a_0) \rightarrow \pi_1(R, a_0)$ is an embedding and $\pi_1(R, a_0) = \ker \lambda \rtimes \pi_1(R_0, a_0)$ (semi-direct product).

First of all, we can naturally extend f to a homeomorphism $f : \overline{S_0} \rightarrow \overline{R_0}$ by Step 1. Further, by use of Step 2, we can extend f to a homeomorphism $\tilde{f} : S \rightarrow R$

in the following way.

$$\tilde{f} = \begin{cases} f & \text{on } \overline{S_0}, \\ h \circ f \circ j & \text{on } S \setminus \overline{S_0}, \end{cases}$$

where j denotes the involution map $S \rightarrow S$ induced by conjugation $J(z) = \bar{z}$. By construction, $\tilde{f}$ can be lifted, that is, there exists a homeomorphism $\tilde{F} : \Omega(\Gamma) \rightarrow \Omega(G)$ such that $p \circ \tilde{F} = \tilde{f} \circ p_0$. By purely topological arguments, it turns out that $\tilde{F}$ can be naturally extended to a homeomorphism $\tilde{F} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. In particular, it is known that D is an image of $\mathbb{H}$ under the self-homeomorphism $\tilde{F}$ of $\hat{\mathbb{C}}$, so D is a Jordan domain.

STEP 3. ∂R_0 is a disjoint union of quasi-analytic curves.

Here, the “quasi-analytic curve” means the quasiconformal image of a circle. For the proof of Step 3, we need just more delicate arguments than in Step 1. By the way, one can prove the following

PROPOSITION. *Let S and R be compact Riemann surfaces and $S_0 \subset S, R_0 \subset R$ be subdomains with quasi-analytic boundaries. Suppose that $\tilde{f} : S \rightarrow R$ is an orientation preserving homeomorphism such that $\tilde{f}(S_0) = R_0$ and the restriction map $\tilde{f}|_{S_0} : S_0 \rightarrow R_0$ is quasiconformal. Then, there exists a quasiconformal map $\tilde{f}_1 : S \rightarrow R$ which is homotopic to $\tilde{f}$ and $\tilde{f}_1 = \tilde{f}$ on R_0 .*

By virtue of this proposition, we can choose a quasiconformal $\tilde{f} : S \rightarrow R$ as the extension of f . Then, a topological extension $\tilde{F} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ of a lift of $\tilde{f}$ is quasiconformal on $\Omega(\Gamma)$, so $\tilde{F} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a quasiconformal self-homeomorphism since $\Lambda(\Gamma) = \hat{\mathbb{C}} \setminus \Omega(\Gamma) \subset \hat{\mathbb{R}}$ is a quasiconformally removable set. Therefore $D = \tilde{F}(\mathbb{H})$ is a quasi-disk, the proof is completed.

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