

Tricanonical map of a certain class of surfaces

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INTRODUCTION. Let S be an projective algebraic surface defined over the complex number field C . We let K_S denote the canonical bundle of S , and mK_S its m -th tensor power. Consider the rational map Φ_{mK_S} associated with the complete linear system $|mK_S|$ (pluricanonical map). S is called of general type if $\Phi_{mK_S}(S)$ is a surface for $m \gg 0$. Putting $R = \sum_{m=0}^{\infty} H^0(S, \mathcal{O}(mK_S))$, the projective scheme $X = \text{Proj}(R)$ is called an (abstract) canonical model of S . It is known that the natural rational map

$$X \rightarrow \Phi_{mK_S}(S) \approx \text{Proj}(\mathcal{S}H^0(S, mK_S))$$

is an isomorphism for $m \gg 0$, and that the rational map $S \rightarrow X$ is a birational morphism.

In this paper, we are concerned with the algebraic surface S whose numerical characters are: $K_S^2 = 1$, $p_g = 0$, where p_g is the geometric genus. We shall prove the following

MAIN THEOREM. Φ_{3K_S} is birational.

Let S be a minimal surface of general type.

By this theorem Bombieri's result about the birationality of pluricanonical maps [3] is sharpened as follows:

Φ_{3K_S} is birational except in the following cases:

- a) $K_S^2 = 1$, $p_g = 2$, where $\Phi_{3K_S}(S)$ is rational;
- b) $K_S^2 = 2$, $p_g = 3$, where $\Phi_{3K_S}(S) = P^2$;
- c) $K_S^2 = 2$, $p_g = 0$. (It is expected that the case c) does not occur.)

1. Generalities. In **this** section we review the well-known results that are used in our proof.

THEOREM A (algebraic index theorem). Let S be an algebraic surface. The intersection numbers for pairs of divisor^s define a quadratic form Q on the numerical divisor group $\text{Num}(S)$. Q is non-degenerate and has one and only one **positive** eigenvalue.

THEOREM B. Let S be a minimal surface of general type. Then for any irreducible curve C on S we have

$$K_S C \geq 0.$$

Moreover, the curves C satisfying $K_S C = 0$ form a finite set and are numerically independent of each other.

Let D be an effective divisor on S . We say that D is numerically connected (or 1-connected) if for any non-trivial decomposition $D = D_1 + D_2$, $D_i > 0$, we have $D_1 D_2 > 0$.

THEOREM C (Ramanujam). If an effective divisor D is 1-connected, then $\dim H^0(D, \mathcal{O}_D) = 1$.

Let S be a minimal surface of general type and X a canonical model of S . The natural map $w_0: S \rightarrow X$ is a minimal resolution of singularities of X . X is a normal surface with a finite number of rational double points. Let $\mathcal{M}$ be the maximal ideal of a rational double point P on X . $w_0^* \mathcal{M}$ is an invertible sheaf that defines a divisor Z . Z is called a fundamental cycle. Z is a sum of irreducible curves C_i such that $C_i K_S = 0$. Conversely such curves are contained in some fundamental cycles.

PROPOSITION 1 (Artin [1] [2]).

(i) An effective divisor Z on S is a fundamental cycle if and only if Z is a maximal cycle with

$$K_S Z = 0, \quad Z^2 = 0.$$

(ii) We have $K_S = \omega_0^* K_X$, where K_X is a line bundle on X .

(iii) For two line bundles δ_1 and δ_2 on a fundamental cycle, δ_1 and δ_2 are isomorphic to each other if $\deg \delta_1 = \deg \delta_2$.

We shall denote the numerical equivalence by the symbol $\sim$.

Thus $D \sim D'$ means that D is numerically equivalent to D' .

Let S be a minimal surface of general type, Z a fundamental cycle, $\omega: \tilde{S} \rightarrow S$ a blowing up, and E the exceptional curve on $\tilde{S}$.

LEMMA 1 (Bombieri's connectedness theorem).

(i) If D is effective and $D \sim mK_S$ ($m \geq 1$), then D is 1-connected.

(ii) If D is effective and $D \sim mK_S$ ($m \geq 2$), then for any decomposition $D = D_1 + D_2$, $D_i > 0$, $D_i K_S > 0$, we have $D_1 D_2 \geq 3$, except if $K_S^2 = 1$ and D_1 or $D_2 \sim K_S$.

(iii) If $D \sim mK_S - Z$ ($m \geq 1$), then D is 1-connected.

(iv) If $D \sim m\omega^* K_S - 2E$ ($m \geq 1$), then D is 1-connected except if $K_S^2 = 1$, $m = 2$, $D = D_1 + D_2$, $D_1 \sim D_2 \sim \omega^* K_S - E$.

PROOF. For the convenience of the reader, we shall give a proof following Bombieri [3]. We discuss in the rational numerical group $\text{Num}_Q(S) = \text{Num}(S) \otimes Q$. We let $t = K_S^2$.

(i) Let $D = D_1 + D_2$, $D_i > 0$ be a decomposition of D . We have

$$0 \leq r = D_1 K_S \leq mt. \quad \text{Hence}$$

$$D_1 = \frac{r}{t} K_S + \xi, \quad \xi \cdot K_S = 0,$$

$$D_2 = \frac{1}{t} (mt - r) K_S - \xi,$$

$$D_1 D_2 = \frac{1}{t} r (mt - r) - \xi^2.$$

From Theorem A we infer $\xi^2 < 0$ except if $\xi \sim 0$. $D_1 D_2 \geq 0$ implies that $r = 0$ or $r = mt$, and $\xi \sim 0$. Therefore we have D_1 or $D_2 \sim 0$, i.e.,

D_1 or $D_2 = 0$.

(ii) In this case $1 \leq r \leq mt-1$. Suppose $2 \leq r \leq mt-2$. Then

$$D_1 D_2 = \frac{1}{t} \cdot r(mt-r) - \xi^2 \geq \frac{1}{t} \cdot 2(mt-2) = 2m - \frac{2}{t}.$$

Since $mt \geq 4$, $2m - \frac{2}{t} \geq 3$. Next consider the case $r = D_1 K_S = 1$.

Then $D_1^2 = \frac{1}{t} + \xi^2 = \text{odd}$. Hence, unless $t=1$ and $\xi \sim 0$, $D_1^2 \leq -1$.

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(iii) Let $D = D_1 + D_2$, $D_i > 0$. ~~If~~ $D_i K_S > 0$ and if $D_i \not\sim K$, we get $D_1(D_2 + Z) \geq 3$, $D_2(D_1 + Z) \geq 3$. Summing up these, we have $2D_1 D_2 + DZ \geq 6$. On the other hand, $DZ = -Z^2 = 2$. Hence $D_1 D_2 \geq 2$. If $D_i K_S > 0$ and if $D_1 \sim K_S$, we have $D_1 D_2 = K_S((m-1)K_S - Z) = (m-1) \geq 1$. If $D_i K_S = 0$, we have $D_1^2 \leq -2$, $D_2 \sim mK_S + \xi$, $\xi^2 \leq 0$. Hence $2D_1 D_2 = (mK_S - Z)^2 - D_1^2 - D_2^2 \geq -\xi^2$. $D_1 D_2 = 0$ implies that $\xi \sim 0$ and that $D_2 \sim mK_S$, $D_1 \sim -Z$, a contradiction.

(iv) Let $D = D_1 + D_2$, $D_i > 0$, and $v = D_1 E$. Then we have

$$D_1 \sim w^* D'_1 - vE, \quad D'_1 \sim \frac{r}{t} K_S + \xi.$$

$$D_2 \sim w^* D'_2 - (2-v), \quad D'_2 \sim (m - \frac{r}{t}) K_S - \xi.$$

Note that D'_i is an effective divisor on S . So $D'_1 D'_2 \geq 1$, and

$$D_1 D_2 \geq 1 - v(v-2)E^2 = 1 + v(v-2).$$

Hence $D_1 D_2 > 0$ unless $v=1$. Suppose $D'_i K_S > 0$ and $v=1$. In this

case $D'_1 D'_2 \geq 3$ unless D'_1 or $D'_2 \sim K_S$. Hence $D_1 D_2 > 1$. Suppose

$D'_1 \sim K_S$. Then $D_1 \sim w^* K_S - E$, $D_2 \sim w^* (m-1)K_S - E$, $D_1 D_2 = (m-1)t - 1$.

Thus $D_1 D_2 = 0$ if and only if $m=2$, $t=K_S^2=1$, $D_1 \sim w^* K_S - E$. Finally

suppose that $D'_i K_S = 0$. Then $D_1^2 \leq -2$, and so $D'_1 D'_2 = -D_1^2 \geq 2$.

Hence, $D_1 D_2 = D'_1 D'_2 + v(v-2) \geq D'_1 D'_2 - 1 \geq 1$.

Q.E.D.

THEOREM D. Let S be as in Lemma 1 and let $\mathcal{L}$ an invertible sheaf such that $\mathcal{L}^n$ is spanned by its global sections and has three algebraically independent sections for $n \gg 0$. Then we have

$$H^1(S, \mathcal{L}^{-1}) = 0.$$

For the proof, see Mumford [1].

COROLLARY. If a divisor $M \sim mK_S$ ($m \geq 2$), then $H^1(S, M) = 0$.

PROOF. For $n \gg 0$, consider the exact sequence

$$0 \rightarrow H^0(\tilde{S}, n\omega^*M - E) \rightarrow H^0(\tilde{S}, n\omega^*M) \rightarrow H^0(E, n\omega^*M) \rightarrow H^1(\tilde{S}, n\omega^*M - E).$$

By the Serre duality theorem we have

$$\dim H^1(\tilde{S}, n\omega^*M - E) = \dim H^1(\tilde{S}, 2E - (n-1)\omega^*M).$$

Since $D \sim (n-1)\omega^*M - 2E$ is 1-connected by Lemma 1, $H^1(\tilde{S}, 2E - (n-1)\omega^*M) = 0$. Thus $H^1(\tilde{S}, n\omega^*M - E) = 0$. Hence $|nM|$ has no base point ($n \gg 0$). Now apply the theorem. Q.E.D.

2. Numerical Godeaux surfaces. We call S a numerical Godeaux surface if it is a minimal surface of general type with numerical characters $K_S^2=1$, $p_g=0$, where p_g denotes the geometric genus $\dim H^2(S, \mathcal{O}_S)$. The following theorem is classical (see [7]).

THEOREM 1. For a surface of general type S , we have $p_g \geq q = \dim H^1(S, \mathcal{O}_S)$.

In what follows, we denote by S a numerical Godeaux surface. As a corollary to Theorem C and Theorem 1, we obtain the following

LEMMA 2. If an effective divisor D is 1-connected, then $H^1(S, \mathcal{O}_S(-D))=0$.

LEMMA 3. If $D \sim K_S$, we have $\dim H^0(S, \mathcal{O}_S(D)) \leq 1$.

PROOF. By the Riemann-Roch theorem and the corollary to Theorem D, we have $\dim H^0(S, \mathcal{O}_S(2D))=2$. Suppose that $\dim H^0(S, \mathcal{O}_S(D)) \geq 2$. Then we have $\dim H^0(S, \mathcal{O}_S(2D)) \geq 3$, a contradiction. Q.E.D.

LEMMA 4. If an effective divisor $D \sim K$, we have $H^1(S, \mathcal{O}_S(D))=0$.

PROOF. We may assume that D is not linearly equivalent to K_S . Since $\dim H^0(S, \mathcal{O}_S(D)) \leq 1$ by Lemma 3 and since $\chi(S, \mathcal{O}_S(D)) = \chi(S, \mathcal{O}_S(K_S)) = 1$, we have

$$\begin{aligned} \dim H^1(S, \mathcal{O}_S(D)) &= -\chi(S, \mathcal{O}_S(D)) + \dim H^0(S, \mathcal{O}_S(D)) + \dim H^2(S, \mathcal{O}_S(D)) \\ &\leq \dim H^2(S, \mathcal{O}_S(D)) = 0. \end{aligned} \quad \text{Q.E.D.}$$

Remark. By the vanishing of q , we know that the linear equivalence coincides with the algebraic equivalence. Hence if $\dim H^0(S, \mathcal{O}_S(D))=1$ ($D > 0$), then there exists no effective divisor

algebraically equivalent to D except D itself. Lemma 4 implies that for any non-zero $\tau \in H^2(S, \mathbb{Z})_{\text{tor}}$, there is one and only one effective divisor D' which is algebraically equivalent to $K_S + \tau$.

LEMMA 5. Let D be an effective divisor and assume that $\dim |D| \geq 1$. Then $DK_S \geq 2$.

PROOF. By Theorem B, we may assume that $|D|$ is fixed part free. Hence we have $D^2 \geq 0$. This implies that $DK_S \geq 2$ or $D \sim K_S$. But the latter case is impossible (see the Remark above).

Q.E.D.

We let M denote the generic member of the moving part of the dicanonical system $|2K_S|$, F the fixed part of $|2K_S|$. Thus $|2K_S| = |M| + F$. From the Riemann-Roch theorem we infer that $|M|$ is composed of a pencil over a projective line P^1 .

LEMMA 6. If M is generically chosen, M is reduced and irreducible. Moreover, M and F satisfy one of the following numerical conditions:

- a) $F=0$;
- b) $FK_S=0$, $F^2=-2$, $M^2=2$, $MF=2$;
- c) $FK_S=0$, $F^2=-4$, $M^2=0$, $MF=4$.

PROOF. First we note that $MK_S \geq 2$ in virtue of Lemma 5; in other words, $FK_S=0$. Suppose generic M admits a non-trivial decomposition $M=M_1+M_2$, $M_i > 0$. Since the M_i can move, we have $M_i K_S \geq 2$, which is a contradiction. From $FK_S=0$ follows $F^2 \leq 0$ unless $F=0$. On the other hand,

$$F^2 = -FM = M^2 - 2MK_S \geq -2MK_S = -4.$$

Q.E.D.

Remark. If D is a divisor numerically equivalent to $2K_S$, then $\dim |D| = 1$. The similar argument in Lemma 6 is valid for $|D|$ in place of $|2K_S|$. Thus the fixed part D_0 of D satisfies $D_0 K_S = 0$, and the generic member of the moving part is an irreducible curve.

From the lemma above we obtain the following

COROLLARY. Let X denote the canonical model of S . $|2K_X|$ has no fixed part and its generic member is irreducible.

Let $\hat{M}$ be a generic member of $|2K_X|$, and consider the natural exact sequence

$$0 \rightarrow \mathcal{O}_X(-2K_X) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\hat{M}} \rightarrow 0.$$

Since X has at most a finite number of rational singularities and $\mathcal{O}_X(-2K_X)$ is an invertible sheaf, we have canonical isomorphisms

$$\begin{aligned} H^i(X, \mathcal{O}_X) &\cong H^i(S, \mathcal{O}_S), \\ H^i(X, \mathcal{O}_X(-2K_X)) &\cong H^i(S, \mathcal{O}_S(-2K_S)). \end{aligned}$$

Hence $\hat{M}$ is an irreducible curve of virtual genus $\pi(\hat{M}) = 4$.

COROLLARY. F is a disjoint union of fundamental cycles.

PROOF. Let Z denote fundamental cycle such that $Z \cap F \neq \emptyset$.

Since $2K_S$ is trivial on Z , we have $\text{supp } F \supset Z$. If $F^2 = -2$, then F is a fundamental cycle (see Proposition 1). Assume that $F^2 = -4$. It is sufficient to prove that F is not connected in this case. Since $M^2 = 0$, M is base point free. If M is generic, M is a non-singular curve of genus 2. If F is connected, $\hat{M} = \omega_0(M)$ has a 4-ple point. Hence $\pi(\hat{M}) = 2 + 3 = 5$. This contradicts the above corollary. Q.E.D.

Now we proceed to the study of the tricanonical system $|3K_S|$.

LEMMA 7 (Bombieri). If $CK_S=0$, then C is not contained in the fixed part of $|3K_S|$.

PROOF. Note that

$\dim H^0(S, \mathcal{O}_S(2K_S - Z)) \geq \dim H^0(S, \mathcal{O}_S(2K_S)) - \dim H^0(Z, \mathcal{O}_Z) = 1$,
and a fortiori $|2K_S - Z|$ contains an effective divisor D . D is numerically connected (see Lemma 1). So $\dim H^1(S, \mathcal{O}_S(Z - 2K_S)) = 0$. If C is contained in the fixed part G of $3K_S$, G must also contain the fundamental cycle Z to which C belongs. Hence we have the canonical isomorphism

$$H^0(S, \mathcal{O}_S(3K_S - Z)) \xrightarrow{\cong} H^0(S, \mathcal{O}_S(3K_S)).$$

This implies that

$$\dim H^1(S, \mathcal{O}_S(3K_S - Z)) = \dim H^1(S, \mathcal{O}_S(Z - 2K_S)) \neq 0.$$

This is absurd.

Q.E.D.

LEMMA 8. $|3K_S|$ is not composed of a pencil.

PROOF. Suppose the contrary. $\Phi_{3K_S}(S)$ is a space curve of $\deg \geq 3$. Let $\hat{S} \xrightarrow{w} S$ be the resolution of the base points of $3K_S$. The moving part of $|3w^*K_S|$ is generically a union of at least 3 components. Hence $|3K_S|$ contains at least 3 irreducible components each of which can move. Therefore $|3K_S|$ admits a decomposition $D_1 + D_2 + D_3$ such that $D_i K_S \geq 2$. This contradicts the equality $3K_S^2 = 3$.

Q.E.D.

PROPOSITION 2. $|3K_S|$ has no fixed part.

PROOF. Suppose the fixed part $G > 0$. From Lemma 7 we infer that $GK_S > 0$. $(3K_S - G)K_S \geq 2$, since otherwise $\dim |3K_S - Z| < 1$.

Note that $3K - G \nmid K_S$. This leads to the inequality

$$(3K_S - G)G \geq 3. \quad \text{Thus}$$

$$(3K_S - G)^2 = (3K_S - G)3K_S - (3K_S - G)G \leq 3.$$

This implies the absurd conclusion that Φ_{3K_S} is a birational map of S onto a quadratic or a cubic hypersurface P^3 .

Q.E.D.

PROPOSITION 3. Let M denote the moving part of $|2K_S|$. If M is generic, M contains no base points of $|3K_S|$.

PROOF. First we consider the case where $F^2=0, -2$ and M is non-singular. In view of the exact sequence

$$0 \rightarrow H^0(S, \mathcal{O}_S(3K_S)) \rightarrow H^0(M, \mathcal{O}_M(3K_S|_M)) \rightarrow 0,$$

we have only to prove that $3K_S|_M = \bar{K}_M + F|_M$ is free from base points. This is, however, a classical fact. Next let us consider the case $F^2=-4$. In this case M is base point free, so generic $\neq M$ does not contain a base point. Finally we consider the case $F=0$ and M has a double point P which is a unique base point of $|M|$. Let $w: \tilde{S} \rightarrow S$ denote the quadric transformation at P and E the associated exceptional curve on $\tilde{S}$. $\tilde{M} = w^*M - 2E$ is a non-singular curve of genus 3. The sequence of sheaves

$$0 \rightarrow \mathcal{O}_{\tilde{S}}(w^*K_S - 2E) \rightarrow \mathcal{O}_{\tilde{S}}(3w^*K_S) \rightarrow \mathcal{O}_{\tilde{M}}(3w^*K_S|_{\tilde{M}}) \rightarrow 0.$$

is exact and we obtain an injection $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3w^*K_S)) \hookrightarrow H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(3w^*K_S|_{\tilde{M}}))$. Since $\dim H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(3w^*K_S|_{\tilde{M}})) = \dim H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3w^*K_S)) = 4$, this is an isomorphism. On the other hand, $|\bar{K}_{\tilde{M}} + E|$ is base point free. Hence $|3w^*K_S|$ is base point free on M . This implies that $|3K_S|$ has no base point on M . Q.E.D.

COROLLARY. Let $\hat{M}$ be the generic member of $|2K_X|$. Then $|3K_X|$ has no base point on $\hat{M}$ and $\Phi_{3K_X}|_{\hat{M}}: \hat{M} \rightarrow \Phi_{3K_X}(\hat{M}) \subset P^3$ is a holomorphic mapping.

Remark. $\Phi_{3K_X}(M)$ is not a plane curve, so $\deg \Phi_{3K_X}(M) = 3$ or 6 .

If $\deg \Phi_x(M)=3$, $\Phi_{3k}(M) \cong P^1$ and $\Phi_{3k}|_M$ is a double covering. If $\deg \Phi_{3k}(M)=6$, $\Phi_{3k}|_M$ is a birational morphism.

PROPOSITION 4. If $\deg \Phi_x(M)=6$, $\Phi_{3k}(M)$ is a complete intersection of type (2,3). Moreover $\Phi_{3k}|_{\hat{M}}$ is an isomorphism.

PROOF. Consider the exact sequences

$$0 \rightarrow \mathcal{O}_X(4K_X) \rightarrow \mathcal{O}_X(6K_X) \rightarrow \mathcal{O}_{\hat{M}}(6K_X) \rightarrow 0,$$

$$0 \rightarrow \mathcal{O}_X(7K_X) \rightarrow \mathcal{O}_X(9K_X) \rightarrow \mathcal{O}_{\hat{M}}(9K_X) \rightarrow 0.$$

By the Riemann-Roch theorem, $\dim H^0(\hat{M}, \mathcal{O}_{\hat{M}}(6K_X))=9$. On the other hand, $\dim H^0(\hat{M}, \mathcal{O}_{\hat{M}}(3K_X))=10$. Hence there exists a quadric Q which contain $\Phi_{3k}(\hat{M})$. Such quadric is unique. In fact, if two quadrics contain a curve C , $\deg C \leq 4$. Next note that $\dim H^0(\hat{M}, \mathcal{O}_{\hat{M}}(9K_X))=15$. Since there are only four independent cubic surfaces which contain Q , there exists a cubic surface R which contains $\Phi_{3k}(\hat{M})$ and does not contain Q . Thus, since $\deg R \cdot Q = \deg \Phi_{3k}(\hat{M})=6$, we have $\Phi_{3k}(\hat{M})=R \cdot Q$. We have

$$\chi(\Phi_{3k}(\hat{M}), \mathcal{O}) = -3.$$

Accordingly $\tau(\Phi_{3k}(\hat{M})) = \tau(\hat{M}) = 4$. Since $\Phi_{3k}|_{\hat{M}}$ is a birational morphism, this means $\Phi_{3k}|_M$ is an isomorphism. Q.E.D.

We shall end this section by the following

LEMMA 9. Let $\mathcal{W}: \tilde{S} \rightarrow S$ be a resolution of the base points of $|3K_S|$ and let $\tilde{\Phi}$ denote the associated holomorphic mapping. If $\tilde{\Phi}$ maps an irreducible curve C onto a point, then C is an exceptional curve on $\tilde{S}$ or an irreducible component of fundamental cycles.

PROOF. Let $\tilde{C}$ be an irreducible curve on $\tilde{S}$ such that $\hat{C} = \mathcal{W}(\tilde{C})$ is an irreducible curve with $\hat{C}K_X > 0$. Suppose that $\tilde{\Phi}(\tilde{C})$ is a point. This is equivalent to the equality

$$\dim H^0(S, \mathcal{O}_S(3K_S - \omega(\tilde{C}))) = 3.$$

From Lemma 5, we infer $\omega(\tilde{C})K_S = 1$. The generic member D of $|3K_S - \omega(\tilde{C})|$ is an effective divisor with $DK_S = 2$, $D^2 \leq 2$.

Note that D is not composed of a pencil. Let $|D'|$ be the moving part of $|D|$. We have $D'K_S = 2$ and $D'^2 = 2$. Thus $\Phi_{D'}: S \rightarrow P^2$ is a double covering, and D' is a non-singular hyperelliptic curve of genus 3. Therefore $K_S|_{D'} \cong D'|_{D'}$.

Consider the exact sequence:

$$0 \rightarrow H^0(S, \mathcal{O}_S(-K_S)) \rightarrow H^0(S, \mathcal{O}_S(D' - K_S)) \rightarrow H^0(D', \mathcal{O}_{D'}(D' - K_S)) \rightarrow 0.$$

Since $\dim H^0(D', \mathcal{O}_{D'}(D' - K_S)) = 1$, there exists an effective curve $D'' \in |D' - K_S|$. D'' satisfies $D''^2 = -1$, $D''K_S = 1$. Assume that D'' is 1-connected. Then $H^1(S, \mathcal{O}_S(-D'')) = 0$, and $H^1(S, \mathcal{O}_S(K_S + D'')) = H^1(S, \mathcal{O}_S(D')) = 0$. This leads to the equality $\dim H^0(S, \mathcal{O}_S(D')) = 1$, a contradiction. Next suppose that D'' is not 1-connected. D'' admits a decomposition $D'' = D_1'' + D_2''$ with $D_1''K_S = 1$, $D_2''K_S = 0$, $D_2''^2 \leq -2$, $D_1''D_2'' = 0$. Then we infer that $D_1'' \sim K_S$. $D' - K_S - D_1''$ is effective. Hence we have

$$\dim H^0(S, \mathcal{O}_S(D')) = \dim H^0(S, \mathcal{O}_S(K_S + D_1'')) = 2.$$

This is a contradiction.

Q.E.D.

3. Proof of the Main Theorem. We let $\Phi = \Phi_{3K_S}: S \rightarrow P^3$ and $\hat{\Phi} = \Phi_{3K_X}: X \rightarrow P^3$. Thus $\hat{\Phi} = \hat{\Phi} \circ \omega$. Let m and d denote the mapping degree of $\hat{\Phi}$ and the degree of the hypersurface $\hat{\Phi}(S) \subset P^3$, respectively. In order to prove our main theorem it suffices to deny each of the following possibilities:

- (a) $d=2$;
- (b) $d=3$ and $m=2$;
- (c) $d=3$ and $m=3$;
- (d) $d=4$ and $m=2$.

We have proved that $\hat{M}$ is an irreducible curve of virtual genus 4 for a generic member $\hat{M} \in |2K_X|$. First we prove the following

LEMMA 10. If $\hat{\Phi}(\hat{M})$ is a projective line of degree 3 embedded in P^3 , then $m \geq 4$.

PROOF. Note that, if $\hat{\Phi}(\hat{M})$ is of degree 3, m is even. Suppose $m=2$. Let $\tilde{X} \xrightarrow{\omega} X$ be a resolution of the base points of $|3K_X|$ and $\tilde{\Phi}: \tilde{X} \rightarrow P^3$ the associated holomorphic mapping. From Lemma 9 we infer that $\tilde{\Phi}^*(\hat{\Phi}(\hat{M})) = \omega^*(\hat{M}) + D$

where D is a divisor whose support lies on the exceptional curves of X . Hence $\tilde{\Phi}^*(\hat{\Phi}(\hat{M}))|_{\omega^*(\hat{M})} \cong \omega^*(\hat{M})|_{\omega^*(\hat{M})}$.

But this is impossible, because

$$1 = \dim H^0(\hat{M}, \mathcal{O}(\hat{M})) \geq \dim H^1(\omega^*(\hat{M}), \mathcal{O}_{\omega^*(\hat{M})}(\omega^*(\hat{M}))) \geq 2.$$

Q.E.D.

LEMMA 11. Case (a) does not occur.

PROOF. Suppose (a). We have a pencil of reducible hyperplane sections $H = L_1' + L_2'$ on $\Phi(S)$. Let $\tilde{\omega}: \tilde{S} \rightarrow S$ be a resolution of the base points of $|3K_S|$, and put $\tilde{\Phi} = \Phi \circ \tilde{\omega}$. $\tilde{\Phi}^* H$ can be decomposed into $\tilde{L}_1 + \tilde{L}_2 + N$, where $\tilde{\Phi}(\tilde{L}_1) = L_1'$. $L_1 = \omega(\tilde{L}_1)$ is a

movable curve on S . So $L_1 K_S \geq 2$. This leads to a contradiction:

$$3 = 3K_S^2 \geq (L_1 + L_2)K_S \geq 4. \quad \text{Q.E.D.}$$

LEMMA 12. Case (b) does not occur.

PROOF. Suppose (b). Let $\tilde{X} \xrightarrow{\pi} X$ be the resolution of the base points and let $|L| + E_1 + E_2 + E_3 = |3\pi^* K_X|$, where $|L|$ is the moving part of $|3\pi^* K_X|$ and the E_i are three distinct exceptional curves. $\tilde{\Phi} = \Phi|_{\tilde{X}}$ is a finite morphism (see Lemma 9). Hence, in virtue of Zariski's Main Theorem, there exists the (holomorphic) involution ι of $\tilde{X}$ induced by $\tilde{\Phi}$. For the generic member $\hat{M}$ of $|2K_X|$, $\iota(\pi^*(\hat{M})) \sim 2K_X$. On the other hand $\tilde{\Phi}(\hat{M}) = 2H$, where H is a hyperplane section of $\Phi(X)$. Hence we have

$$6\pi^* K_X - 2E_1 - 2E_2 - 2E_3 = \tilde{\Phi}^*(\tilde{\Phi}(\hat{M})) = \pi^*(\hat{M}) + \iota(\pi^*(\hat{M})) = 4\pi^* K_X.$$

This is absurd.

Q.E.D.

LEMMA 12. Case (c) does not occur.

PROOF. Suppose (c). If $\Phi(S)$ is not normal, $\Phi(S)$ contains a double line, and so we have a pencil of hyperplane sections $H = L' + L'_0$ where L'_0 is the double line. We put $|3K_S| = \Phi^* H = L + L_1 + L_2 + G$, where $\Phi(L) = L'$ and $\Phi(L_1) = L'_0$. Since L is movable we have $LK_S \geq 2$ and $L_1 K_S \geq 1$. This is impossible.

Next we assume that $\Phi(S)$ is a normal cubic. Let M_1 and M_2 be two generic members of $|2K_S|$. $\Phi(M_i)$ are sections of $\Phi(S)$ by quadric hypersurfaces. Let Λ denote the sublinear system of $|2H|$ generated by $\Phi(M_1)$ and $\Phi(M_2)$. For any $\lambda \in \Lambda$, $\tilde{\Phi}^* \lambda$ contains a divisor $M \in |2K_S|$ which is generated by M_1 and M_2 . This implies that for any $M \in |2K_S|$, $\Phi(M) \in \Lambda$. Since Φ is a finite morphism, the set of base points of Λ lies on

the image of the base points of $|2K_X|$. Take a general point P_0 of $\tilde{\Phi}(M)$. $\tilde{\Phi}^{-1}(P_0) = P + P' + P''$, P being a point of M . Let M' and M'' be two members of $|2K_X|$ which contain P' and P'' respectively. We have $\tilde{\Phi}(M) = \tilde{\Phi}(M') = \tilde{\Phi}(M'')$. In fact, if, say, $\tilde{\Phi}(M) \neq \tilde{\Phi}(M')$, then $P_0 \in \tilde{\Phi}(M) \cdot \tilde{\Phi}(M')$ and P_0 is a base point of Λ . From the discussions above, we see that there exists a natural holomorphic mapping $P^1 \cong |2K_X| \xrightarrow{\sim} |2K_S| \rightarrow P^1 \cong \Lambda$ whose mapping degree = 3. Since 3-sheeted covering $P^1 \rightarrow P^1$ has its ramification locus of degree 4, the ramification locus $\mathcal{R}$ of $\tilde{\Phi}$ satisfies $\mathcal{R} \geq \cancel{4-2K_S} 4M$. On the other hand, since $\tilde{\Phi}(S)$ is a normal cubic surface, we have $K_S \geq \mathcal{R} - \tilde{\Phi}^* H \geq 4M - 3K$. This is a contradiction. Q.E.D.

LEMMA 13. Case (d) does not occur.

PROOF. Suppose (d). $|3K_X|$ has a unique base point P . Let $\tilde{X} \xrightarrow{\alpha} X$ be the quadric transformation at P , E the associated exceptional curve on X , and $\tilde{\Phi}$ the induced holomorphic mapping $\tilde{X} \rightarrow P^3$. Let $\hat{M}$ denote the generic member of $|2K_X|$. We have seen that $\tilde{\Phi}(\hat{M})$ is a complete intersection of type (2,3). Therefore $|2H - \tilde{\Phi}(\hat{M})| \neq \emptyset$ and a fortiori $|2(3K_X - E) - \tilde{\Phi}^*(\hat{M})| = |2K_X - 2E| \neq \emptyset$. Since P is a base point of $|3K_X|$, we have $H^1(X, \mathcal{O}_X(2E - 2K_X)) \neq 0$. This implies that $2K_S - 2E$ is not 1-connected; i.e., there exist effective divisors D_1 and D_2 such that $D_1 \sim D_2 \sim K_S - E$, $D_1 + D_2 \in |2K_S - 2E|$. $\tilde{\Phi}(D_1)$ is a line so $\dim |3K_S - E - D_1| \geq 2$. Let $N \in |3K_S - E - D_1|$. $N \sim 2K_S$. Choosing N generically, we may assume that $\tilde{\Phi}(N)$ is an irreducible plane curve of degree 3. Note that, since $\tilde{\Phi}|_{D_1}$ and $\tilde{\Phi}|_N$ is both double coverings, $\tilde{\Phi}^*(\tilde{\Phi}(D_1)) = D_1$, $\tilde{\Phi}^*(\tilde{\Phi}(N)) = N$. Moreover $\tilde{\Phi}(D_1)$ is a Cartier divisor on H , and $\mathcal{O}_{\tilde{\Phi}(N)}(\tilde{\Phi}(D_1))$ is

an invertible sheaf whose degree = 3. ~~So we have~~

$$\mathcal{O}_N(D_1) = \tilde{\Phi}^* \mathcal{O}_{\Phi(N)}(\tilde{\Phi}(D_1))$$

This implies that $\deg \mathcal{O}_N(D_1) = 6$. But we have

$$ND_1 = 2K_X(K_X - E) = 2.$$

This is a contradiction.

Q.E.D.

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