

NUMERICAL RADII OF OPERATOR MATRICES AND ITS APPLICATION

(作用素行列の数域半径とその応用)

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ABSTRACT. Garcia gave an upper bound of $\|A + iB\|$ for self-adjoint operators A and B . We give a matrix representation of $\|A + iB\|$ and a generalization of Garcia's result. For it we give a numerical range of $\begin{pmatrix} aI & T \\ T^* & bI \end{pmatrix}$ for an operator T and real numbers a and b . On the other hand, Furuta gave a numerical range of $S = \begin{pmatrix} aI & cT \\ dT^* & bI \end{pmatrix}$ for an operator T and nonnegative real numbers a, b, c and d . We pointed out $w(S) = w(\operatorname{Re} S)$ under the condition that a, b, c and d are real numbers with $cd \geq 0$.

1. INTRODUCTION

A capital letter means a bounded linear operator on a complex Hilbert space H . The numerical range $W(T)$ of an operator T is defined by

$$W(T) = \{\langle Tx, x \rangle : \|x\| = 1\}.$$

Toeplitz-Hausdorff's theorem implies that the numerical range $W(T)$ is a convex set on the complex plane (cf. [3]). Moreover the numerical radius $w(T)$ of an operator T is defined by

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup\{|\langle Tx, x \rangle| : \|x\| = 1\}.$$

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It is well known that $w(T) \leq \|T\|$, and $w(T) = \|T\|$ for normal operators T .

In [4], Garcia showed the following theorem:

Theorem A. *If A and B are self-adjoint operators with $m \leq A \leq M$, then*

$$(1.1) \quad \|A + iB\| \leq \frac{1}{2}(M - m) + \frac{1}{2}\sqrt{(M + m)^2 + 4\|B\|^2}.$$

2000 *Mathematics Subject Classification.* 47A30, 47A63, 47B15.

Key words and phrases. numerical range, numerical radius, operator norm, self-adjoint operator, Cartesian decomposition, positive operator, operator inequality.

The norm $\|A + iB\|$ is represented by an operator matrix as follows: $\|A + iB\| = \left\| \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \right\|$ (see Lemma 3.1). So the inequality (1.1) is rewritten as follows:

$$(1.2) \quad w \left(\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \right) = \left\| \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \right\| \leq \frac{1}{2}(M - m) + \frac{1}{2}\sqrt{(M + m)^2 + 4\|B\|^2}.$$

In this note, we shall calculate $w \left(\begin{pmatrix} aI & T \\ T^* & bI \end{pmatrix} \right)$ for an operator T and real numbers a and b (see Theorem 2.2). As a result, we give an upper bound of $w \left(\begin{pmatrix} A_1 & T \\ T^* & -A_2 \end{pmatrix} \right)$ for an operator T and self adjoint operators A_1, A_2 as a generalization of (1.2) (i.e., (1.1)) (see Theorem 3.2). For it, the following equation plays an elementary and essential role

$$w \left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \right) = \left\| \begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \right\| = \sqrt{a^2 + \|T\|^2}$$

for an operator T and $a \in \mathbb{R}$ (see Lemma 2.1(ii)).

On the other hand, Furuta [2] gave the numerical radius $w(S)$ of S which is defined by (1.3). We show that it is sufficiently to obtain the value of $w(\operatorname{Re} S)$ ($= w(\frac{S+S^*}{2})$) for the calculation of $w(S)$ by using Theorem 2.2.

Theorem B. *Let*

$$(1.3) \quad S = \begin{pmatrix} aI & cT \\ dT^* & bI \end{pmatrix}$$

be an operator on a Hilbert space $H = H_1 \oplus H_2$ where T is an operator from H_2 to H_1 and a, b, c and d are nonnegative real numbers. Then

$$(1.4) \quad w(S) = \frac{1}{2}(a + b) + \frac{1}{2}\sqrt{(a - b)^2 + (c + d)^2\|T\|^2}.$$

We calculate $w(S)$ under the condition $a, b, c, d \in \mathbb{R}$ with $cd \geq 0$ (see Theorem 3.3).

2. NUMERICAL RADIUS FOR SELF-ADJOINT OPERATORS

For an operator T and a real number a , we give some properties of $\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix}$ to generalize (1.2) (i.e., (1.1)) in the following lemma:

Lemma 2.1. *Let T be an operator and a be a real number. Then the following holds:*

(i) $W \left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \right)$ is symmetric, i.e.,

$$\alpha \in W \left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \right) \iff -\alpha \in W \left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \right),$$

(ii) $w \left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \right) = \left\| \begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \right\| = \sqrt{a^2 + \|T\|^2},$

$$(iii) \quad \overline{W} \left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \right) = \left[-\sqrt{a^2 + \|T\|^2}, \sqrt{a^2 + \|T\|^2} \right]$$

where $\overline{W}(T)$ is a closure of $W(T)$.

Proof. (i) For $\alpha \in W \left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \right)$, we have

$$\alpha = \left\langle \begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = a\|x\|^2 + 2\operatorname{Re}\langle Ty, x \rangle - a\|y\|^2$$

for some unit vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

On the other hand, since $\begin{pmatrix} \frac{\|y\|}{\|x\|}x \\ -\frac{\|y\|}{\|y\|}y \end{pmatrix}$ is a unit vector, we have

$$\left\langle \begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \begin{pmatrix} \frac{\|y\|}{\|x\|}x \\ -\frac{\|y\|}{\|y\|}y \end{pmatrix}, \begin{pmatrix} \frac{\|y\|}{\|x\|}x \\ -\frac{\|y\|}{\|y\|}y \end{pmatrix} \right\rangle = a\|y\|^2 - 2\operatorname{Re}\langle Ty, x \rangle - a\|x\|^2 = -\alpha$$

and $\left\langle \begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \begin{pmatrix} \frac{\|y\|}{\|x\|}x \\ -\frac{\|y\|}{\|y\|}y \end{pmatrix}, \begin{pmatrix} \frac{\|y\|}{\|x\|}x \\ -\frac{\|y\|}{\|y\|}y \end{pmatrix} \right\rangle \in W \left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \right)$. Hence we have $-\alpha \in W \left(\begin{pmatrix} aI & T \\ T^* & -aI \end{pmatrix} \right)$.

(ii) Since $\begin{pmatrix} aI & T^* \\ T & -aI \end{pmatrix}$ is self-adjoint, we have

$$\left\| \begin{pmatrix} a & T \\ T^* & -a \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} a & T \\ T^* & -a \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} a^2 + TT^* & 0 \\ 0 & a^2 + T^*T \end{pmatrix} \right\| = a^2 + \|T\|^2.$$

(iii) is obvious by (i) and (ii). $\square$

In the following theorem, we give the numerical radius of the self-adjoint operator matrix $\begin{pmatrix} aI & T \\ T^* & bI \end{pmatrix}$:

Theorem 2.2. Let $H = H_1 \oplus H_2$ be a Hilbert space and let T be an operator from H_2 to H_1 . Let a and b be real numbers. Then

$$(2.1) \quad w \left(\begin{pmatrix} aI & T \\ T^* & bI \end{pmatrix} \right) = \left\| \begin{pmatrix} aI & T \\ T^* & bI \end{pmatrix} \right\| = \frac{1}{2}|a+b| + \frac{1}{2}\sqrt{(a-b)^2 + 4\|T\|^2}.$$

Proof. The first equality is obviously. Next we have

$$\begin{aligned} w \left(\begin{pmatrix} aI & T \\ T^* & bI \end{pmatrix} \right) &= w \left(\frac{1}{2}(a+b) + \begin{pmatrix} \frac{a-b}{2}I & T \\ T^* & -\frac{a-b}{2}I \end{pmatrix} \right) \\ &= \frac{1}{2}|a+b| + w \left(\begin{pmatrix} \frac{a-b}{2}I & T \\ T^* & -\frac{a-b}{2}I \end{pmatrix} \right) \quad \text{by Lemma 2.1 (iii)} \\ &= \frac{1}{2}|a+b| + \sqrt{\left(\frac{a-b}{2} \right)^2 + \|T\|^2} \quad \text{by Lemma 2.1 (ii)} \end{aligned}$$

$$= \frac{1}{2}|a+b| + \frac{1}{2}\sqrt{(a-b)^2 + 4\|T\|^2}$$

and hence the second equality holds. $\square$

3. MAIN RESULTS

The following lemma gives matrix representation of $\|A + iB\|$ for self-adjoint operators A and B .

Lemma 3.1. *Let X and Y be operators on H . Then*

$$(3.1) \quad \left\| \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \right\| = \left\| \begin{pmatrix} X & iY \\ iY & X \end{pmatrix} \right\| = \max\{\|X + iY\|, \|X - iY\|\}.$$

In particular, if A and B are self-adjoint operators, then

$$(3.2) \quad \left\| \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \right\| = \left\| \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \right\| = \left\| \begin{pmatrix} A & B \\ iB & A \end{pmatrix} \right\| = \|A + iB\| = \|A - iB\|.$$

Proof. We only prove the equality (3.1). Let I be the identity operator of H . Then we have

$$\begin{aligned} \begin{pmatrix} X + iY & 0 \\ 0 & X - iY \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix} \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \begin{pmatrix} I & -iI \\ -iI & I \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \begin{pmatrix} X & iY \\ iY & X \end{pmatrix} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}. \end{aligned}$$

Since matrix operators $\frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix}$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ are unitary, we have

$$\begin{aligned} \left\| \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \right\| &= \left\| \begin{pmatrix} X & iY \\ iY & X \end{pmatrix} \right\| = \left\| \begin{pmatrix} X + iY & 0 \\ 0 & X - iY \end{pmatrix} \right\| \\ &= \max\{\|X + iY\|, \|X - iY\|\}. \end{aligned}$$

So the desired equalities hold. $\square$

From (3.2), (1.1) in Theorem A can be interpreted as

$$(3.3) \quad w \left(\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \right) \leq \frac{1}{2}(M - m) + \frac{1}{2}\sqrt{(M + m)^2 + 4\|B\|^2}.$$

Thus the following theorem is a generalization of Theorem A:

Theorem 3.2. *Let T be an operator from H_2 to H_1 , and let A_i be self-adjoint operators on H_i with $\sigma(A_i) \subset [m, M]$ where $\sigma(A_i)$ is the spectrum of A_i ($i = 1, 2$). Then*

$$(3.4) \quad w \left(\begin{pmatrix} A_1 & T \\ T^* & -A_2 \end{pmatrix} \right) \leq \frac{1}{2}(M - m) + \frac{1}{2}\sqrt{(M + m)^2 + 4\|T\|^2}.$$

Proof. Since $m \leq A_i \leq M$ ($i = 1, 2$), we have

$$\begin{pmatrix} m & T \\ T^* & -M \end{pmatrix} \leq \begin{pmatrix} A_1 & T \\ T^* & -A_2 \end{pmatrix} \leq \begin{pmatrix} M & T \\ T^* & -m \end{pmatrix}.$$

Here we have by Theorem 2.2

$$\left\| \begin{pmatrix} M & T \\ T^* & -m \end{pmatrix} \right\| = \left\| \begin{pmatrix} m & T \\ T^* & -M \end{pmatrix} \right\| = \frac{1}{2}(M - m) + \frac{1}{2}\sqrt{(M + m)^2 + 4\|T\|^2}.$$

Hence the desired inequality (3.4) holds. $\square$

Next, we pay attention to the fact that $w(S) = w(\operatorname{Re} S)$ for S in (1.3). As a consequence, Theorem 2.2 is applicable to the following theorem. It is a generalization of Theorem B.

Theorem 3.3. *Let $H = H_1 \oplus H_2$ be a Hilbert space and let T be an operator from H_2 to H_1 . Let a, b, c and d be real numbers with $cd \geq 0$. Suppose that $S = \begin{pmatrix} aI & cT \\ dT^* & bI \end{pmatrix}$ be an operator on H . Then*

$$(3.5) \quad w(S) = w(\operatorname{Re} S) = \frac{1}{2}|a + b| + \frac{1}{2}\sqrt{(a - b)^2 + (c + d)^2\|T\|^2}.$$

Proof. We have

$$\begin{aligned} (3.6) \quad w(S) &= \sup \left\{ \left| \left\langle \begin{pmatrix} aI & cT \\ dT^* & bI \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle \right| ; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\} \\ &= \sup \left\{ |a\|x_1\|^2 + b\|x_2\|^2 + c\langle Tx_2, x_1 \rangle + d\overline{\langle Tx_2, x_1 \rangle}| ; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\} \\ &= \sup \left\{ |a\|x_1\|^2 + b\|x_2\|^2| + |(c + d)\langle Tx_2, x_1 \rangle| ; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\}. \end{aligned}$$

The last equality of (3.6) is ensured by $cd \geq 0$. Moreover we have

$$\begin{aligned} w(\operatorname{Re} S) &= \sup \left\{ \left| \left\langle \begin{pmatrix} aI & \frac{c+d}{2}T \\ \frac{c+d}{2}T^* & bI \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle \right| ; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\} \\ &= \sup \left\{ |a\|x_1\|^2 + b\|x_2\|^2 + (c + d)\operatorname{Re}\langle Tx_2, x_1 \rangle| ; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\} \\ &= \sup \left\{ |a\|x_1\|^2 + b\|x_2\|^2| + |(c + d)\langle Tx_2, x_1 \rangle| ; \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| = 1 \right\}. \end{aligned}$$

So the first equality of (3.5) holds.

Next, replacing $\frac{c+d}{2}T$ to T in Theorem 2.2, we have

$$w(\operatorname{Re} S) = w \left(\begin{pmatrix} aI & \frac{c+d}{2}T \\ \frac{c+d}{2}T^* & bI \end{pmatrix} \right) = \frac{1}{2}|a + b| + \frac{1}{2}\sqrt{(a - b)^2 + (c + d)^2\|T\|^2},$$

and hence the second equality holds. $\square$

If $cd < 0$ in Theorem 3.3, then the last equation of (3.6) and so the first equality of (3.5) be not ensured. We confirm this result by using 2×2 real matrix $S = \begin{pmatrix} v & -w \\ w & v \end{pmatrix}$

where $v, w \neq 0$ and $v^2 + w^2 = 1$. Since S is unitary, we have $w(S) = 1$. On the other hand, it follows from $\operatorname{Re} S = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$ that $w(\operatorname{Re} S) = |v| (< 1)$.

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