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Fundamental groups of curves in positive characteristic

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Abstract. We present some recent results (mainly of the author) concerning fundamental groups of curves over algebraically closed fields of positive characteristic.

§1. Main question (cf. [H2,4][T3]).

In this §, we introduce our main question.

Throughout this article, we let k denote an algebraically closed field of characteristic $p>0$, X a smooth curve over k, X^{*} the smooth compactification of X and $\Sigma \stackrel{\mathrm{def}}{=} X^{*}-X$. We define non-negative integers g and n to be the genus of the proper, smooth curve X^{*} and the cardinality of the point set Σ, respectively. Note that X is hyperbolic (resp. affine, resp. projective) if and only if $2-2g-n<0$, i.e., $(g,n)\neq(0,0),(0,1),(0,2),(1,0)$ (resp. $n>0$, resp. $n=0$).

We denote by $k(X)^{\sim}$ the maximal separable algebraic extension of $k(X)$ in which the discrete valuation ring $\mathcal{O}_{X,x}$ is unramified for all $x\in X$, and by $k(X)^{\sim\mathrm{t}}$ the maximal separable algebraic extension of $k(X)$ in which $\mathcal{O}_{X,x}$ is unramified for all $x\in X$ and at most tamely ramified for all $x\in\Sigma$. Then, the fundamental group $\pi_1(X)$ (resp. the tame fundamental group $\pi_1^{\mathrm{t}}(X)$) of X is nothing but the Galois group $\mathrm{Gal}(k(X)^{\sim}/k(X))$ (resp. $\mathrm{Gal}(k(X)^{\sim\mathrm{t}}/k(X))$).

Definition.

(i) For a (discrete) group Γ, we denote by $\Gamma^{\wedge}$ its profinite completion

$$\varprojlim_{\mathrm{N}\lhd\Gamma,\ (\Gamma:\mathrm{N})<\infty} \Gamma/\mathrm{N}.$$

(ii) For a profinite group G, we denote by $G^{p'}$ its maximal pro-prime-to-p quotient

$$\varprojlim_{N\lhd G\text{ closed},\ p\nmid(G:N)<\infty} G/N.$$

(iii) For non-negative integers g and n, we denote by $\Pi_{g,n}$ the topological fundamental group of a compact orientable surface of genus g with n points deleted. More concretely,

$$\Pi_{g,n}=\langle\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g,\gamma_1,\dots,\gamma_n \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\dots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\gamma_1\dots\gamma_n=1\rangle$$

In particular, if $n>0$, $\Pi_{g,n}$ is a free group of rank $2g+n-1$.

The following fact concerning $\pi_1(X)$ and $\pi_1^{\mathrm{t}}(X)$ is more or less well-known:

Lemma.
We have the following three surjections (i)–(iii)*:*

$$\begin{array}{ccccc} & & \pi_1(X) & & \\ & & \downarrow \text{(i)} & & \\ (\Pi_{g,n})^{\wedge} & \overset{\text{(ii)}}{\to} & \pi_1^{\mathrm{t}}(X) & \overset{\text{(iii)}}{\to} & (\Pi_{g,n})^{\wedge p'}. \end{array}$$

Moreover, we have:

$$\text{(i) } \textit{is an isomorphism} \iff n = 0,$$

$$\text{(ii) } \textit{is an isomorphism} \iff (g,n) = (0,0),(0,1),$$

and

$$\text{(iii) } \textit{is an isomorphism} \iff \begin{cases} \textit{either } (g,n) = (0,0),(0,1),(0,2), \\ \textit{or } (g,n) = (1,0) \textit{ and } X \textit{ is supersingular.} \end{cases}$$

Proof. Surjection (i) comes from the definitions of $\pi_1(X)$ and $\pi_1^{\mathrm{t}}(X)$ (or the above descriptions of $\pi_1(X)$ and $\pi_1^{\mathrm{t}}(X)$ in terms of Galois groups). For surjections (ii) and (iii), see [SGA1].

If $n = 0$, i.e., X is projective, then the natural surjection $\pi_1(X) \to \pi_1^{\mathrm{t}}(X)$ is an isomorphism by definition. On the other hand, if $n > 0$, i.e., X is affine, $\pi_1(X)$ is not topologically finitely generated (as its maximal pro-p quotient $\pi_1(X)^p$ is a free pro-p group of rank $|k|$), while $\pi_1^{\mathrm{t}}(X)$ is always topologically finitely generated as a quotient of $(\Pi_{g,n})^{\wedge}$. Thus, in this case, (i) cannot be an isomorphism.

Next, if $(g,n) = (0,0),(0,1)$, then we have $\Pi_{g,n} = \{1\}$. Thus, in this case, surjections (ii) and (iii) must be isomorphisms. If $(g,n) = (0,2)$ (resp. $(g,n) = (1,0)$ and X is supersingular), then $\pi_1^{\mathrm{t}}(X)$ coincides with $(\Pi_{g,n})^{\wedge p'} = \widehat{\mathbb{Z}}^{p'}$ (resp. $\widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'}$). Thus, surjection (iii) is then an isomorphism.

On the other hand, assume that (ii) is an isomorphism. Then, the abelianizations $(\Pi_{g,n})^{\wedge \mathrm{ab}}$ and $\pi_1^{\mathrm{t}}(X)^{\mathrm{ab}}$ are also isomorphic to each other. Observing the pro-p parts, we obtain $2g + n - 1 + b^{(2)} = \gamma$, where $b^{(2)}$ denotes the second Betti number of X (i.e., $b^{(2)} = 1$ for $n = 0$ and $b^{(2)} = 0$ for $n > 0$), and γ denotes the p-rank (or Hasse-Witt invariant) of X. Since $\gamma \leq g$, this equality implies $(g,n) = (0,0),(0,1)$.

Finally, assume that (iii) is an isomorphism, which implies that $\pi_1^{\mathrm{t}}(X)$ has a trivial pro-p-Sylow subgroup. If either $(g,n) = (0,0),(0,1),(0,2)$, or $(g,n) = (1,0)$ and X is supersingular, nothing remains to be proved. If $(g,n) = (1,0)$ and X is ordinary, then $\pi_1^{\mathrm{t}}(X) = \pi_1(X)$ is isomorphic to $\widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'} \times \mathbb{Z}_p$, hence its pro-$p$-Sylow subgroup is non-trivial. Finally, if $(g,n) \neq (0,0),(0,1),(0,2),(1,0)$, i.e., X is hyperbolic, then there exists a tame covering $Y \to X$ such that the genus of Y^* is not less than 2. By [R1], Corollaire 4.3.2, $\pi_1(Y^*)$ has a non-trivial pro-p-Sylow subgroup. Since $\pi_1(Y^*)$ is a subquotient of $\pi_1^{\mathrm{t}}(X)$, this implies that $\pi_1^{\mathrm{t}}(X)$ admits a non-trivial pro-p-Sylow subgroup. This completes the proof. $\square$

Example.
The following are the only cases (for the present) in which we can describe $\pi_1(X)$ or $\pi_1^{\mathrm{t}}(X)$ explicitly.
(i) If $(g,n) = (0,0)$, then we have

$$\pi_1(X) = \pi_1^{\mathrm{t}}(X) = \{1\}.$$

(ii) If $(g,n) = (0,1)$, then we have $\pi_1^{\mathrm{t}}(X) = \{1\}$.
(iii) If $(g,n) = (0,2)$, then we have

$$\pi_1^{\mathrm{t}}(X) \simeq \widehat{\mathbb{Z}}^{p'}.$$

(iv) If $(g,n) = (1,0)$, then we have

$$\pi_1(X) = \pi_1^{\mathrm{t}}(X) \simeq \begin{cases} \widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'} \times \mathbb{Z}_p, & X\text{: ordinary,} \\ \widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'}, & X\text{: supersingular.} \end{cases}$$

Now, our main question is as follows.

Main question.
Exactly what information on the geometry of X does $\pi_1(X)$ (or $\pi_1^{\mathrm{t}}(X)$) carry?

As for this question, the best situation we can expect is:

Hope.
Assume $k = \overline{\mathbb{F}}_p$. (For general k, see [T3]*.)*
(i) *$\pi_1(X)$ determines the isomorphism class of X (as a scheme), unless $(g,n) = (1,0)$.*
(ii) *$\pi_1^{\mathrm{t}}(X)$ determines the isomorphism class of X (as a scheme), unless $(g,n) = (0,0), (0,1), (1,0)$.*

Remark (which shows that our problem is rather subtle).
(i) In characteristic 0, we have $\pi_1(X) = \pi_1^{\mathrm{t}}(X) \simeq (\Pi_{g,n})^{\wedge}$, which carries very little information about X.
(ii) Let $\pi_A(X)$ denote the set of isomorphism classes of finite quotient groups of $\pi_1(X)$. Then, the Abhyankar conjecture (proved by Raynaud [R2] and Harbater [H1]) asserts that, if $n > 0$, we have

$$\pi_A(X) = \{G \mid G^{p'} \text{ is generated by (at most) } 2g+n-1 \text{ elements.}\}/\simeq.$$

Thus, in this case, $\pi_A(X)$ carries very little information about X.
(iii) The geometric Shafarevich conjecture (proved by Harbater [H3] and Pop [P]) asserts that the absolute Galois group $G_{k(X)}$ is a free profinite group of rank $|k|$. Thus, $G_k(X)$, which is an extension group of $\pi_1(X)$, carries very little information about X.

§2. Some results (which support our hope).

In this §, we present some results which support our hope in the last §.

Theorem 0.
(i) ([T1]) $\pi_1(X)$ *determines* (g,n).
(ii) ([T2]) $\pi_1^{\mathrm{t}}(X)$ *determines* (g,n), *unless* $(g,n)=(0,0),(0,1)$.

Theorem 1.
Assume $k=\overline{\mathbb{F}}_p$ *and* $g=0$.
(i) ([T1]) $\pi_1(X)$ *determines the isomorphism class of* X *(as a scheme).*
(ii) ([T2]) $\pi_1^{\mathrm{t}}(X)$ *determines the isomorphism class of* X *(as a scheme), unless* $n=0,1$.

Theorem 2 ([PS], [R4], [T4]).
Assume $k=\overline{\mathbb{F}}_p$.
(i) $\pi_1(X)$ *determines the isomorphism class of* X *up to finite possibilities, unless* $(g,n)=(1,0)$.
(ii) $\pi_1^{\mathrm{t}}(X)$ *determines the isomorphism class of* X *up to finite possibilities, unless* $(g,n)=(1,0)$.

One of the main ingredients of the proofs of the above theorems is Raynaud's theory of theta divisors ([R1], cf. [R3], [M]).

§3. Applications.

In this §, we present two applications of the results in the last §.

Corollary to Theorem 1(ii) (Tamagawa, unwritten yet).
Let L *be a subfield of* $\overline{\mathbb{Q}}$, *and assume that there exist infinitely many rational primes* p, *such that* $I_{\overline{p}}\cap G_L\not\subset I_{\overline{p}}^{\mathrm{w}}$ *holds for some prime* $\overline{p}$ *of* $\overline{\mathbb{Q}}$ *above* p. *(Here,* $I_{\overline{p}}$ *(resp.* $I_{\overline{p}}^{\mathrm{w}}$*) denotes the inertia subgroup (resp. the pro-*p*-Sylow subgroup of the inertia subgroup) at* $\overline{p}$.*) Let* X *be a smooth, hyperbolic curve over* L, *and assume that the smooth compactification of* X *is of genus* 0. *Then, the outer Galois representation* $\rho : G_L\to \mathrm{Out}(\pi_1(X_{\overline{\mathbb{Q}}}))$ *determines the isomorphism class of* X *as an* L*-scheme.*

Corollary to Theorem 2(ii) ([T4]).
Let F *be a function field of one variable over a finite field of characteristic* p. *Let* X *be a smooth, hyperbolic curve over* F, *and assume that* X *is non-isotrivial. Let* ρ^{t} *denote the outer Galois representation* $G_F\to\mathrm{Out}(\pi_1^{\mathrm{t}}(X_{\overline{F}}))$. *Then, for all primes* P *of* F, *we have*

$$\mathrm{Ker}(\rho^{\mathrm{t}})\cap D_P\subset I_P^{\mathrm{w}}.$$

(Here, D_P *(resp.* I_P^{w}*) denotes the decomposition subgroup (resp. the pro-*p*-Sylow subgroup of the inertia subgroup) at* P, *defined up to conjugacy.)*

Remark.
Let F be an arbitrary field of characteristic >0, and X a smooth, affine curve over F. Then, the outer Galois representation $\rho : G_F\to \mathrm{Out}(\pi_1(X_{\overline{F}}))$ is injective (see

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