

Euler 方程式の大域解接続に関する Beale-Kato-Majda の定理とその発展 (Remarks on the result of Beale-Kato-Majda for the Euler equations)

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Abstract

We prove that the *BMO* norm of the vorticity controls the blow-up phenomena of smooth solutions to the Euler equations in the whole space $\mathbf{R}^n$.

Introduction.

In this paper we prove that the *BMO* norm of the vorticity controls the blow-up phenomena of smooth solutions to the Euler equations.

the Euler equations in $\mathbf{R}^n$ ($n \geq 3$) are as follows:

$$(E) \quad \begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, & \operatorname{div} u = 0 \quad \text{in } x \in \mathbf{R}^n, t > 0, \\ u|_{t=0} = a \end{cases}$$

where $u = (u^1(x, t), u^2(x, t), \dots, u^n(x, t))$ and $p = p(x, t)$ denote the unknown velocity vector and the unknown pressure of the fluid at the point $(x, t) \in \mathbf{R}^n \times (0, \infty)$, respectively, while $a = (a^1(x), a^2(x), \dots, a^n(x))$ is the given initial velocity vector.

It is proved by Kato-Lai [3] and Kato-Ponce [4] that for every $a \in W_\sigma^{s,p}$ for $s > n/p + 1$, $1 < p < \infty$, there are $T > 0$ and a unique solution u of (E) on the interval $[0, T)$ in the class

$$(CE)_{s,p} \quad u \in C([0, T); W_\sigma^{s,p}) \cap C^1([0, T); W_\sigma^{s-2,p}),$$

where subindex σ means the divergence free. It is an interesting question whether the solution $u(t)$ really blows up as $t \uparrow T$.

Beale-Kato-Majda [1] proved that under the condition

$$\int_0^T \|\operatorname{rot} u(t)\|_{L^\infty} dt < \infty$$

$u(t)$ can never break down its regularity at $t = T$. (See also [4].) To prove this assertion, in [1] they made use of the logarithmic inequality such as

$$(0.1) \quad \|\nabla u\|_{L^\infty} \leq C (1 + \|\operatorname{rot} u\|_{L^\infty} (1 + \log^+ \|u\|_{W^{s+1,p}}) + \|\operatorname{rot} u\|_{L^2}), \quad sp > n$$

for all vector functions u with $\operatorname{div} u = 0$, where $\log^+ a = \log a$ if $a \geq 1$, $= 0$ if $0 < a < 1$.

The purpose of this paper is to extend these results to BMO which is larger than L^∞ . (It is possible to extend these to more general classes, see [7].)

In a forthcoming paper, we will discuss the blow-up of smooth solutions to the Euler equations in a *bounded* domain.

1 Result.

Before stating our result, we introduce some function spaces. Let $C_{0,\sigma}^\infty$ denote the set of all C^∞ vector functions $\phi = (\phi^1, \phi^2, \dots, \phi^n)$ with compact support in $\mathbf{R}^n$, such that $\operatorname{div} \phi = 0$. L_σ^r is the closure of $C_{0,\sigma}^\infty$ with respect to the L^r -norm $\|\cdot\|_r$; $(\cdot, \cdot)$ denotes the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. L^r stands for the usual (vector-valued) L^r -space over $\mathbf{R}^n$, $1 \leq r \leq \infty$. $W_\sigma^{s,p}$ denotes the closure of $C_{0,\sigma}^\infty$ with respect to the $W^{s,p}$ -norm.

Our result on (E) reads as follows.

Theorem 1 *Let $1 < p < \infty$, $s > n/p + 1$. Suppose that u is the solution of (E) in the class $(CE)_{s,p}$ on $(0, T)$. If either*

$$(1.1) \quad \int_0^T \|\operatorname{rot} u(t)\|_{BMO} dt (\equiv M_0) < \infty$$

or

$$(1.2) \quad \int_0^T \|\operatorname{Def} u(t)\|_{BMO} dt (\equiv M_1) < \infty$$

holds, then u can be continued to the solution in the class $(CE)_{s,p}$ on $(0, T')$ for some $T' > T$.

Here $\operatorname{Def} u$ denotes the deformation tensor of u , i.e., $(\operatorname{Def} u)_{ij} = \partial_i u^j + \partial_j u^i$, $(1 \leq j, k \leq n)$.

Corollary 1 *Let u be the solution of (E) in the class $(CE)_{s,p}$ on $(0, T)$ for $1 < p < \infty$, $s > n/p + 1$. Assume that T is maximal, i.e., u cannot be continued to the solution in the class $(CE)_{s,p}$ on $(0, T')$ for any $T' > T$. Then both*

$$\int_0^T \|\operatorname{rot} u(t)\|_{BMO} dt = \infty \quad \text{and} \quad \int_0^T \|\operatorname{Def} u(t)\|_{BMO} dt = \infty$$

2 Preliminaries.

In what follows we shall denote by C various constants. In particular, $C = C(*, \dots, *)$ denotes constants depending only on the quantities appearing in the parenthesis.

We first recall the Biot-Savart law. By the Biot-Savart law, for solenoidal vectors u , we have the representation

$$(2.1) \quad \frac{\partial u}{\partial x_j} = R_j(R \times \omega), \quad j = 1, \dots, n, \quad \text{where } \omega = \text{rot } u;$$

$$(2.2) \quad \frac{\partial u^l}{\partial x_j} = R_j \left(\sum_{k=1}^n R_k (\text{Def } u)_{kl} \right), \quad j, l = 1, \dots, n, \quad \text{where } (\text{Def } u)_{kl} = \frac{\partial u^k}{\partial x_l} + \frac{\partial u^l}{\partial x_k}.$$

Here $R = (R_1, \dots, R_n)$, and $R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-\frac{1}{2}}$ denote the Riesz transforms. Since R is a bounded operator in BMO , we have by (2.1), (2.2) that

$$(2.3) \quad \|\nabla u\|_{BMO} \leq C \|\text{rot } u\|_{BMO};$$

$$(2.4) \quad \|\nabla u\|_{BMO} \leq C \|\text{Def } u\|_{BMO}.$$

Now we prove the following lemma which is an extension of (0.1).

Lemma 2.1 *Let $1 < p < \infty$ and let $s > n/p$. There is a constant $C = C(n, p, s)$ such that the estimate*

$$(2.5) \quad \|f\|_{\infty} \leq C (1 + \|f\|_{BMO} (1 + \log^+ \|f\|_{W^{s,p}}))$$

holds for all $f \in W^{s,p}$.

Remark. Compared with (0.1), we do not need to add $\|f\|_{L^2}$ to the right hand side of (2.5). This makes it easier to derive an apriori estimate of solutions to the Euler equations than Beale-Kato-Majda [1].

Proof of Lemma 2.1.

We shall make use of the Littlewood-Paley decomposition; there exists a non-negative function $\varphi \in \mathcal{S}$ ($\mathcal{S}$; the Schwartz class) such that $\text{supp } \varphi \subset \{2^{-1} \leq |\xi| \leq 2\}$ and such that $\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1$ for $\xi \neq 0$. See Bergh-Löfström [2, Lemma 6.1.7]. Let us define ϕ_0 and ϕ_1 as

$$\phi_0(\xi) = \sum_{k=1}^{\infty} \varphi(2^k \xi) \quad \text{and} \quad \phi_1(\xi) = \sum_{k=-\infty}^{-1} \varphi(2^k \xi),$$

respectively. Then we have that $\phi_0(\xi) = 1$ for $|\xi| \leq 1/2$, $\phi_0(\xi) = 0$ for $|\xi| \geq 1$ and that $\phi_1(\xi) = 0$ for $|\xi| \leq 1$, $\phi_1(\xi) = 1$ for $|\xi| \geq 2$. It is easy to see that for every positive integer N there holds the identity

$$(2.6) \quad \phi_0(2^N \xi) + \sum_{k=-N}^N \varphi(2^{-k} \xi) + \phi_1(2^{-N} \xi) = 1, \quad \xi \neq 0.$$

Since C_0^∞ is dense in $W^{s,p}$ and since $W^{s,p}$ is continuously embedded in BMO , implied by $s > n/p$, it suffices to prove (2.5) for $f \in C_0^\infty$. For such f we have the representation

$$f(x) = \int_{y \in \mathbf{R}^n} K(x-y) \cdot \nabla f(y) dy \quad \text{with} \quad K(y) = \frac{1}{n\omega_n} \frac{y}{|y|^n},$$

for all $x \in \mathbf{R}^n$, where ω_n denotes the volume of the unit ball in $\mathbf{R}^n$. By (2.6) we decompose f into three parts:

$$\begin{aligned} f(x) &= \int_{y \in \mathbf{R}^n} K(x-y) \times \\ &\quad \times \left(\phi_0(2^N(x-y)) + \sum_{k=-N}^N \varphi(2^{-k}(x-y)) + \phi_1(2^{-N}(x-y)) \right) \cdot \nabla f(y) dy \\ (2.7) \quad &\equiv f_0(x) + g(x) + f_1(x) \end{aligned}$$

for all $x \in \mathbf{R}^n$.

We can show that

$$(2.8) \quad |f_0(x)| \leq C 2^{-\beta N} \|f\|_{W^{s,p}}$$

for all $x \in \mathbf{R}^n$, where $\beta = \beta(n, p, s)$ is a positive constant. For detail, see [6].

By integration by parts we have

$$g(x) = \sum_{k=-N}^N (\operatorname{div} \Psi)_{2^k} * f(x), \quad x \in \mathbf{R}^n,$$

where $\Psi(x) = K(x)\varphi(x)$ and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Since $\Psi \in \mathcal{S}$ with the property that

$$\int_{\mathbf{R}^n} \operatorname{div} \Psi(x) dx = 0,$$

it follows from Stein [9, Chap. IV, 4.3.3] that

$$\begin{aligned} \|g\|_\infty &\leq \sum_{k=-N}^N \|(\operatorname{div} \Psi)_{2^k} * f\|_\infty \\ &\leq \sum_{k=-N}^N \sup_{t>0} \|(\operatorname{div} \Psi)_t * f\|_\infty \\ (2.9) \quad &\leq CN \|f\|_{BMO}, \end{aligned}$$

where $C = C(n)$ is independent of N .

Integrating by parts, we have by a direct calculation

$$\begin{aligned} |f_1(x)| &= \left| \int_{y \in \mathbf{R}^n} \operatorname{div}_y \left(K(x-y) \phi_1(2^{-N}(x-y)) \right) f(y) dy \right| \\ (2.10) \quad &\leq C 2^{-N \cdot \frac{n}{p}} \|f\|_p \end{aligned}$$

for all $x \in \mathbf{R}^n$, where $C = C(n, p)$ is independent of N .

Now it follows from (2.7) and (2.8)-(2.10) that

$$(2.11) \quad \|f\|_\infty \leq C(2^{-\gamma N} \|f\|_{W^{s,p}} + N \|f\|_{BMO})$$

with $\gamma = \text{Min}\{\beta, n/p\}$, where $C = C(n, s, p)$ is independent of N and f . If $\|f\|_{W^{s,p}} \leq 1$, then we may take $N = 1$; otherwise, we take N so large that the first term of the right hand side of (2.11) is dominated by 1, i.e., $N \equiv \left\lceil \frac{\log \|f\|_{W^{s,p}}}{\gamma \log 2} \right\rceil + 1$ ($\lceil \cdot \rceil$; Gauss symbol) and (2.11) becomes

$$\|f\|_\infty \leq C \left\{ 1 + \|f\|_{BMO} \left(\frac{\log \|f\|_{W^{s,p}}}{\gamma \log 2} + 1 \right) \right\}.$$

In both cases, (2.5) holds. This proves Lemma 2.1.

3 Proof of Theorem 5.

We follow the argument of Beale-Kato-Majda [1]. It is proved by Kato-Lai [3] and Kato-Ponce [4] that for the given initial data $a \in W_\sigma^{s,p}$ for $s > 1 + n/p$, the time interval T of the existence of the solution u to (E) in the class $(CE)_{s,p}$ depends only on $\|a\|_{W^{s,p}}$. Hence by the standard argument of continuation of local solutions, it suffices to establish an apriori estimate for u in $W^{s,p}$ in terms of a, T, M_0 or a, T, M_1 according to (1.1) or (1.2). Indeed, we shall show that the solution $u(t)$ in the class $(CE)_{s,p}$ on $(0, T)$ is subject to the following estimate:

$$(3.12) \quad \sup_{0 < t < T} \|u(t)\|_{W^{s,p}} \leq (\|a\|_{W^{s,p}} + e)^{\alpha_j} \exp(CT\alpha_j) \quad \text{with } \alpha_j = e^{CM_j}, \quad j = 0, 1,$$

where $C = C(n, p, s)$ is a constant independent of a and T .

We shall first prove (3.12) under (1.1). It follows from the commutator estimate in L^p given by Kato-Ponce [4, Proposition 4.2] that

$$(3.13) \quad \|u(t)\|_{W^{s,p}} \leq \|a\|_{W^{s,p}} \exp \left(C \int_0^t \|\nabla u(\tau)\|_\infty d\tau \right), \quad 0 < t < T,$$

where $C = C(n, p, s)$.

By the Biot-Savard law (2.1), we have

$$(3.14) \quad \|\nabla u\|_{BMO} \leq C \|\text{rot } u\|_{BMO}$$

with $C = C(n)$. Hence it follows from (3.14) and Lemma 2.1 that

$$(3.15) \quad \|\nabla u(t)\|_\infty \leq C (1 + \|\text{rot } u(t)\|_{BMO} (1 + \log^+ \|u(t)\|_{W^{s,p}}))$$

for all $0 < t < T$ with $C = C(n, p, s)$. Substituting (3.15) to (3.13), we have

$$\begin{aligned} & \|u(t)\|_{W^{s,p}} + e \\ & \leq (\|a\|_{W^{s,p}} + e) \exp \left(C \int_0^t \{1 + \|\text{rot } u(\tau)\|_{BMO} \log(\|u(\tau)\|_{W^{s,p}} + e)\} d\tau \right) \end{aligned}$$

for all $0 < t < T$. Defining $z(t) \equiv \log(\|u(t)\|_{W^{s,p}} + e)$, we obtain from the above estimate

$$z(t) \leq z(0) + CT + C \int_0^t \|\operatorname{rot} u(\tau)\|_{BMO} z(\tau) d\tau, \quad 0 < t < T.$$

Now (1.1) and the Gronwall inequality yield

$$\begin{aligned} z(t) &\leq (z(0) + CT) \exp \left(C \int_0^t \|\operatorname{rot} u(\tau)\|_{BMO} d\tau \right) \\ &\leq (z(0) + CT) \alpha_0 \end{aligned}$$

for all $0 < t < T$ with $C = C(n, p, s)$, which implies (3.12) for $j = 0$.

Similarly we prove (3.12) for $j = 1$ under (1.2). This proves Theorem 5.

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