

Semigroup rings over semiprime ring semigroups

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1 Introduction

The talk presented was based on an on-going joint work with Y.Hirano (Naruto) and B.Solie (ERAU). More detailed and completed version of this manuscript will be submitted elsewhere.

Let R be a ring, and let S be a semigroup. The semigroup ring $R[S]$ simultaneously encodes the semigroup structure of S and the ring structure resulting in an object of great utility in various areas of both ring theory and group theory. The interplay between the structures of R , S , and $R[S]$ is a topic with a long mathematical history full of fascinating results. For instance, much is already known about the structure of $R[G]$, where G is not just a semigroup but a group. Here, we find that the structure of $R[G]$ is exactly characterized by the structure of R and various finiteness properties of G . Maschke's famous theorem states that if G is a finite group and K is a field whose characteristic does not divide $|G|$, then $K[G]$ is semisimple [3]. Further variations include the result that $R[G]$ is prime if and only if R is a prime ring and G has no nontrivial finite normal subgroup [1]. More generally, $R[G]$ is semiprime if and only if R is semiprime and the order of every normal finite subgroup of G is a non-zero-divisor in R [5].

Consequently, one may ask whether the primeness or semiprimeness of $R[S]$ can be characterized in the case where S is an arbitrary semigroup. This is a much more difficult question on which some progress has been made in recent years. For highly structured classes of semigroups, it is again possible to determine the structure of $R[S]$. For example, when S is a cancellative semigroup, then $R[S]$ is semiprime whenever R is semiprime [4]. When F is a field and S is finite,

a version of Maschke's theorem shows that the semisimplicity of $R[S]$ is characterized by the nonsingularity of the structure matrix for S when viewed as a matrix over F [2].

We consider semigroup rings over a particular class of semigroups: those semigroups which arise as the multiplicative semigroup of a ring.

2 Main Results and Examples

Let K be a field and let S be a semigroup. Recall that the *semigroup ring* $K[S]$ consists of the set of all sums $\sum_{s \in S} k_s \hat{s}$ with $k_s = 0$ for all but finitely many $s \in S$. We equip $K[S]$ with the usual addition and multiplication, where $\hat{s}\hat{t} = \widehat{st}$ for all $s, t \in S$.

Given a ring $(R, +, \cdot)$, we may forget addition and thereby obtain a semigroup $(R, \cdot)$ having both 0 and 1. We denote by $K[R]$ the semigroup ring $K[(R, \cdot)]$. Note that as $K[R] = K[0] \oplus \text{Ann}(K[0])$, $K[R]$ is not a prime ring. It is easy to observe that $K[R]$ is a direct sum of two simple rings if and only if every proper nonzero ideal of $K[R]$ is prime. Every prime ideal of $K[R]$ except $\text{Ann}(K[0])$ contains $K[0]$, and every ideal of $K[R]$ contained in $\text{Ann}(K[0])$ does not contain $K[0]$.

Theorem 1 *Let R be a ring and K be a field of characteristic zero. If $K[R]$ is semiprime, then R is semiprime. The converse holds if R is a commutative ring or a domain.*

Theorem 2 *Let R be a ring and let K be a field of characteristic zero. Then $K[R]$ is a semisimple Artinian ring if and only if R is a finite semisimple ring.*

For a ring R , we shall denote the Jacobson radical of R by $J(R)$.

Theorem 3 *Let R be a local ring with finitely many units, let R^* denote the group of units in R and let K be a field of characteristic zero. Then $J(K[R]) = \sum_{r \in J(R)} K(\hat{r} - \hat{0})$ and $K[R]/J(K[R]) \cong K[0] \oplus K[R^*]$.*

Let K be an algebraically closed field of characteristic zero, and let $\mathbb{Z}_n$ denote the ring of integers modulo n . It is immediate that $\mathbb{Z}_n$ is semiprime if and only if n is squarefree, and thus we have the following proposition as a corollary of Theorem 1.

Proposition Let K be an algebraically closed field of characteristic zero, and let $\mathbb{Z}_n$ denote the ring of integers modulo n . Then $K[\mathbb{Z}_n]$ is semiprime if and only if n is squarefree.

We now present a few examples of the structure of $K[R]$ for some finite ring R .

Example 1 It is clear that $K[\mathbb{Z}_2] \cong K \oplus K$.

Let V be a vector space over K . We define a multiplication on the K -linear space $K \oplus V$ by the formula $(a, v) \cdot (b, w) = (ab, aw + bv)$ for any $a, b \in K, v, w \in V$. Then $K \oplus V$ becomes a K -algebra, which we denote by $K \ltimes V$.

Example 2 Consider the ring $K[\mathbb{Z}_4]$ and let $g_i = \hat{i} - \hat{0}$ for $i = 1, 2, 3$. Then $K[\mathbb{Z}_4]$ is the direct sum of two sided ideals $K[0]$ and $S = Kg_1 + Kg_2 + Kg_3$. The identity of the ring S is g_1 . Let us set

$e_1 = \frac{1}{2}(g_1 - g_3), e_2 = \frac{1}{2}(g_1 + g_3)$. Then e_1, e_2 are orthogonal central primitive idempotents of S and $g_1 = e_1 + e_2$. We can easily see that $K[\mathbb{Z}_4] \cong K^2 \oplus (K \ltimes K)$.

Example 3 Consider the ring $K[\mathbb{Z}_6]$ and let $g_i = \hat{i} - \hat{0}$ for $i = 1, 2, \dots, 5$. Then $K[\mathbb{Z}_6]$ is the direct sum of $K[0]$ and $S = Kg_1 + Kg_2 + \dots + Kg_5$. Set $e_1 = \frac{1}{2}(g_1 + g_5)$ and $e_2 = \frac{1}{2}(g_1 - g_5)$. We again have orthogonal central primitive idempotents e_1 and e_2 in S and $g_1 = e_1 + e_2$, and moreover $e_1 S \cong K^3$ and $e_2 S \cong K^2$. Thus we have that $K[\mathbb{Z}_6] \cong K^6$.

Example 4 Consider the ring $K[\mathbb{Z}_8]$ and let $g_i = \hat{i} - \hat{0}$ for $i = 1, 2, \dots, 7$. Then $K[\mathbb{Z}_8]$ is the direct sum of two sided ideals $K[\hat{0}]$ and $S = Kg_1 + Kg_2 + \dots + Kg_7$. The identity of the ring S is g_1 . Let us set

$$e_1 = \frac{1}{4}(g_1 - g_3 - g_5 + g_7)$$

$$e_2 = \frac{1}{4}(g_1 + g_3 - g_5 - g_7)$$

$$e_3 = \frac{1}{4}(g_1 - g_3 + g_5 - g_7)$$

$$e_4 = \frac{1}{4}(g_1 + g_3 + g_5 + g_7).$$

Then e_1, e_2, e_3, e_4 are orthogonal central primitive idempotents of S and $g_1 = e_1 + e_2 + e_3 + e_4$. We can easily see $e_1 S \cong K, e_2 S \cong K, e_3 S \cong K \ltimes K$, and $e_4 S \cong K \ltimes (K \oplus K)$. Therefore we have that $K[\mathbb{Z}_8] \cong K^3 \oplus (K \ltimes K) \oplus (K \ltimes (K \oplus K))$.

Example 5 Consider the ring $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in GF(2) \right\}$ of order $|R| = 2^4 = 16$.

Then $R^* = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a \neq 0, b, c, d \in GF(2) \right\}$. We can easily see that R^* is the dihedral group

D_8 of order 8 and so $K[R^*] \cong K^4 \oplus M_2(K)$. Therefore, by Theorem 3, we have that $K[R]/J(K[R]) \cong K^5 \oplus M_2(K)$.

Example 6 Let $M_2(GF(2))$ denote the ring of 2×2 matrices over the field $GF(2)$. Then we can prove that $Z[M_2(GF(2))]$ is a semiprime ring. Let us set $H = M_2(GF(2)) - GL_2(GF(2))$. Then we can see that $Q[H] \cong Q \oplus M_3(Q)$. In fact, let

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad e_7 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$e_8 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad e_9 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then the elements $E = \hat{O}$, $F_1 = \hat{e}_1 - \hat{O}$, $F_2 = \hat{e}_2 - \hat{O}$,

$$F_3 = (\hat{e}_1 - \hat{O}) + (\hat{e}_2 - \hat{O}) + (\hat{e}_3 - \hat{O}) + (\hat{e}_4 - \hat{O}) - (\hat{e}_5 - \hat{O}) \\ - (\hat{e}_6 - \hat{O}) - (\hat{e}_7 - \hat{O}) - (\hat{e}_8 - \hat{O}) + (\hat{e}_9 - \hat{O})$$

are primitive orthogonal idempotents, and $Q[H] = QE \oplus Q[H](F_1 + F_2 + F_3) \cong Q \oplus M_3(Q)$.

Since $GL_2(GF(2)) \cong S_3$, we have $Q[M_2(GF(2))]/Q[H] \cong Q[S_3]$.

It is easily see that $Q[S_3] \cong Q \oplus Q \oplus M_2(Q)$. Hence $Q[M_2(GF(2))]$ is isomorphic to the semisimple Artinian ring $Q \oplus Q \oplus Q \oplus M_2(Q) \oplus M_3(Q)$.

References

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