

On smoothing properties of Nonlinear Schrödinger Equations

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1 Introduction and Main Results

In this paper, we estimate smoothing properties of local solutions to nonlinear Schrödinger equations with initial data which belong to Sobolev spaces H^s , $s \geq 0$.

We consider the following equation;

$$i\partial_t u = -\Delta u + F(u), \tag{1}$$

where u is a complex-valued function of $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\partial_t = \partial/\partial t$, Δ is the Laplacian on $\mathbb{R}^n$, and $F(u) = F \circ u$ is a local nonlinear operator given by a complex-valued function F on $\mathbb{C}$.

Here we consider the following assumptions on the nonlinear term F in Kato[8];

Assumption (F1) $F \in C^{\{s\}}(\mathbb{C}; \mathbb{C})$, with $F(0) = 0$, where $\{s\} = [s] + 1$ if $s \notin \mathbb{Z}$ and $\{s\} = s$ if $s \in \mathbb{Z}$, and $\{0\} = 1$.

Assumption (F2) If $s > n/2$, no assumption.

If $s \leq n/2$ and if F is a polynomial in $\zeta = \xi + i\eta$ and $\bar{\zeta}$, then degree of F is equal to $k \leq 1 + 4/(n - 2s)$.

If $s \leq n/2$ and if F is not a polynomial,

$$|D^i F(\zeta)| \equiv \max_{0 \leq j \leq i} \{|\partial_{\zeta}^j \partial_{\bar{\zeta}}^{i-j} F|\} \leq M_i |\zeta|^{k-i}, \quad \text{for } |\zeta| \geq 1,$$

for $i = 0, 1, \dots, \{s\}$, where $\partial_{\zeta} = (\partial_{\xi} - i\partial_{\eta})/2$, $\partial_{\bar{\zeta}} = (\partial_{\xi} + i\partial_{\eta})/2$,

and k is a finite number such that

$$\{s\} \leq k \leq 1 + \frac{4}{n-2s}.$$

We denote $\partial_x = (\partial_1, \dots, \partial_n)$, for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, $\partial_x^\alpha = (\partial_1^{\alpha_1}, \dots, \partial_n^{\alpha_n})$ and we often denote ∂_x by ∂ , and use ∂ as ∂^α if $|\alpha| = 1$.

The Cauchy problems of equation (1) with above assumptions were studied by many authors. In Kato[6], Tsutsumi[17], they discussed about local or global wellposedness in the case that the initial data belongs to L^2 , in Ginibre and Velo[3], Kato[5][6], they discussed about local wellposedness in the case that the initial data belongs to H^1 , and in Ginibre and Velo[4], Kato[5][6], about the existence of global solution. Sjölin's result[15] that we noted below was based on the existence of local solutions discussed in Kato[5]. In Kato[6], he discussed about local wellposedness in the case that the initial data belongs to H^2 , and in

Kato[5][6], Tsutsumi[18], about the existence of global solution.

We try to estimate regularity of local solutions to equation (1) with $u(0) = u_0 \in H^s$, for $s \geq 0$, which were obtained by Kato[8].

In Sjölin[14], he obtained the following inequality; For some $C > 0$, depending on $\phi \in C_0^\infty(\mathbb{R}^{n+1})$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |\phi(t, x)(1 - \Delta)^{\frac{1}{4}} e^{it\Delta} f|^2 dx dt \leq C \|f\|_{L^2}^2, \quad \forall f \in L^2. \quad (2)$$

This inequality manifests that the free Schrödinger propagator $e^{it\Delta}$ has the smoothing effect which can improve the properties of differentiability locally in time and space. Independently, Vega[19] was obtained the same as the local smoothing property for the free Schrödinger equation, Constantin and Saut[2] were obtained the same as the one for general dispersive equations in homogeneous or inhomogeneous cases.

Later the similar property for e^{-itH} , where $H = -\Delta + V$ is a self-adjoint operator and $V = V(x)$ or $V(t, x)$ are various scalar

potentials, was studied by many authors (Ben-Artzi[1], Ruiz and Vega[12], etc.). In particular, Kato and Yajima[9] obtained the inequality replacing ϕ in (2) by $(1 + |x|^2)^{-\frac{1}{4}-\varepsilon}$, $\varepsilon > 0$, and Yajima[21] obtained the similar estimate for the propagators of Schrödinger equations with time dependent magnetic and scalar potentials which may increase at infinity $|x| \rightarrow \infty$.

In Sjögren and Sjölin[13], they obtained the extension of (2) in the following form; They defined

$$\mathcal{A} \equiv \{\varphi \in C^\infty(\mathbb{R}^n) \mid \text{There exists } \varepsilon > 0 \text{ such that}$$

$$|\partial^\alpha \varphi(x)| \leq C_\alpha (1 + |x|)^{-\frac{1}{2}-\varepsilon}, \forall \alpha\},$$

where introduced mixed Sobolev spaces $H^{r,\rho} = H^{r,\rho}(\mathbb{R} \times \mathbb{R}^n) = (G_r \otimes G_\rho) * L^2(\mathbb{R}^{n+1})$, where G_r and G_ρ are Bessel kernels in $\mathbb{R}$ and $\mathbb{R}^n$, respectively. If $r \geq 0$, $\rho \geq 0$, then, for each $\varphi \in \mathcal{A}$, $\psi \in C_0^\infty(\mathbb{R})$,

$$\|\psi \varphi e^{-it\overline{H}} u\|_{H^{r,\rho}} \leq C_{\psi\varphi} \|u\|_{H^{mr+\rho-\frac{1}{2}(m-1)}}, \quad (3)$$

for some $C_{\varphi,\psi} > 0$. Here $\overline{H} = -P + V$, where P is a elliptic operators with constant coefficient which degree is $m \geq 2$ and $V = V(x)$ is a real-valued function in C^∞ with $D^\alpha V$ bounded for every α .

In Sjölin[15][16] he adapted these estimates for equation (1), nonlinear Schödinger equation, with H^1 or H^2 -initial data. The author extended in [10] p 's range when the initial data belong to H^2 .

Kato[8] proved that assume (F1) and (F2), if $s \geq n/2$, or if $s < n/2$ and $k < 1 + \max\{4, 2s+2\}/(n-2s)$, and $k < 2/(1-2s)$ if $k = 1$, then there exists a number $T > 0$ and a unique solution $u \in C(I; H^s)$ of equation (1) with $u(0) = u_0 \in H^s$, $s > 0$, where $I = [0, T]$. Moreover $u \in \bar{\mathcal{Y}}_s(I) \equiv \{u \in \bar{\mathcal{X}}(I) | \Lambda^\sigma u \in \bar{\mathcal{X}}(I), 0 \leq \sigma \leq s\}$, where

$$\bar{\mathcal{X}}(I) \equiv \left(\bigcap L^s(I; L^q)\right) \cap C(I; L^2), \quad (4)$$

where $\cap$ is the intersection in (q, s) satisfying $1/q + 2/ns = 1/2$ and $1/2 - 1/n < 1/q \leq 1/2$ and $\Lambda^\sigma u = (-\Delta)^{\sigma/2} u$. In this paper we shall adapt Sjölin's estimates to the solution of equation (1) with H^s -initial values. We obtain the following Theorem.

Theorem 1. Assume (F1) and (F2). If $s > 1$, then the unique solution $u \in C(I; H^s)$ of the equation (1) with $u(0) = u_0 \in H^s$ satisfies $\varphi u \in L^2(I; H^{s+1/2})$ for each $\varphi \in \mathcal{A}$. If $0 \leq s \leq 1$, then, in $1 \leq n \leq 2s + 4$, $\varphi u \in L^2(I; H^{s+1/2})$ for each $\varphi \in \mathcal{A}$. When $n \geq 2s + 5$, under the additional condition $p < 1 + 2/(n - 2s - 2)$ $\varphi u \in L^2(I; H^{s+1/2})$ for each $\varphi \in \mathcal{A}$.

Remark 1 Applying this theorem to Theorem 1 in Yajima[21], we can obtain the regularity estimate of solutions to nonlinear Shrödinger equations with magnetic fields in the case that the initial data belongs to H^s . The author[11] obtained the results in the case of H^1 .

We use the standard notations. We abbreviate $L^p(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n)$ to L^p and H^k , respectively. We denote usual L^p -norm by $\|\cdot\|_p$. For $I = [0, T]$, put $L^{p,r} = L^r(I; L^p)$, where $1 < p < \infty, 1 \leq r \leq \infty$, with its norm denoted by

$$\|f\|_{p,r} \equiv \left(\int_I \|f(t)\|_p^r dt \right)^{\frac{1}{r}}.$$

We denote various constants by C, M , etc. They may differ from line to line.

2 Proof of Theorem

We introduce the following linear operators.

$$(\Gamma\phi)(t) = U(t)\phi = e^{it\Delta}\phi, \quad t \in I, \quad (5)$$

$$(Gf)(t) = \int_0^t U(t-\tau)f(\tau)d\tau, \quad t \in I. \quad (6)$$

The following lemmas is well known. (see, for example, Kato[5][6][7], Yajima[20])

Lemma 2.1 Γ is a bounded operator from L^2 to L^{p_1, r_1} , where (p_1, r_1) satisfies $1/p_1 + 2/nr_1 = 1/2$ and $1/2 - 1/n < 1/p_1 \leq 1/2$. The bound is independent of T , and is uniform for any (p_1, r_1) . Here $L^{2, \infty}$ may be replaced by $C(I; L^2)$.

Lemma 2.2 G is a bounded operator from L^{p_2, r_2} to L^{p_1, r_1} , where (p_1, r_1) and (p_2, r_2) satisfy $1/p_1 + 2/nr_1 = 1/2$ and $1/2 + 1/n < 1/p_1 \leq 1/2$, $1/p_2 + 2/nr_2 = 1/2 + 2/n$ and $1/2 \leq 1/p_2 < 1/2 + 1/n$, respectively. The bound is independent of T , and is uniform for any (p_1, r_1) and (p_2, r_2) . Here $L^{2, \infty}$ may be replaced by $C(I; L^2)$.

For the nonlinear term F , the following lemma is well known.

Lemma 2.3 (Kato[8]) Assume (F1) and (F2) with $s \leq n/2$. If F is a polynomial of degree k , it is obviously the sum of homogeneous polynomials of degree 1 to k . Otherwsie, F can

be written in the form

$$F = F_1 + F_2 + \cdots + F_{\{s\}-1} + F_{\{s\}} + F_k,$$

where the F_j for $j = 1, 2, \dots, \{s\} - 1$ are homogeneous polynomials of degree j , while $F_{\{s\}}$ and F_k are quasi-homogeneous of order $\{s\}$ and of degree $\{s\}$ and k , respectively. If $k = \{s\}$, $F_{\{s\}}$ is redundant and should be omitted.

We say that a function $F : \mathbb{C} \rightarrow \mathbb{C}$ is quasi-homogeneous of degree k and order m , if the following estimate hold;

$$|D^i F(\zeta)| \leq M |\zeta|^{k-i}, \text{ for } 0 \leq i \leq m, \zeta \in \mathbb{C}.$$

Lemma 2.4 (Kato[8]) Let $F \in C^j(\mathbb{C}; \mathbb{C})$, $j \in \mathbb{Z}$. Assume that there is $k \geq j$ such that

$$|D^i F(\zeta)| \leq M |\zeta|^{k-i}, \quad i = 1, 2, \dots, j.$$

If $0 \leq \sigma \leq j$, then

$$\|\Lambda^\sigma F(\phi)\|_r \leq c \|\phi\|_q^{k-1} \|\Lambda^\sigma \phi\|_p, \quad 1/r = 1/p + (k-1)/q, \quad p, q, r \in (1, \infty),$$

where c depends on σ , p , q and r . Moreover,

$$\|\Lambda^\sigma F(u)\|_{r_1, r_2} \leq c \|u\|_{q_1, q_2}^{k-1} \|\Lambda^\sigma u\|_{p_1, p_2}, \text{ for, } 1/r_j = 1/p_j + (k-1)/q_j, \quad j = 1, 2,$$

where c depends on σs , p_j , q_j and r_j . And if $F(\zeta)$ is a polynomial in ζ and $\bar{\zeta}$ of degree $k \geq 1$, then the above inequalities are true for any $\sigma \geq 0$.

Lemma 2.5 Assume (F1) and (F2). Suppose $u \in \bar{\mathcal{Y}}_s(I_0)$, where $I_0 = [0, T_0]$ with $T_0 < \infty$, and $\bar{\mathcal{X}}(I_0)$ is defined in (6) with I_0 instead of I . If $s > 1$, then $F(u) \in L^1(I_0; H^s)$. If $0 \leq s \leq 1$, and $1 \leq n \leq 2s + 4$, then $F(u) \in L^1(I_0; H^s)$. If $0 \leq s \leq 1$, and $n \geq 2s + 5$, then $F(u) \in L^1(I_0; H^s)$ with the additional assumption $k < 1 + 2/(n - 2s - 2)$.

Proof. We may estimate each $F_j(u)$, $j = 1, 2, \dots, \{s\}, k$, in Lemma 2.3. It suffices to find (p_1, r_1) , (p_2, r_2) which satisfy

$$\frac{j-1}{p_1} + \frac{1}{p_2} = \frac{1}{2}, \quad (7)$$

$$\frac{j-1}{r_1} + \frac{1}{r_2} < 1, \quad (8)$$

$$\frac{1}{2} - \frac{s+1}{n} < \frac{1}{p_1} \leq \frac{1}{2}, \quad (9)$$

$$\frac{1}{2} - \frac{1}{n} < \frac{1}{p_2} \leq \frac{1}{2}, \quad (10)$$

and

$$\frac{1}{r_1} = \max\{0, \frac{n}{2}(\frac{1}{2} - \frac{s}{n} - \frac{1}{p_1})/2\}, \quad (11)$$

$$\frac{1}{r_2} = \frac{n}{2}(\frac{1}{2} - \frac{1}{p_2}), \quad (12)$$

Indeed, by (10), (12) and the definition of $\bar{\mathcal{X}}(I_0)$, we have $\bar{\mathcal{X}}(I_0) \subset L^{p_2, r_2}$, and, set $1/p_3 = 1/p_1 + s/n$, then, from (9) and (11), it follows $\bar{\mathcal{X}}(I_0) \subset L^{p_3, r_1}$. Hence, by Sobolev's embedding theorem, we have $\bar{\mathcal{Y}}_s(I_0) \subset L^{p_1, r_1}$. And, if (7) and (8) hold, then we have, by Hölder inequality,

$$\|\Lambda^\sigma u F_j(u)\|_{2,1} \leq MT^{1-1/r_3} \|u\|_{p_1, r_1}^{j-1} \|\Lambda^\sigma u\|_{p_2, r_2}, \quad (13)$$

where $(p-1)/r_1 + 1/r_2 = 1/r_3 < 1$ and for $j = 1, 2, \dots, \{s\}, k$ and $0 \leq \sigma \leq s$. Hence the right hand side of (13) converge,

respectively.

Actually, there are (p_1, r_1) and (p_2, r_2) satisfying the equalities and inequalities from (7) to (12). Solving (7) and (8), we obtain that $1/p_1 = 2/nr_2(j-1)$. By $1/2 - (s+1) < 1/p_1 \leq 1/2$, we have $1/2 - (s+1) < 2/nr_2(j-1) \leq 1/2$, namely $(n-2s-2)(j-1)/4 < 1/r_2 \leq n(j-1)/4$. By $1 \leq j \leq k$, $(n-2s-2)(k-1)/4 < 1/r_2 \leq n(k-1)/4$. (10) and (12) implies that $2 < r_2 \leq \infty$. Since we assume the condition $p < 1 + 2/(n-2s-2)$, there is a desirable r_2 .

Proof of Theorem 1. Recall that $u = \Gamma u_0 - iGF(u)$ is the unique solution of equation (1). Let $\varphi \in \mathcal{A}$. Then $\varphi u = \varphi \Gamma u_0 - i\varphi GF(u)$. Since $\|\varphi \Gamma u_0\|_{L^2(I; H^{s+1/2})} \leq C\|u_0\|_{H^s}$ by (3) with $\overline{H} = -\Delta$, $r = 0$, $\rho = s + 1/2$, it suffices to estimate $\|\varphi GF(u)\|_{L^2(I; H^{s+1/2})}$. But it is easily seen that Theorem 3.1 in Canstantin and Saut[2] holds for $\varphi \in \mathcal{A}$ instead of $\chi \in C(\mathbb{R}^{n+1})$.

We note $\varphi(\check{x}_j) \in L^2(\mathbb{R}_{x_j})$, where $\check{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$, for $j = 1, 2, \dots, n$. Thus it suffices to prove $F(u) \in L^1(I; H^s)$, $s \geq 0$. By Lemma 2.5, we can prove Theorem.

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