

A note on the Rankin-Selberg method for Siegel cusp forms of genus 2

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1 Introduction and Notations

In [K-S] Kohnen and Skoruppa introduced and studied a new type of Dirichlet series, which is associated with the Fourier-Jacobi expansion of a pair F, G of Siegel cusp forms of the same weight and genus 2. The proof is based on the Rankin-Selberg method. In particular, it was shown that this Dirichlet series is equal to the Spinor zeta function attached to F up to constant on condition that F is a Hecke eigenform and G is in the “Maass space”.

In the present note we extend a part of results in [K-S] to the case of *any level*. As an application, we give a new proof of meromorphic continuation of the Spinor zeta function attached to a Siegel cusp form F of any level (on a condition for Fourier coefficients of F), and find certain functional equation satisfied by the Spinor zeta function of any level > 1 . We also prove the Spinor zeta function of F times a simple meromorphic function is entire if F is not in a certain Maass space, which was proved in the level 1 case in [Ev 2], [K-S], [O].

We remark that it is relatively easy to study Kohnen-Skoruppa’s Dirichlet series, even in the case of higher level (or even in the case of half-integral weight), because of its simple integral representation.

Notations. We use standard notations, found in [Ei-Z]. We let $\Gamma^g := \mathrm{Sp}_g(\mathbf{Z})$ be integral symplectic $2g \times 2g$ -matrices and set

$$\Gamma_0^g(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \mid C \equiv O \pmod{N} \right\},$$

where A, B, C, D are $g \times g$ -matrices. We let $\Gamma^{1,J}(N)$ be the semi-direct product of $\Gamma_0^1(N)$ and $\mathbf{Z}^2$ (see [Ei-Za, p.9]), which is called the Jacobi group of level N .

$\mathcal{H}_g$ denotes the Siegel upper half space of genus g consisting of complex $g \times g$ -matrices with positive definite imaginary part. We often write

$$Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_2, \quad X = \mathrm{Re}(Z) = \begin{pmatrix} u & x \\ x & u' \end{pmatrix}, \quad Y = \mathrm{Im}(Z) = \begin{pmatrix} v & y \\ y & v' \end{pmatrix}.$$

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We usually set $|Y| = \det Y$.

Let k be an even integer > 2 . Γ^2 acts on $\mathcal{H}_2$ by

$$\gamma\langle Z \rangle := (AZ + B)(CZ + D)^{-1} \quad \left(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^2, Z \in \mathcal{H}_2 \right),$$

and acts on any function $F(Z)$ on $\mathcal{H}_2$ by

$$F|_k \gamma(Z) := \det(CZ + D)^{-k} F(\gamma\langle Z \rangle).$$

$\Gamma^{1,J}(N)$ acts on any function $\phi(\tau, z)$ on $\mathcal{H}_1 \times \mathbf{C}$ by

$$\phi|_{k,m} \gamma(\tau, z) = \frac{1}{(c\tau + d)^k} \mathbf{e} \left(m \left(\frac{-cz^2}{c\tau + d} + \lambda^2 \frac{a\tau + b}{c\tau + d} + \frac{2\lambda z}{c\tau + d} \right) \right) \phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda(a\tau + b)}{c\tau + d} + \mu \right)$$

$$(\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda, \mu \right) \in \Gamma^{1,J}(N), (\tau, z) \in \mathcal{H}_1 \times \mathbf{C}),$$

where m denotes an integer ≥ 0 .

We write simply $\mathbf{e}(\ast)$ for $\exp(2\pi i \ast)$.

Definition. Let χ be a Dirichlet character modulo N . A *Siegel modular form* of integral weight k , level N and character χ is a holomorphic function on $\mathcal{H}_2$ satisfying

$$(i) \quad F|_k \gamma = \chi(\det D) F \quad (\forall \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^2(N))$$

and the vector space of all such functions F is denoted by $M_k(N, \chi)$. If $F \in M_k(N, \chi)$ satisfies

$$(ii) \quad \Phi(F|_k \gamma) = 0 \quad (\forall \gamma \in \Gamma^2, \Phi \text{ is the Siegel operator, cf. [A, p.75]}),$$

F is called a *Siegel cusp form* and the vector space of all such functions F is denoted by $S_k(N, \chi)$.

A *Jacobi cusp form* ϕ of weight k , level N , character χ and index m is a holomorphic function on $\mathcal{H}_1 \times \mathbf{C}$ satisfying

$$(i)' \quad \phi|_{k,m} \gamma = \chi(d) \phi \quad (\forall \gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right) \in \Gamma^{1,J}(N))$$

$$(ii)' \quad \phi|_{k,0} \gamma = \sum_{\substack{n,r \in \frac{1}{N_\gamma} \mathbf{Z} \\ D = \tau^2 - 4mn < 0}} c(D, \tau) q^n \zeta^r \quad (\forall \gamma \in \Gamma^1, N_\gamma \text{ is a natural number depending on } \gamma)$$

and the vector space of all such functions ϕ is denoted by $J_{k,m}^{\text{cusp}}(N, \chi)$.

The Petersson inner product on these spaces are normalized by

$$\langle F, G \rangle_N := \int_{\Gamma_0^2(N) \backslash \mathcal{H}_2} F(Z) \bar{G}(Z) |Y|^{k-3} dX dY$$

$$(F, G \in M_k(N, \chi), Z = X + iY \in \mathcal{H}_2, \text{ One of } F, G \text{ is in } S_k(N, \chi)),$$

$$\langle \phi, \psi \rangle_N := \int_{\Gamma^{1,J}(N) \backslash \mathcal{H}_1 \times \mathbf{C}} \phi(\tau, z) \bar{\psi}(\tau, z) v^{k-3} \exp \left(-\frac{4\pi my^2}{v} \right) du dv dx dy$$

$$(\phi, \psi \in J_{k,m}^{\text{cusp}}(N, \chi), \tau = u + iv \in \mathcal{H}_1, z = x + iy \in \mathbf{C}).$$

2 Statement of Result

Definition. Take $F \in S_k(N, \chi)$, $G \in M_k(N, \chi)$ and a natural number M which divides N . For $\gamma \in \Gamma^2 = \mathrm{Sp}_2(\mathbf{Z})$, we write the Fourier-Jacobi expansions of $F|_k \gamma$ and $G|_k \gamma$ by

$$F|_k \gamma = \sum_{n \geq 1} \phi_{n, \gamma}(\tau, z) \mathbf{e}\left(\frac{n\tau'}{N}\right) \quad \text{and} \quad G|_k \gamma = \sum_{n \geq 1} \psi_{n, \gamma}(\tau, z) \mathbf{e}\left(\frac{n\tau'}{N}\right).$$

Then we define a *Dirichlet series* $D_{F, G, M}(s)$ as $\zeta(2s - 2k + 4)$ times

$$\sum_{n \geq 1} \left\{ \int_{\mathcal{F}} \sum_{\gamma \in \Gamma_0^2(N) \backslash \Gamma_0^2(M)} \phi_{n, \gamma}(\tau, z) \bar{\psi}_{n, \gamma}(\tau, z) \exp\left(-\frac{4\pi n y^2}{vN}\right) v^{k-3} du dv dx dy \right\} n^{-s}, \quad (1)$$

on the assumption that $D_{F, G, M}(s)$ converges for sufficiently large $\mathrm{Re}(s)$, where $\mathcal{F}$ is a fundamental domain $\Gamma^{1, J}(M) \backslash \mathcal{H}_1 \times \mathbf{C}$. We define its gamma factor by

$$D_{F, G, M}^*(s) := (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) D_{F, G, M}(s).$$

In a special case of $M = N$, the Dirichlet series above is an obvious generalization of Rankin's Dirichlet series in the case of genus 1 (cf. [R]). In fact, if we write the Fourier-Jacobi expansions of F and G by

$$F(Z) = \sum_{n \geq 1} \phi_n(\tau, z) \mathbf{e}(n\tau') \quad \text{and} \quad G(Z) = \sum_{n \geq 1} \psi_n(\tau, z) \mathbf{e}(n\tau'),$$

then

$$D_{F, G, N}(s) = \frac{1}{N^s} \zeta(2s - 2k + 4) \sum_{n \geq 1} \frac{\langle \phi_n, \psi_n \rangle_N}{n^s}.$$

On the other hand, if $F(Z) \in S_k(N, \chi)$ is a Hecke eigenform with

$$T(n)F = \lambda_F(n)F$$

for all the Hecke operators $T(n)$ with $(n, N) = 1$, we can associate with F the *Spinor zeta function* $Z_F(s)$ which has an Euler product of the form

$$Z_F(s) := \prod_{\substack{p: \text{prime} \\ (p, N)=1}} Q_{F, p}(\bar{\chi}(p)p^{-s}) \quad (\mathrm{Re}(s) \gg 0),$$

$$Q_{F, p}(t) := \{1 - \lambda_F(p)t + (\lambda_F(p)^2 - \lambda_F(p^2) - \chi(p^2)p^{2k-4})t^2 - \chi(p^2)\lambda_F(p)p^{2k-3}t^3 + \chi(p^4)p^{4k-6}t^4\}^{-1}, \quad (2)$$

see [A, (4.3.35), Proposition 3.3.35, Exercise 3.3.38 and (4.4.21)]. We define its gamma factor by

$$Z_F^*(s) = (2\pi)^{-2s} \Gamma(s) \Gamma(s - k + 2) Z_F(s).$$

Note that the gamma factor of $D_{F, G, M}(s)$ coincides with that of $Z_F(s)$.

The modular forms which play an important role in relating (1) to (2) are Poincaré series. First, for a negative discriminant $D = r^2 - 4n$, we define the D -th *Jacobi Poincaré series* $P_{D,N}(\tau, z)$ of level N and index 1 by

$$\lambda_{k,D} P_{D,N}(\tau, z) := \sum_{\gamma \in \Gamma^{1,J}(\infty) \backslash \Gamma^{1,J}(N)} \bar{\chi}(\gamma) \mathbf{e}(n\tau + rz)|_{k,1} \gamma \in J_{k,1}^{\text{cusp}}(N, \chi), \quad (3)$$

where we write $\lambda_{k,D} := \frac{1}{2} \Gamma(k - \frac{3}{2}) (\pi|D|)^{-k+3/2}$, $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda, \mu \right) \in \Gamma^{1,J}(N)$ and $\Gamma^{1,J}(\infty) := \left(\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}, 0, \mu \right) \subset \Gamma^{1,J}(N)$. Next, we define a *Siegel modular form* $\mathcal{P}_{D,N}(Z) \in M_k(N, \chi)$ as the image of $P_{D,N}(\tau, z)$ under the Maass lifting (for the definition, see (6) in the section 3).

Now let us state our main result.

Theorem. *Let F be a Siegel cusp form in $S_k(N, \chi)$ (k : even integer > 2). For a natural number M dividing N such that χ is defined modulo M , we define a trace of F by*

$$\text{Tr}_M^N(F) := \sum_{\gamma \in \Gamma_0^2(N) \backslash \Gamma_0^2(M)} F|_k \gamma(Z) \in S_k(M, \chi).$$

Suppose that $\text{Tr}_M^N(F)$ is a non-zero Hecke eigenform. Then for any negative fundamental discriminant D and a Siegel modular form $\mathcal{P}_{D,M}(Z) \in M_k(M, \chi)$ defined above, we have a relation

$$d_{F, \mathcal{P}_{D,M}, M}(s) = d_{\text{Tr}_M^N(F), D}(s) Z_{\text{Tr}_M^N(F)}(s). \quad (4)$$

Here for $\text{Tr}_M^N(F)(Z) = \sum_{Q>0} \tilde{A}(Q) \mathbf{e}(\text{tr} QZ)$, by writing the indices of Fourier coefficients by integral ideals of some order in quadratic fields, we define a Dirichlet series

$$d_{\text{Tr}_M^N(F), D}(s) := \frac{1}{N^s} \sum_{\mathfrak{S} | M^\infty} \tilde{A}(\mathfrak{S}) N\mathfrak{S}^{-(s-k+2)} \quad (\text{Re}(s) \gg 0), \quad (5)$$

where $\mathfrak{S}$ runs through all integral ideals of the maximal order in $\mathbf{Q}(\sqrt{D})$ such that each of the prime ideals which divides $\mathfrak{S}$ also divides M and $N\mathfrak{S}$ denotes the norm of $\mathfrak{S}$. This Dirichlet series is also defined by a following meromorphic function on the whole s -plane:

$$d_{\text{Tr}_M^N(F), D}(s) := \frac{1}{N^s h(D)} \sum_{\xi} \prod_{\mathfrak{p} | M} \left(1 - \frac{\bar{\xi}(\mathfrak{p})}{N\mathfrak{p}^{s-k+2}} \right)^{-1} \sum_{i=1}^{h(D)} \xi(\mathfrak{S}_i) \tilde{A}(\mathfrak{S}_i),$$

where $h(D)$ denotes the class number of $\mathbf{Q}(\sqrt{D})$, $\mathfrak{p}$ runs through all prime ideals dividing M of the maximal order in $\mathbf{Q}(\sqrt{D})$, $\{\mathfrak{S}_i\}_{i=1, \dots, h(D)}$ denotes a set of representatives of ideal class group and ξ runs through all ideal class characters.

We shall write down our relation (4) in the special case of $M = N$. Let

$$F(Z) = \sum_{T>0} A(T) \mathbf{e}(\text{tr} TZ) = \sum_{m>0} \phi_m(\tau, z) \mathbf{e}(m\tau') \in S_k(N, \chi)$$

be a non-zero Hecke eigenform for all the Hecke operators $T(n)$ with $(n, N) = 1$, then for any negative fundamental discriminant D we have an explicit relation

$$\zeta(2s - 2k + 4) \sum_{n \geq 1} \frac{\langle \phi_n, P_{D,N} | V_n \rangle_N}{n^s} = \sum_{\mathfrak{S} | N^\infty} \frac{A(\mathfrak{S})}{N\mathfrak{S}^{s-k+2}} \times Z_F(s),$$

where V_n denotes the n -th Hecke operator which maps $J_{k,1}^{\text{cusp}}(N, \chi)$ to $J_{k,n}^{\text{cusp}}(N, \chi)$ (see below).

3 Proof

The proof proceeds along the lines of the second proof of [K-S], which uses the “Maass lifting” of Jacobi Poincaré series and “Andrianov’s formula”.

We generalize Maass lifting as follows:

Theorem-Definition ((Saito-Kurokawa-)Maass lifting). (cf. [Ei-Za] and [M-Ra-V]) Let $\phi(\tau, z)$ be a Jacobi cusp form of index 1 in $J_{k,1}^{\text{cusp}}(N, \chi)$. Then we have a lifting map from $J_{k,1}^{\text{cusp}}(N, \chi)$ to $M_k(N, \chi)$ via

$$\phi(\tau, z) \mapsto \text{Lift}(\phi) := \sum_{m \geq 1} \phi|V_m(\tau, z) \mathbf{e}(m\tau'),$$

where V_m is the m -th Hecke operator which maps $J_{k,1}^{\text{cusp}}(N)$ to $J_{k,m}^{\text{cusp}}(N)$ and defined by

$$(\phi|V_m)(\tau, z) := m^{k-1} \sum_{\substack{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^1(N) \backslash \mathbf{M}_2(\mathbf{Z}) \\ ad-bc=m, \ c|N, \ (a, N)=1}} \chi(a)(c\tau+d)^{-k} \mathbf{e}\left(\frac{-mcz^2}{c\tau+d}\right) \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{mz}{c\tau+d}\right).$$

We call this map the *Maass lifting*. We call the image $\text{Lift}(J_{k,1}^{\text{cusp}}(N, \chi))$ the *Maass space* of level N and character χ .

Before the proof, we give a definition.

Definition. We define the *Jacobi subgroup* of level N of $\Gamma_0^2(N)$ by

$$C_{2,1}(N) := \left\{ \begin{pmatrix} a & 0 & b & \mu \\ \lambda' & 1 & \mu' & \kappa \\ c & 0 & d & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_0^2(N) \right\}, \quad (\lambda', \mu') = (\lambda, \mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which is a central extension of $\Gamma^{1,J}(N)$ by $\mathbf{Z}$.

Proof. The proof is a direct generalization of [Ei-Z, Theorem 6.2 and Theorem 4.2]. By straightforward calculations, we see $\phi|V_m$ transforms like a Jacobi form of index m . Therefore

$$\phi|V_m(\tau, z) \mathbf{e}(m\tau')$$

transforms like a Siegel modular form under the action of $C_{2,1}(N)$, hence a sum $\text{Lift}(\phi)$ also does.

On the other hand, if we write the Fourier expansion of ϕ by

$$\phi(\tau, z) = \sum_{\substack{n, r \in \mathbf{Z} \\ r^2 - 4n < 0}} c(r^2 - 4n, r) q^n \zeta^r \quad (q := \mathbf{e}(\tau), \zeta = \mathbf{e}(z)),$$

then a standard calculation shows

$$\phi|V_m(\tau, z) = \sum_{\substack{n, r \in \mathbf{Z} \\ r^2 - 4mn < 0}} \left(\sum_{a|(n, r, m)} \chi(a) a^{k-1} c\left(\frac{r^2 - 4mn}{a^2}, \frac{r}{a}\right) \right) q^n \zeta^r,$$

hence we have

$$\text{Lift}(\phi) \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} = \sum_{\substack{n & r/2 \\ r/2 & m}}_{>0} \left(\sum_{a|(n,r,m)} \chi(a) a^{k-1} c \left(\frac{r^2 - 4mn}{a^2}, \frac{r}{a} \right) \right) q^n \zeta^r p^m \quad (p := \mathbf{e}(\tau')).$$

Also we can easily see $\text{Lift}(\phi)$ is symmetric in n and m , so we deduce that $\text{Lift}(\phi)$ transforms like a Siegel modular form with respect to the matrix

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore $\text{Lift}(\phi)$ satisfies the transformation law of Siegel modular forms by using Lemma 1 below on generators for $\Gamma_0^2(N)$.

□

Remark. We have not succeeded in proving $\text{Lift}(\phi)$ is a cusp form $\in S_k(N, \chi)$ in general.

Lemma 1. $\Gamma_0^2(N)$ is generated by $C_{2,1}(N)$ (the Jacobi subgroup of level N) and the element

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proof. Any integral primitive vector $X = {}^t(x_1, x_2, x_3, x_4)$ could be reduced by the left multiplication by the element of type

$$M(x, y, z) = \begin{pmatrix} 1 & 0 & 0 & x \\ -y & 1 & x & z \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to a vector with $\text{g.c.d.}(x_2, x_4) = 1$. Next using the element of type

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad c \equiv 0 \pmod{N},$$

we may reduce the primitive vector X with $N|x_3, x_4$ to $X = {}^t(x_1, x_2, x_3, 0)$. Moreover X reduces to $(x_1, 1, x_3, 0)$ by using a matrix of type $M(x, y, z)$, and then by the left multiplication by the element of type

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad c \equiv 0 \pmod{N},$$

X could be reduced to $X = (x_1, 1, 0, 0)$ (note that $\text{g.c.d.}(x_1, x_3) = 1$ and $N|x_3$).

For any element $\gamma = (X_1, X_2, X_3, X_4) \in \Gamma_0^2(N)$, we reduce the 2-th column vector X_2 to the form ${}^t(x_1, 1, 0, 0)$ and multiplying an element $VM(x, y, z)V$ finally to $(0, 1, 0, 0)$. It is easily shown that this type matrix belongs to the parabolic subgroup $C_{2,1}(N)$, so Lemma 1 is proved. $\square$

We define a Siegel modular form as the Maass lifting of Jacobi Poicaré series defined in (3), i.e.

$$\mathcal{P}_{D,M}(Z) := \text{Lift}(P_{D,M}) = \sum_{m \geq 1} (P_{D,M}|V_m)(\tau, z) \mathbf{e}(m\tau') \in M_k(M, \chi). \quad (6)$$

Now, we recall an important property of Jacobi Poincaré series:

Lemma 2. $P_{D,N}(\tau, z)$ (the D -th Jacobi Poincaré series in $J_{k,1}^{\text{cusp}}(N, \chi)$ defined in (3)) is characterized by

$$\langle \phi, P_{D,N} \rangle_N = c(D, r) \quad (\forall \phi \in J_{k,1}^{\text{cusp}}(N)),$$

where $c(D, r)$ denotes the (D, r) -th Fourier coefficient of ϕ , i.e.

$$\phi(\tau, z) = \sum_{\substack{n, r \in \mathbf{Z} \\ D=r^2-4n < 0}} c(D, r) q^n \zeta^r \quad (q := \mathbf{e}(\tau), \zeta := \mathbf{e}(z)).$$

(Note that $c(D, r)$ depend only on $D = r^2 - 4n$ and $r \pmod{2}$).

Proof. This is proved using the unfolding trick, in the same way of [G-K-Z, p.520]. $\square$

For a half integral symmetric matrix $T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ with $D := b^2 - 4ac$, we can associate with T a binary quadratic form

$$Q(x, y) = [a, b, c](x, y) = ax^2 + bxy + cy^2$$

of discriminant D , and a proper $\mathfrak{o}$ -ideal of some order $\mathfrak{o}$ of the quadratic field $\mathbf{Q}(\sqrt{D})$:

$$\mathfrak{S} = a\mathbf{Z} + \frac{-b + \sqrt{D}}{2}\mathbf{Z}.$$

We occasionally write $A(Q)$, $A(a, b, c)$ or $A(\mathfrak{S})$ instead of $A(T)$ for Fourier coefficients of Siegel modular forms.

Proof of Theorem. We put the assumption that $D_{F, \mathcal{P}_{D,M}, M}(s)$ converges sufficiently large $\text{Re}(s)$ and put forward calculations, and later will remove the assumption by the convergence of Spinor zeta functions. Write the Fourier and the Fourier-Jacobi expansion of $\text{Tr}_M^N(F)$ by

$$\text{Tr}_M^N(F)(Z) = \sum_{T > 0} \tilde{A}(T) \mathbf{e}(\text{tr}TZ) = \sum_{m > 0} \tilde{\phi}_m(\tau, z) \mathbf{e}(m\tau')$$

respectively, where T runs over all positive definite half integral matrices.

We recall the definition (6) of the Siegel modular form $\mathcal{P}_{D,N}(Z) \in M_k^*(N, \chi)$. We note that for any $\gamma \in \Gamma_0^2(M)$

$$\mathcal{P}_{D,M}|_k \gamma(Z) = \mathcal{P}_{D,M}(Z) = \sum_{m>0} P_{D,M}|V_m(\tau, z) e(m\tau'),$$

so in the notations of (2) in Definition

$$\psi_{n,\gamma} = \begin{cases} 0 & n \text{ is not divisible by } N \\ P_{D,M}|V_m & \text{if } n = Nm \end{cases}.$$

Therefore the Nm -th coefficient of $\zeta(2s - 2k + 4)^{-1} D_{F, \mathcal{P}_{D,M}, M}(s)$ is equal to

$$\begin{aligned} \int_{\mathcal{F}} \sum_{\gamma \in \Gamma_0^2(M) \setminus \Gamma_0^2(N)} \phi_{Nm, \gamma}(\tau, z) \bar{\psi}_{Nm, \gamma}(\tau, z) \exp\left(\frac{-4\pi m y^2}{v}\right) v^{k-3} du dv dx dy \\ = \langle \sum_{\gamma} \phi_{Nm, \gamma}, P_{D,M}|V_m \rangle_M. \end{aligned}$$

We remark that $\sum_{\gamma} \phi_{Nm, \gamma}(\tau, z) = \tilde{\phi}_m(\tau, z)$ is nothing but the m -th Fourier-Jacobi coefficient of $\text{Tr}_M^N(F)$ and it is a Jacobi form of index m and level M . Hence we can rewrite the above as

$$\langle \tilde{\phi}_m, P_{D,M}|V_m \rangle_M = \langle \tilde{\phi}_m | V_m^*, P_{D,M} \rangle_M,$$

where $V_m^* : J_{k,m}^{\text{cusp}}(M, \chi) \rightarrow J_{k,1}^{\text{cusp}}(M, \chi)$ denotes the adjoint operator of $V_m : J_{k,1}^{\text{cusp}}(M, \chi) \rightarrow J_{k,m}^{\text{cusp}}(M, \chi)$. Now we must calculate the action of V_m^* on Fourier coefficients explicitly.

Proposition 1. *Let $V_m^* : J_{k,m}^{\text{cusp}}(N, \chi) \rightarrow J_{k,1}^{\text{cusp}}(N, \chi)$ be the adjoint operator of $V_m : J_{k,1}^{\text{cusp}}(N, \chi) \rightarrow J_{k,m}^{\text{cusp}}(N, \chi)$ with respect to the Petersson inner products. Then we have*

$$\begin{aligned} \sum_{\substack{D < 0, r \in \mathbf{Z} \\ D \equiv r^2 \pmod{4m}}} c(D, r) e\left(\frac{r^2 - D}{4m} \tau + rz\right) |V_m^* \\ = \sum_{\substack{D < 0, r \in \mathbf{Z} \\ D \equiv r^2 \pmod{4}}} \left(\sum_{d|m} \bar{\chi}(m/d) d^{k-2} \sum_{\substack{s \pmod{2d} \\ s^2 \equiv D \pmod{4d}}} c\left(\frac{m^2}{d^2} D, \frac{m}{d} s\right) \right) e\left(\frac{r^2 - D}{4} \tau + rz\right). \end{aligned}$$

(Here, $c(D, r)$ denotes the Fourier coefficient of a Jacobi form of index m and note that $c(D, r)$ depends only on D and $r \pmod{2m}$.)

Proof. In our general case (i.e. level $N \geq 1$ and with character χ), we can proceed along the same calculation on [K-S, p.554-557]. □

Using Proposition 1 and the characterization of $P_{D,M}$ in Lemma 2, we have

$$\langle \tilde{\phi}_m | V_m^*, P_{D,M} \rangle_M = \sum_{d|m, (m/d, M)=1} \bar{\chi}(m/d) d^{k-2} \sum_{s \pmod{2d}, s^2 \equiv D \pmod{4d}} \tilde{A}\left(\frac{m}{d} \left(\frac{s^2 - D}{4d}, s, d\right)\right),$$

where $\tilde{A}(\ast)$ denotes the Fourier coefficients of $\text{Tr}_M^N(F)$. Let $\{Q_i\}_{i=1,\dots,h}$ be a set of representatives of binary quadratic forms of discriminant $r^2 - 4n$ and let

$$n(Q_i; d) := \#\left\{s \pmod{2d} \mid s^2 \equiv D \pmod{4d}, \left[\frac{s^2 - D}{4d}, s, d\right] \sim Q_i\right\}$$

be the number of $s \pmod{2d}$ such that $s^2 \equiv D \pmod{4d}$ and the quadratic form $Q(x, y) = \frac{s^2 - D}{4d}x^2 + sxy + dy^2$ is equivalent to Q_i . Then we have

$$\langle \tilde{\phi} | V_m^*, P_{D,M} \rangle_M = \sum_{i=1}^h \sum_{d|m} \bar{\chi}(m/d) d^{k-2} n(Q_i; d) \tilde{A}\left(\frac{m}{d} Q_i\right).$$

By [Z, Proposition 3 (i)] we can see

$$\sum_{n \geq 1} n(Q_i; n) n^{-s} = \zeta_{Q_i}(s) \zeta(2s)^{-1},$$

where $\zeta_{Q_i}(s)$ is the (partial) zeta function of the class of Q_i (= the zeta function of the ideal class of $\mathbf{Q}(\sqrt{D})$ corresponding in the usual way to the class of Q_i), so we obtain

$$D_{F, \mathcal{P}_{D,M}, M}(s) = N^{-s} \sum_{i=1}^h \zeta_{Q_i}(s - k + 2) R_{Q_i, \text{Tr}_M^N(F), M}(s), \quad (7)$$

with

$$R_{Q_i, \text{Tr}_M^N(F), M}(s) := \sum_{n \geq 1, (n, M) = 1} \bar{\chi}(n) \tilde{A}(nQ_i) n^{-s}.$$

We now recall *Andrinov's formula*, which is mentioned in [A, Theorem 4.3.16] in a most general form. Take any negative fundamental discriminant D and any Hecke eigenform $F(Z) = \sum_Q A(Q) \mathbf{e}(\text{tr} QZ) \in S_k(M, \chi)$. Then for any class character ξ of the class group $H(D)$ and any completely multiplicative function ω on $N_{(M)} := \{n \in \mathbf{N} \mid (n, M) = 1\}$, it holds that

$$\begin{aligned} A_\xi(s) & \prod_{\substack{\wp: \text{prime ideal} \\ (\wp, M) = 1}} \left(1 - \frac{\chi(N\wp) \omega(N\wp) \xi(\wp)}{(N\wp)^{s-k+2}}\right) \prod_{\substack{p: \text{prime} \\ (p, M) = 1}} Q_{F,p}(\omega(p) p^{-s}) \\ & = \sum_{i=1}^{h(D)} \xi(Q_i) \sum_{n \in N_{(M)}} \frac{\omega(n) A(nQ_i)}{n^s}, \end{aligned}$$

with

$$A_\xi(s) := \sum_{i=1}^{h(D)} \xi(Q_i) A(Q_i),$$

where $h = h(D) = \#H(D)$ is the class number of discriminant D . Inverting this,

$$\begin{aligned} & \sum_{n \in N_{(M)}} \frac{\omega(n) A(nQ_i)}{n^s} \\ & = \frac{1}{h} \prod_{(p, M) = 1} Q_{F,p}(\omega(p) p^{-s}) \sum_{\xi} \bar{\xi}(Q_i) A_\xi(s) \prod_{\substack{\wp: \text{prime ideal} \\ (\wp, M) = 1}} \left(1 - \frac{\chi(N\wp) \omega(N\wp) \xi(\wp)}{(N\wp)^{s-k+2}}\right). \end{aligned}$$

Instituting this formula for $F = \text{Tr}_M^N(F)$, $\omega = \bar{\chi}$ in (7), we have

$$\begin{aligned} D_{F, \mathcal{P}_{D, M}, M}(s) &= \frac{Z_{\text{Tr}_M^N(F)}(s)}{N^s h} \sum_{i=1}^h \zeta_{Q_i}(s-k+2) \sum_{\xi} \bar{\xi}(Q_i) \tilde{A}_{\xi}(s) \prod_{\substack{\wp: \text{prime ideal} \\ (\wp, M)=1}} \left(1 - \frac{\bar{\xi}(\wp)}{(N\wp)^{s-k+2}}\right). \\ &= \frac{Z_{\text{Tr}_M^N(F)}(s)}{N^s h} \sum_{\xi} \prod_{\wp|M} \left(1 - \frac{\bar{\xi}(\wp)}{N\wp^{s-k+2}}\right)^{-1} \tilde{A}_{\xi}(s), \end{aligned}$$

since, by writing the above Euler product by $L(s, \xi)$, it holds $L(s, \bar{\xi}) = L(s, \xi)$. We note that

$$d_{F, D}(s) := \frac{1}{N^s h} \sum_{\xi} \prod_{\wp|M} \left(1 - \frac{\bar{\xi}(\wp)}{N\wp^{s-k+2}}\right)^{-1} \tilde{A}_{\xi}(s)$$

is a meromorphic function on the whole s -plane. Expanding the right hand side we get

$$D_{F, \mathcal{P}_{D, M}, M}(s) = \frac{Z_{\text{Tr}_M^N(F)}(s)}{N^s h} \sum_{i=1}^h \tilde{A}(Q_i) \sum_{\xi} \sum_{\mathfrak{S}} \frac{\xi(\mathfrak{S} Q_i^{-1})}{N^{\mathfrak{S} s-k+2}},$$

and summing up for ξ 's, we have the relation (4) and the expression (5).

Now we can remove the assumption on convergence of $D_{F, \mathcal{P}_{D, M}, M}(s)$ for sufficiently large $\text{Re}(s)$ by using convergence of $Z_F(s)$. This completes the proof of Theorem. $\square$

4 Applications

We summarize the known facts about the analytic properties for $D_{F, G, M}(s)$'s.

We define *Eisenstein series of Klingen-Siegel type* of weight 0 and level N by

$$E_{s, N}(Z) := \sum_{\gamma \in C_{2,1}(N) \backslash \Gamma_0^2(N)} \left(\frac{\det \text{Im } \gamma \langle Z \rangle}{\text{Im } \gamma \langle Z \rangle_1} \right)^s,$$

where $C_{2,1}(N)$ stands for the Jacobi subgroup of level N (see Definition in the section 3) and Z_1 denotes the left upper entry of $Z \in \mathcal{H}_2$. We define its gamma factor by

$$E_{s, N}^*(Z) := \pi^{-s} \Gamma(s) \zeta(2s) \prod_{p|N} \left(1 - \frac{1}{p^{2s}}\right) E_{s, N}(Z).$$

In this last section, for Siegel modular forms $F \in S_k(N, \chi)$, $G \in M_k(N, \chi)$ and a natural number M dividing N , we put

$$D_{F, G; M}(s) := \prod_{p|M} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) D_{F, G, M}(s), \quad D_{F, G; M}^*(s) := \prod_{p|M} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) D_{F, G, M}^*(s),$$

$$Z_{F; N}(s) := \prod_{p|N} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) Z_F(s), \quad Z_{F; N}^*(s) := \prod_{p|N} \left(1 - \frac{1}{p^{2(s-k+2)}}\right) Z_F^*(s).$$

Then $D_{F, G; M}(s)$ has a following integral representation:

Lemma 3 ([H 1, Lemma 2]). *We have*

$$N^s D_{F,G;M}^*(s) = \pi^{-k+2} \langle F E_{s-k+2;M}^*, G \rangle_N.$$

Also we can prove functional equations of Eisenstein series $E_{s,N}(Z)$ for arbitrary level:

Lemma 4. *Let N be a natural number. Then, the function $E_{s,N}^*(Z)$ has a meromorphic continuation to $\mathbf{C}$ with possible simple poles at $s = 0, 2$ and satisfies a functional equation*

$$\frac{1}{N^{2-s}} \sum_{d|N} d^{2(2-s)} E_{2-s,d}^*(Z) = \frac{1}{N^s} \sum_{d|N} d^{2s} E_{s,d}^*(Z),$$

or equivalently

$$E_{2-s,N}^*(Z) = \frac{1}{N^2} \sum_{e|N} e^{2s} \prod_{p|N/e} (1 - p^{2s-2}) E_{s,e}^*(Z).$$

Proof. (For details, see [H 3].) We will prove for any natural numbers m and N the formula

$$N^s E_{s,m}(NZ) = \sum_{\substack{(m,N)|d|N \\ (m,N/d)=1}} \prod_{p|m} (p^{2s} - 1) \prod_{\substack{p|d \\ p|m}} p^{2s} \prod_{\substack{p^{f+1}|d \\ f \geq 1}} p^{2fs} E_{s,md}(Z) \quad (8)$$

by induction on N , and later specialize the formula (8) to $m = 1$.

By the reduction method found in [H 1, section 4], we can easily prove

$$N^s E_{s,m}(NZ) = - \sum_{1 \neq M|N} \mu(M) \sum_{d|M} \mu(d) (N/M)^s E_{s,\text{l.c.m.}(m,d)}((N/M)Z) + N^{2s} E_{s,mN}(Z),$$

where $\mu(*)$ denotes the Möbius function. We note that for a square-free number M with $(m, M) > 1$

$$\sum_{d|M} \mu(d) E_{s,\text{l.c.m.}(m,d)}((N/M)Z) = \sum_{d_1|M/(m,M)} \mu(d_1) \sum_{d_2|(m,M)} \mu(d_2) E_{s,md_1}((N/M)Z) = 0,$$

then we have

$$N^s E_{s,m}(NZ) = - \sum_{\substack{1 \neq M|N \\ (m,M)=1}} \mu(M) \sum_{d|M} \mu(d) (N/M)^s E_{s,md}((N/M)Z) + N^{2s} E_{s,mN}(Z).$$

Now by using the assumption of induction on N , we have

$$\begin{aligned} & N^s E_{s,m}(NZ) \\ &= - \sum_{\substack{1 \neq M|N \\ (m,M)=1}} \mu(M) \sum_{d|M} \mu(d) \sum_{\substack{(md,N/M)|e|N/M \\ (md,N/(Me))=1}} \prod_{\substack{p|e \\ p|md}} (p^{2s} - 1) \prod_{\substack{p|e \\ p|md}} p^{2s} \prod_{\substack{p^{f+1}|e \\ f \geq 1}} p^{2fs} E_{s,mde}(Z) \\ &\quad + N^{2s} E_{s,mN}(Z) \\ &= - \sum_{\substack{M|N \\ (m,M)=1}} \mu(M) \sum_{d|M} \mu(d) \sum_{\substack{(md,N/M)|e|N/M \\ (md,N/(Me))=1}} \prod_{\substack{p|e \\ p|md}} p^{2s} \prod_{\substack{p^{f+1}|e \\ f \geq 1}} p^{2fs} E_{s,mde}(Z) \\ &\quad + \sum_{\substack{(m,N)|e|N \\ (m,N/e)=1}} \prod_{\substack{p|e \\ p|m}} (p^{2s} - 1) \prod_{\substack{p|e \\ p|m}} p^{2s} \prod_{\substack{p^{f+1}|e \\ f \geq 1}} p^{2fs} E_{s,me}(Z) \\ &\quad + N^{2s} E_{s,mN}(Z). \end{aligned}$$

Now we can see the sum of the first and third lines on the RHS is equal to 0 by using the following Claim and get the formula (8).

Claim. We fix natural numbers d, e, m and N such that $de|N$, $(d, m) = 1$ and d is square-free, then we have

$$\sum_{\substack{M \in \mathbf{N}, d|M|N, (m, M)=1 \\ (md, N/M)|e|N/M, (md, N/(Me))=1}} \mu(M) = \begin{cases} \mu(d) & \text{if } de = N \\ 0 & \text{if } de < N \end{cases}.$$

Then the assertions for meromorphic continuation and poles are obvious by (8) and induction on N , and the symmetric functional equation follows by specializing (8) to the case $m = 1$ and using the functional equation $E_{2-s,1}^*(Z) = E_{s,1}^*(Z)$ (cf. [K-S, Main Lemma]). We can easily prove the other functional equation from the symmetric one. □

By Lemma 3 and 4, we can deduce

Proposition 2 ([H 1, Proposition 1 and the section 4] and [H 3]). All $D_{F,G;M}(s)$'s with $M|N$ have a meromorphic continuation to $\mathbf{C}$, are entire if $\langle F, G \rangle_N = 0$ and otherwise has a simple pole at $s = k$ as its only singularity with the residue

$$\text{Res}_{s=k} D_{F,G;M}(s) = \frac{4^k \pi^{k+2}}{(k-1)! N^k M^2} \prod_{p|M} \left(1 - \frac{1}{p^2}\right) \langle F, G \rangle_N.$$

Furthermore there exists a functional equation

$$N^{2(k-s)} D_{F,G;N}^*(2k-2-s) = \sum_{M|N} M^{2(s-k+2)} \prod_{p|N/M} \left(1 - p^{2(s-k+1)}\right) D_{F,G;M}^*(s).$$

□

Using Proposition 2 and Theorem in the case of $M = N$ we have

Cororally 1. Let $F \in S_k(N, \chi)$ be a non-zero Hecke eigenform of level N . Suppose that $d_{F,D}(s)$ defined by (5) is not identically zero for some fundamental discriminant D . Then $Z_{F;N}(s)$ has a meromorphic continuation to the whole s -plane, the possible poles of $d_{F,D}(s) Z_{F;N}(s)$ are $s = k$. If $d_{F,D}(k) \langle F, \mathcal{P}_{N,D} \rangle_N \neq 0$, then we have

$$\frac{1}{\pi^{k+2} \langle F, \mathcal{P}_{N,D} \rangle_N} \text{Res}_{s=k} Z_{F;N}(s) = \frac{4^k}{(k-1)! N^{k+2} d_{F,D}(k)} \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \in \mathbf{Q}(F, \mathbf{e}(1/h(D))),$$

where $\mathbf{Q}(F, \mathbf{e}(1/h(D)))$ is the field generated by the Fourier coefficients of F and a primitive $h(D)$ -th root of unity over $\mathbf{Q}$.

Furthermore there exists a functional equation satisfied by the Spinor zeta function $Z_{F;N}(s)$ and the Dirichlet series $D_{F,\mathcal{P}_{M,D},M}(s)$'s with $M|N$. Explicitly, it holds

$$\begin{aligned} & N^{2(k-s)} d_{F,D}(2k-2-s) Z_{F;N}^*(2k-2-s) \\ &= N^{2(s-k+2)} d_{F,D}(s) Z_{F;N}^*(s) + \sum_{\substack{M|N \\ M \neq N}} M^{2(s-k+2)} \prod_{p|N/M} \left(1 - p^{2(s-k+1)}\right) D_{F,G;M}^*(s). \end{aligned}$$

□

Remark. Similar results of Corollary 1 are given in [Ma] by the different method. For principal congruence subgroups. Similar results of Corollary 1 are reported in [Ev 1, English transl. p.457] (without proof).

Corollary 2. (cf. [Ev 2], [K-S], [O].) *Let $F \in S_k(N, \chi)$ be a non-zero Hecke eigenform. Suppose F is in the orthogonal complement of $\text{Lift}(J_{k,1}^{\text{cusp}}(N, \chi))$ (the Maass space, see the section 3), then $d_{F,D}(s)Z_{F,N}(s)$ is holomorphic for all s .*

□

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