

ON THE OSCILLATION OF SOLUTIONS OF 4-DIMENSIONAL EMDEN-FOWLER DIFFERENTIAL SYSTEMS

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1. Introduction

In this paper we consider the first order 4-dimensional Emden-Fowler differential system

$$(A) \quad \begin{cases} u_1' = a_1(t)|u_2|^{\lambda_1} \operatorname{sgn} u_2, \\ u_2' = a_2(t)|u_3|^{\lambda_2} \operatorname{sgn} u_3, \\ u_3' = a_3(t)|u_4|^{\lambda_3} \operatorname{sgn} u_4, \\ u_4' = -a_4(t)|u_1|^{\lambda_4} \operatorname{sgn} u_1, \end{cases}$$

where λ_i are positive constants, and $a_i(t)$ are continuous functions on $[0, \infty)$ and $a_i(t) > 0$ for $t \geq 0$, $i = 1, 2, 3, 4$.

A vector function $\mathbf{u} = (u_1, u_2, u_3, u_4)$ on an interval $J \subset [0, \infty)$ is called a solution of (A) on J if the components u_i of $\mathbf{u}$ are defined and C^1 -class on J and satisfy the system (A) for each $t \in J$. In the case where J is an infinite interval, if some component of a solution $\mathbf{u}$ of (A) is oscillatory [resp. nonoscillatory], then the other components of $\mathbf{u}$ are oscillatory [resp. nonoscillatory]. A nontrivial solution $\mathbf{u}$ of (A) on an infinite interval is called oscillatory [resp. nonoscillatory] if all components of $\mathbf{u}$ are oscillatory [resp. nonoscillatory].

In this paper we restrict our attention to the solutions $\mathbf{u} = (u_1, u_2, u_3, u_4)$ of (A) existing on infinite intervals of the form $[T, \infty)$, where T may depend on each solution $\mathbf{u}$, and we discuss the oscillation and nonoscillation of solutions of (A) in detail. The oscillatory and nonoscillatory properties of solutions of (A) are influenced heavily by the growth conditions near the infinity of the functions $a_1(t), a_2(t)$ and $a_3(t)$. In the present paper we consider the following two important cases:

$$(H_1) \quad \int_0^\infty a_i(t) dt = \infty, \quad i = 1, 2, 3;$$

$$(H_2) \quad \begin{cases} \int_0^\infty a_1(t) dt = \infty, & \int_0^\infty a_2(t) dt < \infty, & \int_0^\infty a_3(t) dt = \infty, \\ \int_0^\infty a_1(t) \left(\int_t^\infty a_2(s) ds \right)^{\lambda_1} dt = \infty, & \int_0^\infty a_2(t) \left(\int_0^t a_3(s) ds \right)^{\lambda_2} dt = \infty. \end{cases}$$

The condition (H_1) or (H_2) will be assumed throughout the paper.

In Section 2 we classify nonoscillatory solutions $\mathbf{u} = (u_1, u_2, u_3, u_4)$ of (A) according to the signs of $u_i(t)$, $i = 1, 2, 3, 4$, and take up several important classes of nonoscillatory solutions $\mathbf{u}$ for which the components u_i of $\mathbf{u}$ are regulated by specific asymptotic conditions as $t \rightarrow \infty$. In Section 3 we establish necessary and sufficient conditions for (A) to have nonoscillatory solutions in the important classes mentioned above. In Section 4 we show that, for some special classes of (A), the situation that all solutions of (A) are oscillatory can be completely characterized.

The system (A) contains a number of important differential equations and differential systems. For example, the fourth order scalar differential equation

$$(B) \quad (|y''|^{\alpha} \operatorname{sgn} y'')'' + q(t)|y|^{\beta} \operatorname{sgn} y = 0$$

can be rewritten in the form of (A). Therefore, making use of the general results in Sections 2-4, we can derive oscillation and nonoscillation results for (B).

2. Classification of Nonoscillatory Solutions

We begin by investigating the signs of the components of nonoscillatory solutions $\mathbf{u}$ of (A).

Lemma 2.1. *Assume (H_1) or (H_2) holds. If $\mathbf{u} = (u_1, u_2, u_3, u_4)$ is a nonoscillatory solution of (A), then one of the following cases holds:*

$$(2.1) \quad u_1(t)u_2(t) > 0, \quad u_1(t)u_3(t) > 0, \quad u_1(t)u_4(t) > 0;$$

$$(2.2) \quad u_1(t)u_2(t) > 0, \quad u_1(t)u_3(t) < 0, \quad u_1(t)u_4(t) > 0.$$

Now we discuss the asymptotic behavior as $t \rightarrow \infty$ of nonoscillatory solutions $\mathbf{u} = (u_1, u_2, u_3, u_4)$ of (A). By Lemma 2.1, a nonoscillatory solution $\mathbf{u} = (u_1, u_2, u_3, u_4)$ satisfies either (2.1) or (2.2). Let us begin with the case where $\mathbf{u}$ satisfies (2.1). We may assume that $u_1(t)$ is eventually positive. Then

$$(2.3) \quad u_1(t) > 0, \quad u_2(t) > 0, \quad u_3(t) > 0, \quad u_4(t) > 0 \quad \text{for } t \geq T,$$

where T is chosen sufficiently large. Integrating (A), we get

$$(2.4) \quad \begin{cases} u_1(t) = u_1(T) + \int_T^t a_1(s)[u_2(s)]^{\lambda_1} ds, \\ u_2(t) = u_2(T) + \int_T^t a_2(s)[u_3(s)]^{\lambda_2} ds, \\ u_3(t) = u_3(T) + \int_T^t a_3(s)[u_4(s)]^{\lambda_3} ds, \\ u_4(t) = u_4(\infty) + \int_t^{\infty} a_4(s)[u_1(s)]^{\lambda_4} ds, \quad t \geq T, \end{cases}$$

where $u_4(\infty) = \lim_{t \rightarrow \infty} u_4(t) \geq 0$.

We define the functions $A_i(t)$, $i = 1, 2, 3, 4$, by

$$(2.5) \quad \begin{cases} A_4(t) = 1, \\ A_3(t) = \int_0^t a_3(s) ds, \\ A_2(t) = \int_0^t a_2(s) [A_3(s)]^{\lambda_2} ds, \\ A_1(t) = \int_0^t a_1(s) [A_2(s)]^{\lambda_1} ds, \end{cases} \quad t \geq 0.$$

Similarly, we define the functions $B_i(t)$, $i = 1, 2, 3, 4$, by

$$(2.6) \quad \begin{cases} B_3(t) = 1, \\ B_2(t) = \int_0^t a_2(s) ds, \\ B_1(t) = \int_0^t a_1(s) [B_2(s)]^{\lambda_1} ds, \\ B_4(t) = \int_t^\infty a_4(s) [B_1(s)]^{\lambda_4} ds, \end{cases} \quad t \geq 0.$$

We have

$$\lim_{t \rightarrow \infty} \frac{A_i(t)}{B_i(t)} = \infty \quad (i = 1, 2, 3, 4).$$

Proposition 2.1. (i) Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ be a nonoscillatory solution of (A) satisfying (2.3). Then we have

$$(2.7) \quad \lim_{t \rightarrow \infty} \frac{u_i(t)}{A_i(t)} \text{ exists and is positive} \quad (i = 1, 2, 3, 4)$$

or

$$(2.8) \quad \lim_{t \rightarrow \infty} \frac{u_i(t)}{A_i(t)} = 0 \quad (i = 1, 2, 3, 4).$$

(ii) Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ be a nonoscillatory solution of (A) satisfying (2.3). Then, the function $B_4(t)$ is always well-defined. For the case where (H_1) holds, we have

$$(2.9) \quad \lim_{t \rightarrow \infty} \frac{u_i(t)}{B_i(t)} = \infty \quad (i = 1, 2, 3, 4)$$

or

$$(2.10) \quad \lim_{t \rightarrow \infty} \frac{u_i(t)}{B_i(t)} \text{ exists and is positive} \quad (i = 1, 2, 3, 4).$$

For the case where (H_2) holds, if $\mathbf{u} = (u_1, u_2, u_3, u_4)$ has the additional property that

$$(2.11) \quad \lim_{t \rightarrow \infty} u_3(t) \text{ exists and is positive,}$$

then we have (2.10), and moreover, for any nonoscillatory solution $\mathbf{v} = (v_1, v_2, v_3, v_4)$ of (A) satisfying (2.3), the limit

$$\lim_{t \rightarrow \infty} \frac{u_i(t)}{v_i(t)}$$

exists and is finite ($i = 1, 2, 3, 4$).

By Proposition 2.1 we see that if $\mathbf{u} = (u_1, u_2, u_3, u_4)$ is a nonoscillatory solution of (A) satisfying (2.3), then

$$\beta_i B_i(t) \leq u_i(t) \leq \alpha_i A_i(t) \quad (i = 1, 2, 3, 4)$$

for all large t , where α_i and β_i ($i = 1, 2, 3, 4$) are positive constants. Consequently, in the set of all nonoscillatory solutions of (A) satisfying (2.3), a solution $\mathbf{u} = (u_1, u_2, u_3, u_4)$ of (A) which satisfies the asymptotic condition (2.7) [resp. (2.10)] is regarded as a “maximal” [resp. “minimal”] solution.

Definition 2.1. Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ be a nonoscillatory solution of (A). We say that $\mathbf{u}$ is a solution of the type (I) if it satisfies (2.7), and that $\mathbf{u}$ is a solution of the type (II) if it satisfies (2.10).

Next we consider the case that a nonoscillatory solution $\mathbf{u} = (u_1, u_2, u_3, u_4)$ satisfies (2.2). We may assume that $u_1(t)$ is eventually positive:

$$(2.12) \quad u_1(t) > 0, \quad u_2(t) > 0, \quad u_3(t) < 0, \quad u_4(t) > 0 \quad \text{for } t \geq T.$$

Here, T is taken sufficiently large. In this case, the second equation and the third equation of (A) can be rewritten in the forms of

$$u_2'(t) = -a_2(t)|u_3(t)|^{\lambda_2} \quad \text{and} \quad |u_3(t)|' = -a_3(t)[u_4(t)]^{\lambda_3},$$

respectively. Then we can get

$$(2.13) \quad \begin{cases} u_1(t) = u_1(T) + \int_T^t a_1(s)[u_2(s)]^{\lambda_1} ds, \\ u_2(t) = u_2(\infty) + \int_t^\infty a_2(s)|u_3(s)|^{\lambda_2} ds, \\ |u_3(t)| = |u_3(\infty)| + \int_t^\infty a_3(s)[u_4(s)]^{\lambda_3} ds, \\ u_4(t) = \int_t^\infty a_4(s)[u_1(s)]^{\lambda_4} ds, \quad t \geq T, \end{cases}$$

where $u_i(\infty) = \lim_{t \rightarrow \infty} u_i(t)$, $i = 2, 3$. Furthermore, for the case where (H_1) holds, we find that $u_3(\infty) = 0$ in (2.13).

Now, for the case where (H_1) holds, we define the functions $C_i(t)$, $i = 1, 2, 3, 4$, by

$$(2.14) \quad \begin{cases} C_2(t) = 1, \\ C_1(t) = \int_0^t a_1(s) ds, \\ C_4(t) = \int_t^\infty a_4(s) [C_1(s)]^{\lambda_4} ds, \\ C_3(t) = \int_t^\infty a_3(s) [C_4(s)]^{\lambda_3} ds, \quad t \geq 0. \end{cases}$$

For the case where (H_2) holds, we replace $C_3(t)$ in (2.14) by

$$C_3(t) = 1 + \int_t^\infty a_3(s) [C_4(s)]^{\lambda_3} ds, \quad t \geq 0.$$

We also define the functions $D_i(t)$, $i = 1, 2, 3, 4$, by

$$(2.15) \quad \begin{cases} D_1(t) = 1, \\ D_4(t) = \int_t^\infty a_4(s) ds, \\ D_3(t) = \int_t^\infty a_3(s) [D_4(s)]^{\lambda_3} ds, \\ D_2(t) = \int_t^\infty a_2(s) [D_3(s)]^{\lambda_2} ds, \quad t \geq 0. \end{cases}$$

It is easily seen that

$$\lim_{t \rightarrow \infty} \frac{C_i(t)}{D_i(t)} = \infty \quad (i = 1, 2, 3, 4).$$

Proposition 2.2. (i) Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ be a nonoscillatory solution of (A) satisfying (2.12). Assume moreover that the following additional condition is satisfied:

$$(2.16) \quad \begin{cases} u_2(\infty) > 0 & \text{in the case where } (H_1) \text{ holds,} \\ u_2(\infty) > 0 \text{ and } u_3(\infty) < 0 & \text{in the case where } (H_2) \text{ holds.} \end{cases}$$

Then the functions $C_i(t)$ ($i = 3, 4$) are well-defined and

$$(2.17) \quad \begin{cases} \lim_{t \rightarrow \infty} \frac{u_i(t)}{C_i(t)} \text{ exists and is positive } (i = 1, 2, 4), \\ \lim_{t \rightarrow \infty} \frac{u_3(t)}{C_3(t)} \text{ exists and is negative.} \end{cases}$$

Furthermore, if a nonoscillatory solution $\mathbf{u} = (u_1, u_2, u_3, u_4)$ of (A) satisfies (2.12) and (2.17), then, for any nonoscillatory solution $\mathbf{v} = (v_1, v_2, v_3, v_4)$ of (A) satisfying (2.12), the limit

$$\lim_{t \rightarrow \infty} \frac{|v_i(t)|}{|u_i(t)|}$$

exists and is finite ($i = 1, 2, 3, 4$).

(ii) Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ be a nonoscillatory solution of (A) satisfying (2.12). Then the functions $D_i(t)$ ($i = 2, 3, 4$) are always well-defined. If $\mathbf{u} = (u_1, u_2, u_3, u_4)$ satisfies the additional condition

$$(2.18) \quad \begin{cases} u_2(\infty) = 0 & \text{in the case where } (H_1) \text{ holds,} \\ u_2(\infty) = 0 \text{ and } u_3(\infty) = 0 & \text{in the case where } (H_2) \text{ holds,} \end{cases}$$

then we have either

$$(2.19) \quad \begin{cases} \lim_{t \rightarrow \infty} \frac{u_i(t)}{D_i(t)} \text{ exists and is positive } (i = 1, 2, 4), \\ \lim_{t \rightarrow \infty} \frac{u_3(t)}{D_3(t)} \text{ exists and is negative,} \end{cases}$$

or

$$(2.20) \quad \lim_{t \rightarrow \infty} \frac{u_i(t)}{D_i(t)} = \infty \quad (i = 1, 2, 4) \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{u_3(t)}{D_3(t)} = -\infty.$$

If a nonoscillatory solution $\mathbf{u} = (u_1, u_2, u_3, u_4)$ of (A) satisfies (2.12) and (2.19), then, for any nonoscillatory solution $\mathbf{v} = (v_1, v_2, v_3, v_4)$ of (A) satisfying (2.12), the limit

$$\lim_{t \rightarrow \infty} \frac{|u_i(t)|}{|v_i(t)|}$$

exists and is finite ($i = 1, 2, 3, 4$).

By (2.13) it is seen that if $C_i(t)$ and $D_i(t)$ ($i = 1, 2, 3, 4$) are well-defined, then, for a nonoscillatory solution $\mathbf{u} = (u_1, u_2, u_3, u_4)$ of (A) satisfying (2.12), there are positive constants δ_i and γ_i ($i = 1, 2, 3, 4$) such that

$$\delta_i D_i(t) \leq |u_i(t)| \leq \gamma_i C_i(t) \quad (i = 1, 2, 3, 4)$$

for all large t . Proposition 2.2 implies that a nonoscillatory solution $\mathbf{u} = (u_1, u_2, u_3, u_4)$ satisfying (2.17) [resp. (2.19)] is "maximal" [resp. "minimal"] in the set of all nonoscillatory solutions of (A) satisfying (2.12).

Definition 2.2. Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ be a nonoscillatory solution of (A). We say that $\mathbf{u}$ is a solution of the type (III) if it satisfies (2.17), and that $\mathbf{u}$ is a solution of the type (IV) if it satisfies (2.19).

In the following section we give necessary and sufficient conditions for the existence of nonoscillatory solutions of (A) with the special types (I), (II), (III) and (IV).

3. Existence of Solutions of the Special Types

In this section we deal with the existence of nonoscillatory solutions of (A) with the special types (I), (II), (III) and (IV).

Theorem 3.1. *Suppose (H_1) or (H_2) holds. A necessary and sufficient condition for (A) to have a nonoscillatory solution of the type (I) is that*

$$(3.1) \quad \int_0^\infty a_4(t) [A_1(t)]^{\lambda_4} dt < \infty.$$

Theorem 3.2. *Suppose (H_1) or (H_2) holds. A necessary and sufficient condition for (A) to have a nonoscillatory solution of the type (II) is that*

$$(3.2) \quad \int_0^\infty a_4(t) [B_1(t)]^{\lambda_4} dt < \infty, \quad \text{and}$$

$$(3.3) \quad \int_0^\infty a_3(t) [B_4(t)]^{\lambda_3} dt < \infty.$$

Theorem 3.3. *Suppose (H_1) or (H_2) holds. A necessary and sufficient condition for the existence of a nonoscillatory solution of (A) with the type (III) is that*

$$(3.4) \quad \int_0^\infty a_4(t) [C_1(t)]^{\lambda_4} dt < \infty,$$

$$(3.5) \quad \int_0^\infty a_3(t) [C_4(t)]^{\lambda_3} dt < \infty, \quad \text{and}$$

$$(3.6) \quad \int_0^\infty a_2(t) [C_3(t)]^{\lambda_2} dt < \infty.$$

Theorem 3.4. *Suppose (H_1) or (H_2) holds. A necessary and sufficient condition for the existence of a nonoscillatory solution of (A) with the type (IV) is that*

$$(3.7) \quad \int_0^\infty a_4(t) dt < \infty,$$

$$(3.8) \quad \int_0^\infty a_3(t) [D_4(t)]^{\lambda_3} dt < \infty,$$

$$(3.9) \quad \int_0^\infty a_2(t) [D_3(t)]^{\lambda_2} dt < \infty, \quad \text{and}$$

$$(3.10) \quad \int_0^\infty a_1(t) [D_2(t)]^{\lambda_1} dt < \infty.$$

4. Oscillation Theorems

This section is devoted to the discussion of the oscillation of all solutions of (A). However, under the general framework, it is difficult to characterize the oscillation of all solutions of (A). Therefore we require the following additional condition on $a_1(t)$ and $a_3(t)$:

$$(H_3) \quad 0 < \liminf_{t \rightarrow \infty} \frac{a_3(t)}{a_1(t)} \leq \limsup_{t \rightarrow \infty} \frac{a_3(t)}{a_1(t)} < \infty.$$

We can prove the following theorems.

Theorem 4.1. *Suppose that either (H_1) or (H_2) , and (H_3) hold. Suppose moreover that*

$$(4.1) \quad \lambda_i \leq 1 \quad (i = 1, 2, 3) \quad \text{and} \quad \lambda_4 < 1.$$

Then, all solutions of (A) are oscillatory if and only if

$$(4.2) \quad \int_0^\infty a_4(t) \left(\int_0^t a_1(s) \left(\int_0^s a_2(r) \left(\int_0^r a_3(\xi) d\xi \right)^{\lambda_2} dr \right)^{\lambda_1} ds \right)^{\lambda_4} dt = \infty.$$

Theorem 4.2. *Suppose that either (H_1) or (H_2) , and (H_3) hold. Suppose moreover that*

$$(4.3) \quad \lambda_i \geq 1 \quad (i = 1, 2) \quad \text{and} \quad \lambda_3 \lambda_4 > 1.$$

Then, all solutions of (A) are oscillatory if and only if one of the following conditions is satisfied:

$$(4.4) \quad \int_0^\infty a_4(t) dt = \infty;$$

$$(4.5) \quad \begin{cases} (4.5-1) & (4.4) \text{ does not hold, and} \\ (4.5-2) & \int_0^\infty a_3(t) \left(\int_t^\infty a_4(s) ds \right)^{\lambda_3} dt = \infty; \end{cases}$$

$$(4.6) \quad \begin{cases} (4.6-1) & \text{neither (4.4) nor (4.5-2) holds, and} \\ (4.6-2) & \int_0^\infty a_2(t) \left(\int_t^\infty a_3(s) \left(\int_s^\infty a_4(r) dr \right)^{\lambda_3} ds \right)^{\lambda_2} dt = \infty; \end{cases}$$

$$(4.7) \quad \begin{cases} (4.7-1) & \text{none of the conditions (4.4), (4.5-2) and (4.6-2) holds, and} \\ (4.7-2) & \int_0^\infty a_1(t) \left(\int_t^\infty a_2(s) \left(\int_s^\infty a_3(r) \left(\int_r^\infty a_4(\xi) d\xi \right)^{\lambda_3} dr \right)^{\lambda_2} ds \right)^{\lambda_1} dt = \infty. \end{cases}$$

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