

# Induced Norm from $L_2$ to $L_\infty$ in SISO Sampled-Data Systems

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**Abstract**—This paper investigates the maximum amplitude (i.e., the  $L_\infty$  norm) of the output for the worst input with a unit energy (i.e., a unit  $L_2$  norm) in single-input/single-output (SISO) linear time-invariant (LTI) sampled-data systems, by which we mean the generalized plant and the controller are both LTI. It is known that the induced norm from  $L_2$  to  $L_\infty$  coincides with the  $H_2$  norm in SISO LTI systems. To highlight the arguments tailored to (SISO) sampled-data systems in this paper, we first see how this induced norm reduces to  $H_2$  norms in the continuous-time and discrete-time cases. Through the lifting-based arguments, we next give a closed-form representation of the induced norm from  $L_2$  to  $L_\infty$  in SISO LTI sampled-data systems. We further exploit the associated arguments to compare this induced norm with two existing definitions of the  $H_2$  norm for sampled-data systems, and show that the induced norm coincides with neither of them in SISO LTI sampled-data systems. We further develop a more sophisticated closed-form representation for the induced norm and give an approximate but asymptotically exact method for its computation.

## I. INTRODUCTION

The  $L_2$  norm can be used for evaluating the energy of signals, and the  $L_2$ -induced norm of a continuous-time LTI system corresponds to the  $H_\infty$  norm of the system. Hence, the study associated with the treatment of the  $L_2$ -induced norm has been called the  $H_\infty$  problem. Furthermore, there have been a number of studies on the continuous-time or discrete-time  $H_\infty$  problem [1]–[7] since this system norm has been used as a typical measure in the sensitivity reduction problem and robust control problem.

The  $L_\infty$  norm can be used for considering the maximum amplitude of signals, and the  $L_\infty$ -induced norm of a continuous-time LTI system corresponds to the  $L_1$  norm of the impulse response of the continuous-time system. Thus, the study associated with the treatment of the  $L_\infty$ -induced norm has been named the  $L_1$  problem. There have been a number of studies on the  $L_1$  problem [8]–[14] because evaluating the maximum amplitude of the output is very important in practice and this problem is pertinent to bounded persistent disturbances often encountered in control systems.

On the other hand, even when the performance analysis for decaying disturbances such as those in  $L_2$  is considered, evaluating the maximum amplitude of the output rather than its  $L_2$  norm may equally play an important role. In other words, computing the induced norm from  $L_2$  to  $L_\infty$  could play very important roles in control system analysis. This is indeed true particularly because this induced norm admits an alternative interpretation as the  $H_2$  norm in the single-input/single-output (SISO) LTI case, both for continuous-time

and discrete-time systems, [15]–[18], even though the  $H_2$  norm (of a multi-input/multi-output LTI system) is usually defined in the frequency domain and related to the power of the output for a white noise input. Another well-known interpretation of the  $H_2$  norm is related to the  $L_2$  norm of the output for impulse disturbances.

In view of such relevant studies, one could naturally raise a question whether or not the induced norm from  $L_2$  to  $L_\infty$  in SISO sampled-data systems coincides with either of the two (conceptually) different definitions for the  $H_2$  norm of LTI sampled-data systems [19]–[22]. The induced norm from  $L_2$  to  $L_\infty$  in sampled-data systems was analytically formulated first in [23] by using the idea of the lifting technique [24]–[26], but no explicit computation method for the induced norm was provided in that study. This is partly because the treatment of the induced norm in that study involves an infinite summation, whose explicit computation was not discussed. More importantly, the study is interested in the synthesis of the optimal controller minimizing the induced norm and thus it does not give an exact characterization for analyzing the norm. An explicit computation method for the induced norm without the lifting arguments was developed in [27]. However, its comparison with two definitions for the  $H_2$  norm of sampled-data systems was not discussed there. This paper employs the lifting arguments and deals with the induced norm from  $L_2$  to  $L_\infty$  in SISO LTI sampled-data systems directly, in such a way that the comparison of the induced norm with two existing definitions for the  $H_2$  norm of sampled-data systems is easy. As it turns out, the arguments in this paper give a negative answer to the aforementioned question on their mutual relation and thus could be interpreted as giving yet another definition of the  $H_2$  norm of SISO LTI sampled-data systems. These discrepancies of the present norm from the existing  $H_2$  norms can be regarded as stemming from another aspect of the hybrid continuous-time/discrete-time nature of sampled-data systems, even though such a nature has already been studied intensively in [20]–[22] in the context of extending the  $H_2$  problems to sampled-data systems.

In the following, we use the notations  $\mathbb{N}$  and  $\mathbb{R}^\nu$  to denote the set of positive integers and the set of  $\nu$ -dimensional real vectors, respectively. We further use the notation  $\mathbb{N}_0$  to imply  $\mathbb{N} \cup \{0\}$ . The notation  $\|\cdot\|_\infty$  is used to mean either the  $L_\infty$  norm of a function, i.e.,

$$\|f(\cdot)\|_\infty := \operatorname{ess\,sup}_{0 \leq t < \infty} |f(t)| \quad (1)$$

or the  $l_\infty$  norm of a sequence, i.e.,

$$\|g(\cdot)\|_\infty := \sup_{k \in \mathbb{N}_0} |g(k)| \quad (2)$$

or the  $\infty$ -norm of a finite-dimensional matrix (induced from the vector  $\infty$ -norm), whose distinction will be clear from the context. On the other hand, the notation  $\|\cdot\|_2$  is used to mean either the  $L_2$  norm of a function, i.e.,

$$\|f(\cdot)\|_2 := \left( \int_0^\infty f^2(t) dt \right)^{1/2} \quad (3)$$

or the  $l_2$  norm of a sequence, i.e.,

$$\|g(\cdot)\|_2 := \left( \sum_{k=0}^\infty g^2(k) \right)^{1/2} \quad (4)$$

or the 2-norm of a finite-dimensional matrix (induced from the vector Euclidean norm), whose distinction will be also clear from the context. Let  $\mathbf{T}$  be an operator from  $L_2$  to  $L_\infty$  (or from  $l_2$  to  $l_\infty$ ). Then, the notation  $\|\cdot\|_{\infty/2}$  is used to denote the induced norm from  $L_2$  to  $L_\infty$  (or from  $l_2$  to  $l_\infty$ ), i.e.,

$$\|\mathbf{T}\|_{\infty/2} := \sup_{\|w\|_2 \leq 1} \|\mathbf{T}w\|_\infty \quad (5)$$

Furthermore, we call the induced norms from  $L_2$  to  $L_\infty$  and from  $l_2$  to  $l_\infty$  the  $L_\infty/L_2$ -induced norm and  $l_\infty/l_2$ -induced norm, respectively, for simplicity.

## II. CONTINUOUS-TIME AND DISCRETE-TIME CASES

As mentioned in the preceding section, the  $L_\infty/L_2$ -induced norm of SISO continuous-time LTI systems and the  $l_\infty/l_2$ -induced norm of SISO discrete-time LTI systems have been known to coincide with the continuous-time  $H_2$  norm and the discrete-time  $H_2$  norm, respectively. However, this fact has been stated only without proof in [8] and [28], while [15]–[18] deal also with relevant topics and explicit proofs are given only for such topics. Recovering the proof of the above fact from the relevant proofs may not necessarily be extremely hard but not straightforward, either, while giving an explicit proof is expected to be helpful in highlighting the arguments of the present paper tailored to (SISO) sampled-data systems. Hence, this section is devoted to such an explicit proof.

### A. Continuous-Time Case

We first consider the continuous-time case. Let us consider the stable continuous-time SISO FDLTI system

$$G_c : \begin{cases} \frac{dx}{dt} &= A_c x + B_c w \\ z &= C_c x \end{cases} \quad (6)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $w(t) \in \mathbb{R}$  is the input and  $z(t) \in \mathbb{R}$  is the output. Throughout the paper, we assume that  $x(0) = 0$ . Then,

$$\begin{aligned} z(t) &= \int_0^t C_c \exp(A_c(t-\tau)) B_c w(\tau) d\tau \\ &=: (\mathbf{T}_c w)(t) \quad (0 \leq t < \infty) \end{aligned} \quad (7)$$

$\mathbf{T}_c$  defined above for the system (6) is known to be an operator from  $L_2$  to  $L_\infty$ . By noting that  $\mathbf{T}_c w$  is a continuous

function and the system (6) is LTI, it is easy to see that the  $L_\infty/L_2$ -induced norm of the system (6) can be described by

$$\begin{aligned} \|\mathbf{T}_c\|_{\infty/2} &:= \sup_{\|w\|_2 \leq 1} \|\mathbf{T}_c w\|_\infty \\ &= \sup_{\|w\|_2 \leq 1} \sup_t |(\mathbf{T}_c w)(t)| = \sup_t \sup_{\|w\|_2 \leq 1} |(\mathbf{T}_c w)(t)| \\ &= \sup_t \sup_{\|w\|_2 \leq 1} \left| \int_0^t C_c \exp(A_c(t-\tau)) B_c w(\tau) d\tau \right| \\ &= \sup_t \sup_{\|w\|_2 \leq 1} \left| \int_0^t C_c \exp(A_c\theta) B_c w(t-\theta) d\theta \right| \\ &=: \sup_t \sup_{\|u\|_2 \leq 1} \left| \int_0^t C_c \exp(A_c\theta) B_c u(\theta) d\theta \right| \\ &=: \sup_t \sup_{\|u\|_2 \leq 1} |(\mathbf{F}_c u)(t)|_\infty = \lim_{t \rightarrow \infty} \sup_{\|u\|_2 \leq 1} |(\mathbf{F}_c u)(t)| \\ &=: \|\mathbf{F}_c\|_{\infty/2} \end{aligned} \quad (8)$$

*Remark 1:*  $\mathbf{F}_c$  defined above is also regarded as an operator from  $L_2$  to  $L_\infty$ . In the following, we compute the  $L_\infty/L_2$ -induced norm  $\|\mathbf{T}_c\|_{\infty/2}$  by computing the  $L_\infty/L_2$ -induced norm  $\|\mathbf{F}_c\|_{\infty/2}$  instead because of some simplicities in the following arguments.

Here, we review the continuous-time Cauchy-Schwarz inequality, with the functions  $f$  and  $g$ , given by

$$\left( \int_0^t f(\theta) g(\theta) d\theta \right)^2 \leq \int_0^t f^2(\theta) d\theta \cdot \int_0^t g^2(\theta) d\theta \quad (9)$$

where the equality holds if and only if  $f = \lambda g$  on  $[0, t]$  for a constant  $\lambda$ . By this inequality, we have

$$\begin{aligned} \|\mathbf{F}_c\|_{\infty/2} &= \lim_{t \rightarrow \infty} \sup_{\|u\|_2 \leq 1} \left| \int_0^t C_c \exp(A_c\theta) B_c u(\theta) d\theta \right| \\ &= \left( \int_0^\infty C_c \exp(A_c\theta) B_c B_c^T \exp(A_c^T\theta) C_c^T d\theta \right)^{1/2} \end{aligned} \quad (10)$$

because

$$\begin{aligned} &\left( \int_0^t C_c \exp(A_c\theta) B_c u(\theta) d\theta \right)^2 \\ &\leq \int_0^t C_c \exp(A_c\theta) B_c B_c^T \exp(A_c^T\theta) C_c^T d\theta \cdot \int_0^t u^2(\theta) d\theta \end{aligned} \quad (11)$$

Thus, by the Plancherel theorem, we can see from (10) that the  $L_\infty/L_2$ -induced norm  $\|\mathbf{F}_c\|_{\infty/2}$  coincides with the  $H_2$  norm associated with (the transfer function of) the SISO continuous-time LTI system (6).

### B. Discrete-Time Case

We next consider the discrete-time case. Let us consider the stable discrete-time SISO FDLTI system

$$G_d : \begin{cases} x(k+1) &= A_d x(k) + B_d w(k) \\ z(k) &= C_d x(k) + D_d w(k) \end{cases} \quad (12)$$

where  $x(k) \in \mathbb{R}^n$  is the state,  $w(k) \in \mathbb{R}$  is the input and  $z(k) \in \mathbb{R}$  is the output. Assuming that  $x(0) = 0$ ,

$$\begin{aligned} z(k) &= \sum_{i=0}^{k-1} C_d A_d^i B_d w(k-1-i) + D_d w(k) \\ &=: (\mathbf{T}_d w)(k) \quad (k \in \mathbb{N}_0) \end{aligned} \quad (13)$$

$\mathbf{T}_d$  defined above is known to be an operator from  $l_2$  to  $l_\infty$ . Similarly to the continuous-time case, the  $l_\infty/l_2$ -induced norm of the system (12) can be given by

$$\begin{aligned}
\|\mathbf{T}_d\|_{\infty/2} &= \sup_{\|w\|_2 \leq 1} \|\mathbf{T}_d w\|_\infty \\
&= \sup_{\|w\|_2 \leq 1} \sup_k |(\mathbf{T}_d w)(k)| = \sup_k \sup_{\|w\|_2 \leq 1} |(\mathbf{T}_d w)(k)| \\
&= \sup_k \sup_{\|w\|_2 \leq 1} \left| \sum_{i=0}^{k-1} C_d A_d^i B_d w(k-1-i) + D_d w(k) \right| \\
&=: \sup_k \sup_{\|u\|_2 \leq 1} \left| \sum_{i=1}^k C_d A_d^{i-1} B_d u(i) + D_d u(0) \right| \\
&=: \sup_k \sup_{\|u\|_2 \leq 1} |(\mathbf{F}_d u)(k)| = \lim_{k \rightarrow \infty} \sup_{\|u\|_2 \leq 1} |(\mathbf{F}_d u)(k)| \\
&=: \|\mathbf{F}_d\|_{\infty/2} \tag{14}
\end{aligned}$$

*Remark 2:* Similarly to the continuous-time case, we compute the  $l_\infty/l_2$ -induced norm  $\|\mathbf{T}_d\|_{\infty/2}$  by computing the  $l_\infty/l_2$ -induced norm  $\|\mathbf{F}_d\|_{\infty/2}$  instead.

Now, the discrete-time Cauchy-Schwarz inequality, with the sequences  $f(i)$  and  $g(i)$ , states that

$$\left( \sum_{i=0}^k f(i)g(i) \right)^2 \leq \sum_{i=0}^k f^2(i) \cdot \sum_{i=0}^k g^2(i) \tag{15}$$

where the equality holds if and only if  $f(i) = \lambda g(i)$ ,  $i = 0, \dots, k$  for a constant  $\lambda$ . Hence, it readily follows that

$$\begin{aligned}
\|\mathbf{F}_d\|_{\infty/2} &= \lim_{k \rightarrow \infty} \sup_{\|u\|_2 \leq 1} \left| \sum_{i=1}^k C_d A_d^{i-1} B_d u(i) + D_d u(0) \right| \\
&= \left( \sum_{i=0}^{\infty} C_d A_d^i B_d B_d^T (A_d^T)^i C_d^T + D_d D_d^T \right)^{1/2} \tag{16}
\end{aligned}$$

because

$$\begin{aligned}
&\left( \sum_{i=1}^k C_d A_d^i B_d u(i) + D_d u(0) \right)^2 \\
&\leq \left( \sum_{i=1}^k C_d A_d^i B_d B_d^T (A_d^T)^i C_d^T + D_d D_d^T \right) \cdot \sum_{i=0}^k u^2(i) \tag{17}
\end{aligned}$$

Thus, we can see from (16) that the  $l_\infty/l_2$ -induced norm  $\|\mathbf{F}_d\|_{\infty/2}$  coincides with the  $H_2$  norm associated with (the transfer function of) the SISO discrete-time LTI system (12).

### III. $L_\infty/L_2$ -INDUCED NORM OF SISO SAMPLED-DATA SYSTEMS

In the preceding section, we gave an explicit proof of the fact that the  $L_\infty/L_2$ -induced norm  $\|\mathbf{F}_c\|_{\infty/2}$  and the  $l_\infty/l_2$  induced norm  $\|\mathbf{F}_d\|_{\infty/2}$  coincide with the continuous-time and discrete-time  $H_2$  norms, respectively, where the main idea was the application of the Cauchy-Schwarz inequalities. Since these induced norms of SISO continuous-time and discrete-time LTI systems coincides with the continuous-time and discrete-time  $H_2$  norms, respectively, it is interesting to ask whether the same is true for SISO LTI sampled-data

systems; more precisely, since there are two different definitions for the  $H_2$  norm of LTI sampled-data systems [19]–[22], whether the  $L_\infty/L_2$ -induced norm coincides with either of the two definitions. In this regard, it is also of interest to see whether or not the Cauchy-Schwarz inequalities can be directly applied to the sampled-data case. This section is devoted to such arguments and gives a negative answer to the question on the relationship with the  $H_2$  norm.

#### A. $L_\infty/L_2$ -Induced Norm and Its Lifting-Based Treatment

Let us consider the stable sampled-data system  $\Sigma_{SD}$  shown in Fig. 1, where  $P$  denotes the continuous-time LTI generalized plant, while  $\Psi$ ,  $\mathcal{H}$  and  $\mathcal{S}$  denote the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period  $h$  in a synchronous fashion. Solid lines and dashed lines in Fig. 1 are used to represent continuous-time signals and discrete-time signals, respectively. Suppose that  $P$  and  $\Psi$  are described respectively by

$$P : \begin{cases} \frac{dx}{dt} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x \end{cases} \tag{18}$$

$$\Psi : \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k \\ u_k = C_\Psi \psi_k + D_\Psi y_k \end{cases} \tag{19}$$

where  $x(t) \in \mathbb{R}^n$ ,  $w(t) \in \mathbb{R}$ ,  $u(t) \in \mathbb{R}^{n_u}$ ,  $z(t) \in \mathbb{R}$ ,  $y(t) \in \mathbb{R}^{n_y}$ ,  $\psi_k \in \mathbb{R}^{n_\Psi}$ ,  $y_k = y(kh)$  and  $u(t) = u_k$  ( $kh \leq t < (k+1)h$ ). Note that we have assumed ‘ $D_{11} = 0$ ’ and ‘ $D_{21} = 0$ ’ in the description (18) of the continuous-time generalized plant  $P$ . This is necessary (and sufficient by the stability of  $\Sigma_{SD}$ ) for the  $L_\infty/L_2$ -induced norm  $\sup_{\|w\|_2 \leq 1} \|z\|_\infty$  of the sampled-data system  $\Sigma_{SD}$  to be bounded/well-defined.

Because the sampled-data system  $\Sigma_{SD}$  is a hybrid continuous-time/discrete-time system, this system viewed in continuous-time is (periodically) time-varying. To deal with  $\Sigma_{SD}$  as a time-invariant system, we apply the lifting technique [24]–[26]. That is, given  $f \in L_\infty$  or  $f \in L_2$ , its lifting  $\{f_k\}_{k=0}^\infty$  with  $\hat{f}_k \in L_\infty[0, h)$  or  $L_2[0, h)$  ( $k \in \mathbb{N}_0$ ) (with sampling period  $h$ ) is defined as follows [24]–[26]:

$$\hat{f}_k(\theta) = f(kh + \theta) \quad (0 \leq \theta < h) \tag{20}$$

By applying lifting to  $w \in L_2$  and  $z \in L_\infty$ , the lifted representation of the sampled-data system  $\Sigma_{SD}$  is described by

$$\begin{cases} \xi_{k+1} = \mathcal{A} \xi_k + \mathcal{B} \hat{w}_k \\ \hat{z}_k = \mathcal{C} \xi_k + \mathcal{D} \hat{w}_k \end{cases} \tag{21}$$

with  $\xi_k := [x_k^T \ \psi_k^T]^T$  ( $x_k := x(kh)$ ), the matrix

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d} D_\Psi C_{2d} & B_{2d} C_\Psi \\ B_\Psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}^{n+n_\Psi} \rightarrow \mathbb{R}^{n+n_\Psi} \tag{22}$$

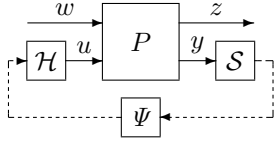


Fig. 1. Sampled-data system  $\Sigma_{SD}$ .

and the operators

$$\mathcal{B} = J\mathbf{B}_1 : L_2[0, h) \rightarrow \mathbb{R}^{n+n_\Psi} \quad (23)$$

$$\mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}^{n+n_\Psi} \rightarrow L_\infty[0, h) \quad (24)$$

$$\mathcal{D} = \mathbf{D}_{11} : L_2[0, h) \rightarrow L_\infty[0, h) \quad (25)$$

where

$$A_d := \exp(Ah), \quad B_{2d} := \int_0^h \exp(A\theta) B_2 d\theta, \quad C_{2d} := C_2 \quad (26)$$

$$J := \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+n_\Psi) \times n}, \quad C_\Sigma := \begin{bmatrix} I & 0 \\ D_\Psi C_{2d} & C_\Psi \end{bmatrix} \quad (27)$$

$$\mathbf{B}_1 w = \int_0^h \exp(A(h-\theta)) B_1 w(\theta) d\theta \quad (28)$$

$$\left( \mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix} \right) (\theta) = C_0 \exp(A_2 \theta) \begin{bmatrix} x \\ u \end{bmatrix} \quad (29)$$

$$A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \quad C_0 := [C_1 \quad D_{12}] \quad (30)$$

$$(\mathbf{D}_{11} w)(\theta) = \int_0^\theta C_1 \exp(A(\theta-\tau)) B_1 w(\tau) d\tau \quad (31)$$

From the stability assumption of  $\Sigma_{SD}$ ,  $\mathcal{A}$  is stable, i.e., has all its eigenvalues in the open unit disc.

Once the discrete-time LTI representation (21) of the sampled-data system  $\Sigma_{SD}$  is obtained by applying lifting (for simplicity, we say the sampled-data system  $\Sigma_{SD}$  is LTI for the existence of such a representation), one may consider that its  $L_\infty/L_2$ -induced norm could be easily computed through some technique similar to that employed in the computation of the  $l_\infty/l_2$ -induced norm of discrete-time systems. However, (21) is actually quite different from the state equation (12) for discrete-time systems because  $\mathcal{B}$ ,  $\mathcal{C}$  and  $\mathcal{D}$  are operators. Hence, the discrete-time Cauchy-Schwarz inequality, which is the key technique in showing the equivalence of the  $l_\infty/l_2$ -induced norm and the  $H_2$  norm in SISO discrete-time LTI systems, cannot be directly applied to the lifted representation (21). Consequently, we need to develop a method specific to (SISO) sampled-data systems. The details of the numerical computation method will be discussed in Section IV, and we restrict our attention in this section to a possible relationship of the  $L_\infty/L_2$  induced norm with the two existing definitions of the  $H_2$  norm of sampled-data systems. More specifically, this subsection is devoted to a preliminary consideration, which is then exploited in the following subsection to study such a possible relationship in more detail.

To give an alternative characterization of the  $L_\infty/L_2$ -induced norm of the SISO LTI sampled-data system  $\Sigma_{SD}$  in the lifting-based framework, we first note (21) and describe

the closed-loop relation between  $\hat{w}_k$  and  $\hat{z}_k$  ( $k = 0, \dots, \infty$ ) as follows:

$$\begin{bmatrix} \hat{z}_0 \\ \hat{z}_1 \\ \hat{z}_2 \\ \hat{z}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{D} & 0 & \cdots & \cdots \\ \mathcal{CB} & \mathcal{D} & 0 & \cdots \\ \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots \\ \mathcal{CA}^2\mathcal{B} & \mathcal{CAB} & \mathcal{CB} & \mathcal{D} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \hat{w}_0 \\ \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \\ \vdots \end{bmatrix} \quad (32)$$

Because the lifting is norm-preserving in both  $L_\infty$  and  $L_2$ , the  $L_\infty/L_2$ -induced norm of the sampled-data system  $\Sigma_{SD}$  coincides with the  $L_\infty/L_2$ -induced norm of the above operator in the right hand side. Furthermore, since this operator has a Toeplitz structure (and thus every row is an extension of the previous row), it readily follows that the  $L_\infty/L_2$ -induced norm of  $\Sigma_{SD}$  coincides with the  $L_\infty/L_2$ -induced norm of its “last” block row, i.e., (after reordering without affecting the  $L_\infty/L_2$ -induced norm)

$$\mathcal{F} := [\mathcal{D} \quad \mathcal{CB} \quad \mathcal{CAB} \quad \mathcal{CA}^2\mathcal{B} \quad \cdots] \quad (33)$$

The  $L_\infty/L_2$ -induced norm  $\|\mathcal{F}\|_{\infty/2}$  is defined as

$$\begin{aligned} \|\mathcal{F}\|_{\infty/2} &:= \sup_{\|\hat{w}\|_2 \leq 1} \|(\mathcal{F}\hat{w})(\cdot)\|_\infty \\ &= \sup_{\|\hat{w}\|_2 \leq 1} \sup_{0 \leq \theta < h} |(\mathcal{F}\hat{w})(\theta)| = \sup_{0 \leq \theta < h} \sup_{\|\hat{w}\|_2 \leq 1} |(\mathcal{F}\hat{w})(\theta)| \end{aligned} \quad (34)$$

where  $\hat{w} =: [\hat{w}_0, \hat{w}_1, \dots]^T$ . For a fixed  $\theta \in [0, h)$ , we have

$$\begin{aligned} (\mathcal{F}\hat{w})(\theta) &= (\mathcal{D}\hat{w}_0)(\theta) + (\mathcal{CB}\hat{w}_1)(\theta) + (\mathcal{CAB}\hat{w}_2)(\theta) + \cdots \\ &= \int_0^\theta D_\theta(\tau) \hat{w}_0(\tau) d\tau + \int_0^h C_\theta B_h(\tau) \hat{w}_1(\tau) d\tau \\ &\quad + \int_0^h C_\theta A B_h(\tau) \hat{w}_2(\tau) d\tau + \cdots \end{aligned} \quad (35)$$

with the matrix functions

$$B_h(\tau) = J \exp(A(h-\tau)) B_1 \quad (36)$$

$$D_\theta(\tau) = C_1 \exp(A(\theta-\tau)) B_1 \quad (37)$$

and the matrix  $C_\theta = C_0 \exp(A_2 \theta) C_\Sigma$ . Applying the continuous-time Cauchy-Schwarz inequality to (35) leads to

$$\begin{aligned} |(\mathcal{F}\hat{w})(\theta)| &\leq \left( \int_0^\theta D_\theta^2(\tau) d\tau \right)^{1/2} \cdot \left( \int_0^\theta \hat{w}_0^2(\tau) d\tau \right)^{1/2} \\ &\quad + \left( \int_0^h (C_\theta B_h(\tau))^2 d\tau \right)^{1/2} \cdot \left( \int_0^h \hat{w}_1^2(\tau) d\tau \right)^{1/2} + \cdots \end{aligned} \quad (38)$$

Furthermore, by applying the discrete-time Cauchy-Schwarz inequality to (38), it easily follows that

$$\begin{aligned} \sup_{\|\hat{w}\|_2 \leq 1} |(\mathcal{F}\hat{w})(\theta)| &\leq \left( \int_0^\theta D_\theta^2(\tau) d\tau + \int_0^h (C_\theta B_h(\tau))^2 d\tau + \int_0^h (C_\theta A B_h(\tau))^2 d\tau \right. \\ &\quad \left. + \int_0^h (C_\theta A^2 B_h(\tau))^2 d\tau + \cdots \right)^{1/2} =: F(\theta) \end{aligned} \quad (39)$$

**Remark 3:** The infinite series (39) is convergent by the stability assumption of  $\mathcal{A}$ .

In particular, if we construct  $\hat{w}$  as

$$\hat{w}_0(\tau) = \begin{cases} \lambda D_\theta(\tau) & (0 \leq \tau < \theta) \\ 0 & (\theta \leq \tau < h) \end{cases} \quad (40)$$

$$\hat{w}_i(\tau) = \lambda C_\theta \mathcal{A}^i B_h(\tau) \quad (0 \leq \tau < h, i \in \mathbb{N}) \quad (41)$$

where  $\lambda := 1/F(\theta)$ , we then easily see that  $\|\hat{w}\|_2 = 1$  and the equalities hold both in (38) and (39). This immediately implies that  $\sup_{\|\hat{w}\|_2 \leq 1} |(\mathcal{F}\hat{w})(\theta)| = F(\theta)$ . Thus, by (34), the  $L_\infty/L_2$ -induced norm  $\|\mathcal{F}\|_{\infty/2}$  can be given by

$$\|\mathcal{F}\|_{\infty/2} = \sup_{0 \leq \theta < h} F(\theta) \quad (42)$$

### B. Relationship between the $L_\infty/L_2$ -Induced Norm and Existing Definitions of the $H_2$ Norm of Sampled-Data Systems

Based on the alternative characterization (42) of the  $L_\infty/L_2$ -induced norm of SISO LTI sampled-data systems, this subsection is devoted to discussing whether this induced norm coincides with either of the two existing definitions of the  $H_2$  norm of sampled-data systems. This is a natural question because this induced norm does coincide with the  $H_2$  norm for SISO continuous-time LTI systems (and the  $l_\infty/l_2$ -induced norm coincides with the  $H_2$  norm for SISO discrete-time LTI systems). We begin by reviewing the two definitions for the  $H_2$  norm of (SISO) LTI sampled-data systems. The first definition [19] considers the  $L_2$  norm of the regulated output  $z(t)$  for the impulse input  $w(t) = \delta(t)$  occurring at  $t = 0$ , an instant at which the sampler takes its action. The second definition [20]–[22], on the other hand, considers the root mean square of the  $L_2$  norms of different responses of  $z(t)$  for the impulse inputs  $w(t)$  occurring at any instants in  $[0, h)$ . The precise definitions are as follows.

1)  $H_2$  norm definition through a single impulse input [19]: When  $w(t) = \delta(t)$ , we can formally regard that its lifted representation is given by

$$\begin{cases} \hat{w}_0 = \delta(\theta) \\ \hat{w}_i = 0 \quad (i \in \mathbb{N}) \end{cases} \quad (43)$$

By evaluating the  $L_2$  norm of the corresponding output, the  $H_2$  norm of the (SISO) LTI sampled-data system  $\Sigma_{\text{SD}}$ , denoted by  $\|\Sigma_{\text{SD}}\|_{H_2}^{[0]}$ , is defined as

$$\begin{aligned} \|\Sigma_{\text{SD}}\|_{H_2}^{[0]} &:= \left\| \begin{bmatrix} \mathcal{D}\delta & \mathcal{C}B\delta & \mathcal{C}A\mathcal{B}\delta & \cdots \end{bmatrix}^T \right\|_2 \\ &= \left( \int_0^h D_h(\theta)^2 d\theta + \int_0^h (C_\theta B_h(0))^2 d\theta + \int_0^h (C_\theta A B_h(0))^2 d\theta \right. \\ &\quad \left. + \int_0^h (C_\theta A^2 B_h(0))^2 d\theta + \cdots \right)^{1/2} \end{aligned} \quad (44)$$

where  $\|\cdot\|_2$  in (44) denotes the  $L_2[0, h)$  norm of an infinite-dimensional vector function on  $[0, h)$ , i.e.,  $\|f\|_2 := (\int_0^h f^T(\theta)f(\theta)d\theta)^{1/2}$ . It is easy to see from (39) and (45) that the variables of integration in (39) are different from those in (45), and this is expected to lead to  $F(\theta)$  different from  $\|\Sigma_{\text{SD}}\|_{H_2}^{[0]}$ , for all  $\theta \in [0, h)$ . This suggests that the  $L_\infty/L_2$ -induced norm  $\|\mathcal{F}\|_{\infty/2}$  is intrinsically different from the  $H_2$  norm  $\|\Sigma_{\text{SD}}\|_{H_2}^{[0]}$  in [19].

2)  $H_2$  norm definition through averaging about impulse inputs [20]–[22]: By considering the impulse inputs  $w(t) = \delta_\tau(t) := \delta(t-\tau)$  for all  $\tau \in [0, h)$ , another  $H_2$  norm, denoted by  $\|\Sigma_{\text{SD}}\|_{H_2}^{[0,h]}$ , is defined as the root mean square of the  $L_2$  norms of  $z(t)$  for these impulse inputs as

$$\begin{aligned} \|\Sigma_{\text{SD}}\|_{H_2}^{[0,h]} &:= \left( \frac{1}{h} \int_0^h \left\| \begin{bmatrix} \mathcal{D}\delta_\tau & \mathcal{C}B\delta_\tau & \mathcal{C}A\mathcal{B}\delta_\tau & \cdots \end{bmatrix}^T \right\|_2^2 d\tau \right)^{1/2} \\ &= \frac{1}{\sqrt{h}} \left( \int_0^h \int_0^\theta (D_\theta(\tau))^2 d\tau d\theta + \int_0^h \int_0^h (C_\theta B_h(\tau))^2 d\tau d\theta \right. \\ &\quad \left. + \int_0^h \int_0^h (C_\theta A B_h(\tau))^2 d\tau d\theta + \cdots \right)^{1/2} \end{aligned} \quad (46)$$

It is very interesting to see from (39) and (46) that

$$\|\Sigma_{\text{SD}}\|_{H_2}^{[0,h]} = \left( \frac{1}{h} \int_0^h F^2(\theta) d\theta \right)^{1/2} \quad (47)$$

while the  $L_\infty/L_2$ -induced norm  $\|\mathcal{F}\|_{\infty/2}$  is described by  $\sup_{0 \leq \theta < h} F(\theta)$  as shown in (42). Hence,

$$\|\Sigma_{\text{SD}}\|_{H_2}^{[0,h]} \leq \|\mathcal{F}\|_{\infty/2} \quad (48)$$

follows immediately and it is suggested that  $L_\infty/L_2$ -induced norm  $\|\mathcal{F}\|_{\infty/2}$  is intrinsically different also from the  $H_2$  norm  $\|\Sigma_{\text{SD}}\|_{H_2}^{[0,h]}$ .

Summarizing the above arguments, we could conclude that the  $L_\infty/L_2$ -induced norm  $\|\mathcal{F}\|_{\infty/2}$  of SISO LTI sampled-data systems may not be characterized by either of the two  $H_2$  norms of sampled-data systems given so far in [19]–[22].

*Remark 4:* When we consider SISO continuous-time LTI systems as a special class of sampled-data systems,  $F(\theta)$  in (39) becomes a constant function on  $[0, h)$ .

### IV. COMPUTATION METHOD OF THE $L_\infty/L_2$ -INDUCED NORM IN SISO LTI SAMPLED-DATA SYSTEMS

This section gives methods for computing  $F(\theta)$  in (39) and the  $L_\infty/L_2$ -induced norm  $\|\mathcal{F}\|_{\infty/2} = \sup_{0 \leq \theta < h} F(\theta)$ . For a fixed  $\theta \in [0, h]$ , we first consider the controllability Gramian

$$W_\theta := \int_0^\theta \exp(A(\theta - \tau)) B_1 B_1^T \exp(A^T(\theta - \tau)) d\tau \quad (49)$$

Then, it is easy to see that

$$\int_0^\theta D_\theta^2(\tau) d\tau = C_1 W_\theta C_1^T \quad (50)$$

$$\int_0^h (C_\theta A^i B_h(\tau))^2 d\tau = C_\theta A^i \begin{bmatrix} W_h & 0 \\ 0 & 0 \end{bmatrix} (A^T)^i C_\theta^T \quad (i \in \mathbb{N}_0) \quad (51)$$

Hence

$$F^2(\theta) = C_1 W_\theta C_1^T + C_\theta \left( \sum_{i=0}^{\infty} A^i \begin{bmatrix} W_h & 0 \\ 0 & 0 \end{bmatrix} (A^T)^i \right) C_\theta^T \quad (52)$$

and by solving the discrete-time Lyapunov equation

$$\mathcal{A}X_h\mathcal{A}^T - X_h + \begin{bmatrix} W_h & 0 \\ 0 & 0 \end{bmatrix} = 0 \quad (53)$$

we readily have

$$F^2(\theta) = C_1W_\theta C_1^T + C_\theta X_h C_\theta^T \quad (54)$$

Hence by (42), we immediately have the following result.

*Theorem 1:* The  $L_\infty/L_2$ -induced norm  $\|\mathcal{F}\|_{\infty/2}$  associated with the SISO LTI sampled-data system  $\Sigma_{SD}$  is given by

$$\|\mathcal{F}\|_{\infty/2} = \sup_{0 \leq \theta < h} (C_1W_\theta C_1^T + C_\theta X_h C_\theta^T)^{1/2} \quad (55)$$

Even though Theorem 1 gives an almost direct method for the computation of the  $L_\infty/L_2$ -induced norm  $\|\mathcal{F}\|_{\infty/2}$  of SISO sampled-data systems, taking the supremum over  $[0, h)$  precisely is bothersome. Regarding this issue, the following result for approximate computation follows readily.

*Theorem 2:* Let  $M \in \mathbb{N}$  and  $h' := h/M$ . Then,

$$\max_{\theta \in \{0, h', \dots, (M-1)h'\}} (C_1W_\theta C_1^T + C_\theta X_h C_\theta^T)^{1/2} \rightarrow \|\mathcal{F}\|_{\infty/2} \quad (56)$$

as  $M \rightarrow \infty$ .

## V. CONCLUSION

This paper tackled the problem of characterizing the induced norm from  $L_2$  to  $L_\infty$  in single-input/single-output (SISO) LTI sampled-data systems. Behind the interest in this problem lied the two facts that (i) this induced norm coincides with the  $H_2$  norm when we confine ourselves to SISO continuous-time LTI systems as a special class of systems under consideration, while (ii) there exist two conceptually different definitions for the  $H_2$  norm of sampled-data systems [19]–[22]. We first gave a closed-form expression for the induced norm and argued that it coincides with neither of the two existing definitions for the  $H_2$  norm. In particular, we showed that it is at least as large as (a more commonly used) one of the two definitions. We then gave a more sophisticated closed-form expression, by which we established an approximate but asymptotically exact method for computing the induced norm. These results are believed to shed a new light on the consequences of the hybrid and periodically time-varying nature of sampled-data systems. Finally, we would like to remark that the induced norm studied in this paper can be regarded as a new definition of the  $H_2$  norm of sampled-data systems, and the optimal controller synthesis problem of minimizing the induced norm may be an interesting future topic.

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