

Generic 構造の安定性について II (The Stability Spectrum of Generic Graphs)

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Abstract

Generic 構造でその理論が superstable であるが ω -stable でないものがあるか、という問題がある (Baldwin の問題). この問題に関して次の部分的結果が得られた. 定理: $\mathbf{K}$ が部分グラフに関して閉じているとき, $\mathbf{K}$ -generic グラフの理論は strictly stable かあるいは ω -stable になる.

1 Introduction

Our notation and definition are similar to those of [8] in this volume. We do not explain all of those details here. For further details, see [2, 3, 4, 5, 6, 9]. Our aim is to give a partial solution for the following problem.

Problem 1 (Baldwin [1]) Is there a generic structure that is superstable but not ω -stable ?

This problem is still open. In fact, all of known stable generic structures is strictly stable or ω -stable. In this paper, we will consider the problem under the following assumption.

Assumption 2 $\mathbf{K}$ is a subclass of $\mathbf{K}_\alpha$ that is closed under substructures. M is a saturated $\mathbf{K}$ -generic graph. (It follows that $\mathbf{K}$ has the amalgamation property.)

For the stability of $\mathbf{K}$ -generic graphs, the following fact is well-known.

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Fact 3 ([3], [9]) Let $T = \text{Th}(M)$. Then

- (i) T is stable;
- (ii) If α is rational, then T is ω -stable.

2 Triviality of $\mathbf{K}$

For each $n \in \omega$, I_n denotes a graph of size n with no relations.

Lemma 4 $I_n \in \mathbf{K}$ for each $n \in \omega$.

Proof If $m \leq n$ and $I_n \in \mathbf{K}$, then $I_m \in \mathbf{K}$ since $\mathbf{K}$ is closed under substructures. So we can assume $n \geq 3$. Then we can take $k \in \omega$ such that $\left\{ k \cdot \binom{k-2}{n-2} \right\} / \binom{k}{n} < \alpha$. Take any $A \in \mathbf{K}$ of size k . Then it is enough to see that A contains a copy of I_n . This can be shown as follows: If not, then $r(A) \leq \binom{k}{n} / \binom{k-2}{n-2}$. So we have $\delta(A) \leq k - \left\{ \binom{k}{n} / \binom{k-2}{n-2} \right\} \alpha < 0$. This is a contradiction.

Lemma 5 If Ab is a finite graph with $A \in \mathbf{K}$ and $r(A, b) = 0$, then $Ab \in \mathbf{K}$.

Proof Let $|A| = k$. Take $n \in \omega$ with $n > k/\alpha$. By lemma 4, we have $I_n \in \mathbf{K}$. So we can assume that $I_n A (\in \mathbf{K})$ is an amalgamation of I_n and A over $\emptyset$. It is enough to show that there is $b' \in I_n - A$ with $b' \cong_A b$. This can be shown as follows: If $A \subset I_n$, then we easily get $b' \in I_n - A$ with $b' \cong_A b$. So we can assume $A \not\subset I_n$. If not, then $r(A - I_n, I_n) \geq n$. Then $\delta(A/I_n) = \delta(A - I_n) - \alpha \cdot r(A - I_n, I_n) \leq k - \alpha \cdot n < 0$. A contradiction.

Definition 6 $\mathbf{K}$ is said to be *trivial*, if there is some $n \in \omega$ such that if $A \in \mathbf{K}$ is connected then $|A| \leq n$.

Lemma 7 If $\mathbf{K}$ is trivial, then $\text{Th}(M)$ is ω -stable.

Proof Take any countable $A \leq M$ and $\bar{b} \in M$. To show that $\text{Th}(M)$ is ω -stable, it is enough to see that $S(A)$ is countable. Since $\mathbf{K}$ is trivial, there is finite $A_0 \leq A$ with $d(\bar{b}/A) = d(\bar{b}/A_0)$. Let $B = \text{cl}(\bar{b}A_0)$ and $A_1 = B \cap A$. Note that $A_1 \leq M$.

Claim: $\text{tp}(\bar{b}/A_1) \vdash \text{tp}(\bar{b}/A)$.

Proof: Take any $\bar{c} \in M$ such that $\text{tp}(\bar{c}/A_1) = \text{tp}(\bar{b}/A_1)$ and $d(\bar{c}/A_1) = d(\bar{b}/A_1)$. Let $C = \text{cl}(\bar{c}A_1)$. Then we have $B \cong_{A_1} C$. From proposition 13 it follows that B and A are free over A_1 and $BA \leq M$, and that C and A are free over A_1 and

$CA \leq M$. In particular we have $B \cong_A C$, and so $\text{tp}(B/A) = \text{tp}(C/A)$. Hence $\text{tp}(\bar{b}/A) = \text{tp}(\bar{c}/A)$. (End of Proof of Claim)

Since there is a countable saturated model, $\text{Th}(M)$ is small, and hence the number of the types over a given finite set is countable. It follows that $|S(A)| \leq \aleph_0 \cdot \aleph_0 = \aleph_0$.

For a finite graph A and $e \in A$, we denote $\deg_A(e) = \max\{|B| : \forall b \in B, R(e, b)\}$.

Lemma 8 Suppose that $\mathbf{K}$ is non-trivial. Then for any $n \in \omega$ the following condition $(*)_n$ holds:

$(*)_n$ There is $A \in \mathbf{K}$ and $a \in A$ with $\deg_A(a) \geq n$.

Proof For each $m \in \omega$, let L_m denote a finite graph $a_0 a_1 \dots a_m$ with the relations $R(a_0, a_1), R(a_1, a_2), \dots, R(a_{m-1}, a_m)$. We divide into two cases.

Case 1: $L_m \notin \mathbf{K}$ for some $m \in \omega$.

Since $\mathbf{K}$ is non-trivial, $(*)_n$ clearly holds for each $n \in \omega$. (mousukosi seikaku nil!)

Case 2: $L_m \in \mathbf{K}$ for any $m \in \omega$.

We prove by induction. By induction hypothesis, we assume that $\deg_A(a) \geq n$ for some $A \in \mathbf{K}$.

Subcase 2.1: $\alpha \leq \frac{1}{2}$.

Let acd be a graph with the relations $R(a, c)$ and $R(c, d)$. Since $L_2 \in \mathbf{K}$ and $\alpha < \frac{1}{2}$, we have $ad \leq acd \in \mathbf{K}$. On the other hand, we can assume $r(A, d) = 0$. By lemma 5, we have $ad \leq Ad \in \mathbf{K}$. So we can assume that $cdA (\in \mathbf{K})$ is an amalgamation of acd and Ad over ad . Note that $c \notin A$ since $r(A, c) = 0$. Hence $\deg_{Ac}(a) \geq n + 1$.

Subcase 2.2: $\alpha > \frac{1}{2}$.

Let $m = \min\{k : (k - 1) - k\alpha > 0\}$. Note that $m \geq 3$ since $\alpha > \frac{1}{2}$. Then we have $L_m = a_0 a_1 \dots a_m \in \mathbf{K}$. We can assume $a = a_0$ and $r(A, a_m) = 0$. By lemma 5, we have $aa_m \leq Aa_m \in \mathbf{K}$. By the definition of m , we have $aa_m \leq L_m \in \mathbf{K}$. So we can assume that $AL_m (\in \mathbf{K})$ is an amalgamation of Aa_m and L_m over aa_m . Then we see $a_1 \notin A$. (Proof: If not, then we have $aa_1 a_m \leq L_m$, and so $\delta(L_m/aa_1 a_m) = (m - 2) - (m - 1)\alpha > 0$. This contradicts the definition of m .) Therefore $\deg_{Aa_1}(a) \geq n + 1$.

For each $n \in \omega$, S_n denote a finite graph $aa_1 a_2 \dots a_n$ with the relations $R(a, a_i)$ for every $i = 1, 2, \dots, n$.

Lemma 9 Suppose that $\mathbf{K}$ is non-trivial. Then $S_n \in \mathbf{K}$ for each $n \in \omega$.

Proof Take sufficiently large k . By lemma, there is $A \in \mathbf{K}$ with $\deg_A(a) \geq k$ for some $a \in A$. We can assume that $R(a, b)$ for any $b \in A - a$. Then it is enough to show that there is $I_n \subset A - \{a\}$. (If not, then $r(A - a) \geq \binom{k}{n} / \binom{k-2}{n-2}$, so we have $\delta(A) = \delta(A - a) + \delta(a/A - a) = \delta(A - a) + 1 - k\alpha \leq \left\{ \binom{k}{n} / \binom{k-2}{n-2} \right\} \alpha + 1 - k\alpha < 0$. A contradiction.)

Lemma 10 Let $\mathbf{K}$ be non-trivial. Suppose that Ab is a finite graph such that there is $a \in A$ with $a \leq A \in \mathbf{K}$ and $r(A, b) = r(a, b) = 1$. Then we have $Ab \in \mathbf{K}$.

Proof Let $|A| = k$. Take $n \in \omega$ with $n > k(1 + 1/\alpha)$. By lemma 9, we have $S_n \in \mathbf{K}$. We assume that $a \leq S_n$, and that $R(a, c)$ for any $c \in S_n - \{a\}$. By our assumption, we have $a \leq A \in \mathbf{K}$. So we can assume that $AS_n \in \mathbf{K}$ is an amalgamation of A and S_n over a . To show our lemma, it is enough to see that there is $b' \in S_n - A$ with $b' \cong_A b$. This can be shown as follows: If $A \subset S_n$, then we can easily pick b' as required. If $A \not\subset S_n$, we can take $b' \in S_n - A$ with $r(b', A) = r(b', a) = 1$. (If not, then we have $\delta(A/S_n) = \delta(A - S_n) - r(A - S_n, S_n) \cdot \alpha \leq k - (n - k)\alpha < 0$. On the other hand, we have $S_n \leq AS_n$ since AS_n is an amalgamation. A contradiction.)

Proposition 11 Suppose that $\mathbf{K}$ is non-trivial. Then any finite graph with no cycles belongs to $\mathbf{K}$.

Proof Let B be a finite graph with no cycles. We will prove by induction on $|B|$. Since B has no cycles, we can take $b \in B$ such that there are no distinct $c, d \in B$ with $R(c, b) \wedge R(d, b)$. Let $A = B - \{b\}$. By induction hypothesis, we have $A \in \mathbf{K}$. If $r(b, A) = 0$, then we have $B = Ab \in \mathbf{K}$ by lemma 10. Therefore we assume $r(b, A) = 1$. Since A has no cycles, we have $a \leq A \in \mathbf{K}$. Take $n \in \omega$ with $n > |A|/\alpha$. By lemma 9, $S_n \in \mathbf{K}$. We can assume that $a \in S_n$ with $R(a, c)$ for any $c \in S_n - \{a\}$. Hence we have $a \leq S_n \in \mathbf{K}$.

3 Proof of Theorem

The following proposition was proved in [7] to show that there is no $\mathbf{K}$ -generic pseudoplane that is stable but not ω -stable.

Proposition 12 ([7]) If α is irrational, then there is an infinite graph D with an element e and finite subgraphs $B_1, B_2, \dots$ with the following properties:

- (1) $D = \text{cl}_D(eB_1B_2 \dots)$ has no cycles;
- (2) $d_D(e/B_1) > d_D(e/B_2) > \dots$;
- (3) $B_1 \leq B_2 \leq \dots \leq D$.

In [8], we studied algebraic types of $\mathbf{K}$ -generic graphs, where $\mathbf{K}$ is closed under subgraphs. As a corollary, we have the following proposition.

Proposition 13([8]) Assume that $\mathbf{K}$ is closed under subgraphs. Let A, B, C be finite such that $B, C \leq M$ and $A = B \cap C$. Then the following are equivalent.

- (i) $d(B/A) = d(B/C)$;
- (ii) B and C are free over A , and $BC \leq M$;
- (iii) $\text{tp}(B/C)$ does not fork over A .

Using proposition 12 and 13, we obtain the following theorem.

Theorem Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$ that is closed under subgraph and M a saturated $\mathbf{K}$ -generic graph. Then $\text{Th}(M)$ is strictly stable or ω -stable.

Proof of Theorem If $\mathbf{K}$ is trivial or α is rational, then, by fact 3 and lemma 7, $\text{Th}(M)$ is ω -stable. Thus we can assume that $\mathbf{K}$ is non-trivial and α is irrational. By fact 3 again, $\text{Th}(M)$ is stable. So we have to show that $\text{Th}(M)$ is not superstable: Since α is irrational, by proposition 12, there is an infinite graph D with an element e and finite subgraphs $B_1, B_2, \dots$ such that (i) $D = \text{cl}_D(eB_1B_2\dots)$ has no cycles; (ii) $d_D(e/B_1) > d_D(e/B_2) > \dots$; (iii) $B_1 \leq B_2 \leq \dots \leq D$. Since $\mathbf{K}$ is non-trivial and D has no cycles, by proposition 11, any finite subset of D belongs to $\mathbf{K}$, and so we can assume $D \leq M$. It follows that $d_M(e/B_1) > d_M(e/B_2) > \dots$. By proposition 13, we have $\text{tp}(e/B_1) \subset_f \text{tp}(e/B_2) \subset_f \dots$. Hence $\text{Th}(M)$ is not superstable.

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