Admissible homomorphisms and equivariant relations between weighted projective lines

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- $\mathbf{p}=(p_1,p_2,\cdots,p_t)$: a sequence consisting of positive integers, $p=$ l.c.m.$(p_1,\cdots,p_t)$.
- $\mathbb{L}(\mathbf{p})=\langle\vec{x}_1,\vec{x}_2,\cdots,\vec{x}_t|p_1\vec{x}_1=p_2\vec{x}_2=\cdots=p_t\vec{x}_t\rangle$.
- $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\cdots,\lambda_t)$: a sequence of pairwise distinct points on the projective line $\mathbb{P}^1_{\mathbf{k}}$, normalized such that $\lambda_1=\infty,\lambda_2=0,\lambda_3=1$.
- $S(\mathbf{p};\boldsymbol{\lambda})=\mathbf{k}[X_1,X_2,\cdots,X_t]/I$, where the ideal I is generated by $X_i^{p_i}-(X_2^{p_2}-\lambda_iX_1^{p_1})$ for $3\leq i\leq t$. $S(\mathbf{p};\boldsymbol{\lambda})$ is $\mathbb{L}(\mathbf{p})$-graded by means of $\deg X_i=\vec{x}_i$ for $1\leq i\leq t$.
- By [Geigle-Lenzing], there has an equivalence

$$\frac{\mathrm{mod}^{\mathbb{L}(\mathbf{p})}\text{-}S(\mathbf{p};\boldsymbol{\lambda})}{\mathrm{mod}_0^{\mathbb{L}(\mathbf{p})}\text{-}S(\mathbf{p};\boldsymbol{\lambda})}\xrightarrow{\sim}\mathrm{coh}\text{-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}).$$

Trichotomy for WPLs

- *dualizing element* of $\mathbb{L}(\mathbf{p})$: $\vec{\omega}=(t-2)\vec{c}-\sum_{i=1}^t\vec{x}_i$;
- a group homomorphism: $\delta\colon\mathbb{L}(\mathbf{p})\to\mathbb{Z};\quad\vec{x}_i\mapsto\frac{p}{p_i}$.

Then we have the following trichotomy for WPLs according to $\delta(\vec{\omega})<0,=0$ or >0 respectively:

- domestic: $(),(p),(p_1,p_2),(2,2,p_3),(2,3,3),(2,3,4),(2,3,5)$;
- tubular: $(2,3,6),(3,3,3),(2,4,4),(2,2,2,2)$;
- wild: all other weight types.

Motivation

- $\mathrm{Aut}(\mathrm{coh}\text{-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))$: the automorphism group of $\mathrm{coh}\text{-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda})$;
- $\mathrm{Aut}(\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))$: the subgroup of $\mathrm{Aut}(\mathrm{coh}\text{-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))$ fixing the structure sheaf.

By [Lenzing-Meltzer], there is a split-exact sequence

$$1\to\mathbb{L}(\mathbf{p})\to\mathrm{Aut}(\mathrm{coh}\text{-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))\to\mathrm{Aut}(\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))\to 1.$$

According to [Lenzing], there is a dominance graph for domestic type

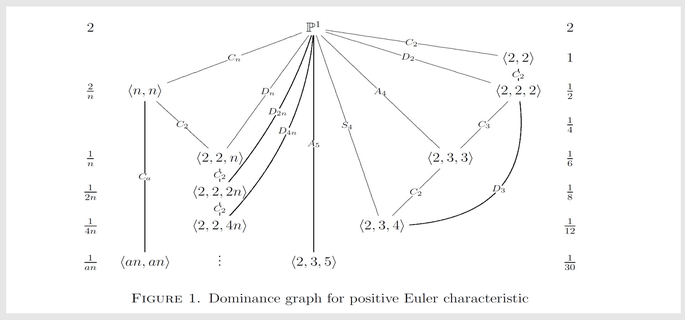


FIGURE 1. Dominance graph for positive Euler characteristic

- For each $1\leq i\leq t$, define

$$\pi_i\colon\mathbb{L}(\mathbf{p})\to\mathbb{Z}/p_i\mathbb{Z};\ \vec{x}_j\mapsto\delta_{i,j}\bar{1}.$$

An infinite subgroup $H\subseteq\mathbb{L}(\mathbf{p})$ is effective if $\pi_i(H)=\mathbb{Z}/p_i\mathbb{Z}$ for each i.

- For each element $\vec{x}=\sum_{i=1}^t l_i\vec{x}_i+l\vec{c}$ in its normal form, that is $0\leq l_i<p_i$ and $l\in\mathbb{Z}$, set

$$\mathrm{mult}(\vec{x}):=\max\{l+1,0\}.$$

Definition ([Chen-Chen]). A group homomorphism $\pi\colon\mathbb{L}(\mathbf{p})\to\mathbb{L}(\mathbf{q})$ is admissible provided that the following conditions are satisfied

1. the subgroup $\mathrm{im}\pi\subseteq\mathbb{L}(\mathbf{q})$ is effective;
2. for each $\vec{z}\in\mathrm{im}\pi$, we have $\sum_{\vec{x}\in\pi^{-1}(\vec{z})}\mathrm{mult}(\vec{x})=\mathrm{mult}(\vec{z})$.

A finite subgroup H of $\mathbb{L}(\mathbf{p})$ is called of *Cyclic type* if

$$H=\langle\frac{p_i}{n}\vec{x}_i-\frac{p_j}{n}\vec{x}_j\rangle;\ \text{ for some } i\neq j \text{ and } n\,|\,\mathrm{g.c.d.}(p_i,p_j);$$

it is called of *Klein type* if

$$H=\langle\frac{p_i}{2}\vec{x}_i-\frac{p_j}{2}\vec{x}_j,\frac{p_i}{2}\vec{x}_i-\frac{p_k}{2}\vec{x}_k\rangle;\ \text{ for some pairwise distinct } i,\ j,\ k.$$

Main Theorem

Let H be a finite subgroup of $\mathbb{L}(\mathbf{p})$. Then the following statements are equivalent:

(1) there exists an admissible homomorphism $\pi\colon\mathbb{L}(\mathbf{p})\to\mathbb{L}(\mathbf{q})$ with $\ker\pi=H$.

(2) for any parameter sequence $\boldsymbol{\lambda}$, there exists a parameter sequence $\boldsymbol{\mu}$, s.t.

$$(\mathrm{coh}\text{-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))^H\xrightarrow{\sim}\mathrm{coh}\text{-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{q};\boldsymbol{\mu}).$$

Moreover, $\ker\pi$ in (1) is either of *Cyclic type* or of *Klein type*. More precisely, up to a permutation isomorphism, one of the following holds:

- $\ker\pi=\langle\frac{p_1}{n}\vec{x}_1-\frac{p_2}{n}\vec{x}_2\rangle$ with $n\,|\,\mathrm{g.c.d.}(p_1,p_2)$; in this case,

$$\mathbf{q}=(\frac{p_1}{n},\frac{p_2}{n},\underbrace{p_3,\cdots,p_3}_{n\text{ times}},\cdots,\underbrace{p_t,\cdots,p_t}_{n\text{ times}});$$

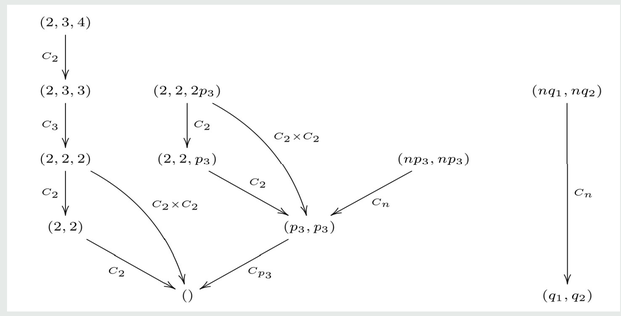
- $\ker\pi=\langle\frac{p_1}{2}\vec{x}_1-\frac{p_2}{2}\vec{x}_2,\frac{p_1}{2}\vec{x}_1-\frac{p_3}{2}\vec{x}_3\rangle$ with p_1,p_2,p_3 even; in this case,

$$\mathbf{q}=(\frac{p_1}{2},\frac{p_1}{2},\frac{p_2}{2},\frac{p_2}{2},\frac{p_3}{2},\frac{p_3}{2},\underbrace{p_4,\cdots,p_4}_{4\text{ times}},\cdots,\underbrace{p_t,\cdots,p_t}_{4\text{ times}}).$$

Admissible homomorphisms between domestic types

$\mathbf{p}$	$\mathbf{q}$	$\ker\pi$	$\pi(\vec{x}_1,\vec{x}_2,\cdots,\vec{x}_t)$
(nq_1,nq_2)	(q_1,q_2)	$\langle q_1\vec{x}_1-q_2\vec{x}_2\rangle$	$(\vec{z}_1,\vec{z}_2)$
$(2,2,p_3)$	(p_3,p_3)	$\langle\vec{x}_1-\vec{x}_2\rangle$	$(\vec{d},\vec{d},\vec{z}_1+\vec{z}_2)$
$(2,2,2p_3)$	$(2,2,p_3)$	$\langle\vec{x}_2-p_3\vec{x}_3\rangle$	$(\vec{z}_1+\vec{z}_2,\vec{d},\vec{z}_3)$
$(2,2,2p_3)$	(p_3,p_3)	$\langle\vec{x}_1-\vec{x}_2,\vec{x}_1-p_3\vec{x}_3\rangle$	$(2\vec{d},2\vec{d},\vec{z}_1+\vec{z}_2)$
$(2,3,3)$	$(2,2,2)$	$\langle\vec{x}_2-\vec{x}_3\rangle$	$(\vec{z}_1+\vec{z}_2+\vec{z}_3,\vec{d},\vec{d})$
$(2,3,4)$	$(2,3,3)$	$\langle\vec{x}_1-2\vec{x}_3\rangle$	$(\vec{d},\vec{z}_2+\vec{z}_3,\vec{z}_1)$

Equivariant relations between domestic types



Here, a weight symbol (a,b,c) stands for the (isoclass of the) weighted projective line $\mathbb{P}^1_{\mathbf{k}}(a,b,c)$; an arrow $\mathbb{P}^1_{\mathbf{k}}(\mathbf{p})\xrightarrow{H}\mathbb{P}^1_{\mathbf{k}}(\mathbf{q})$ stands for an equivalence $(\mathrm{coh}\text{-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p}))^H\xrightarrow{\sim}\mathrm{coh}\text{-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{q})$; and the symbol C_m (resp. $C_2\times C_2$) stands for a finite subgroup of $\mathbb{L}(\mathbf{p})$ of Cyclic type with order m (resp. of Klein type).

Admissible homomorphisms between tubular types

$\mathbf{p}$	$\mathbf{q}$	$\ker\pi$	$\pi(\vec{x}_1,\vec{x}_2,\cdots,\vec{x}_t)$
$(2,2,2,2)$	$(2,2,2,2)$	$\langle\vec{x}_1-\vec{x}_2\rangle$	$(\vec{d},\vec{d},\vec{z}_1+\vec{z}_2,\vec{z}_3+\vec{z}_4)$
		$\langle\vec{x}_1-\vec{x}_2,\vec{x}_1-\vec{x}_3\rangle$	$(2\vec{d},2\vec{d},2\vec{d},\vec{z}_1+\vec{z}_2+\vec{z}_3+\vec{z}_4)$
$(3,3,3)$	$(3,3,3)$	$\langle\vec{x}_1-\vec{x}_2\rangle$	$(\vec{d},\vec{d},\vec{z}_1+\vec{z}_2+\vec{z}_3)$
$(4,4,2)$	$(4,4,2)$	$\langle2\vec{x}_1-\vec{x}_3\rangle$	$(\vec{z}_3,\vec{z}_1+\vec{z}_2,\vec{d})$
	$(2,2,2,2)$	$\langle\vec{x}_1-\vec{x}_2\rangle$	$(\vec{d},\vec{d},\vec{z}_1+\vec{z}_2+\vec{z}_3+\vec{z}_4)$
		$\langle2\vec{x}_1-2\vec{x}_2\rangle$	$(\vec{z}_1,\vec{z}_2,\vec{z}_3+\vec{z}_4)$
		$\langle2\vec{x}_1-2\vec{x}_2,2\vec{x}_1-\vec{x}_3\rangle$	$(\vec{z}_1+\vec{z}_2,\vec{z}_3+\vec{z}_4,2\vec{d})$
$(6,3,2)$	$(3,3,3)$	$\langle3\vec{x}_1-\vec{x}_3\rangle$	$(\vec{z}_1,\vec{z}_2+\vec{z}_3,\vec{d})$
	$(2,2,2,2)$	$\langle2\vec{x}_1-\vec{x}_2\rangle$	$(\vec{z}_1,\vec{d},\vec{z}_2+\vec{z}_3+\vec{z}_4)$

Equivariant relations between tubular types

Assume $\mathbb{L}(\mathbf{p})$ and $\mathbb{L}(\mathbf{q})$ are both of tubular type. Let H be a finite subgroup of $\mathbb{L}(\mathbf{p})$. Then all the equivalences of the form $(\mathrm{coh}\text{-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{p};\boldsymbol{\lambda}))^H\xrightarrow{\sim}\mathrm{coh}\text{-}\mathbb{P}^1_{\mathbf{k}}(\mathbf{q};\boldsymbol{\mu})$ for some $\boldsymbol{\lambda},\boldsymbol{\mu}$ are classified as the following table.

$(\mathbf{p};\boldsymbol{\lambda})$	$(\mathbf{q};\boldsymbol{\mu})$	H	μ
$(2,2,2,2;\lambda)$	$(2,2,2,2;\mu)$	$\langle\vec{x}_1-\vec{x}_2\rangle$ or $\langle\vec{x}_3-\vec{x}_4\rangle$	$\Gamma((\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1})^2)$
		$\langle\vec{x}_1-\vec{x}_3\rangle$ or $\langle\vec{x}_2-\vec{x}_4\rangle$	$\Gamma((\frac{\sqrt{1-\lambda}+1}{\sqrt{1-\lambda}-1})^2)$
		$\langle\vec{x}_1-\vec{x}_4\rangle$ or $\langle\vec{x}_2-\vec{x}_3\rangle$	$\Gamma((\frac{\sqrt{\lambda}+\sqrt{\lambda-1}}{\sqrt{\lambda}-\sqrt{\lambda-1}})^2)$
		$\langle\vec{x}_i-\vec{x}_j,\vec{x}_i-\vec{x}_k\rangle$	$\Gamma(\lambda)$
$(3,3,3)$	$(3,3,3)$	$\langle\vec{x}_i-\vec{x}_j\rangle$	
$(4,4,2)$	$(4,4,2)$	$\langle2\vec{x}_1-\vec{x}_3\rangle,\langle2\vec{x}_2-\vec{x}_3\rangle$	
	$(2,2,2,2;\mu)$	$\langle\vec{x}_1-\vec{x}_2\rangle$	
		$\langle2\vec{x}_1-2\vec{x}_2\rangle$	$\Gamma(-1)$
		$\langle2\vec{x}_1-2\vec{x}_2,2\vec{x}_1-\vec{x}_3\rangle$	
$(6,3,2)$	$(3,3,3)$	$\langle3\vec{x}_1-\vec{x}_3\rangle$	
	$(2,2,2,2;\mu)$	$\langle2\vec{x}_1-\vec{x}_2\rangle$	$\Gamma(\omega)$

Here, $\Gamma(\lambda)=\{\lambda,\frac{1}{\lambda},1-\lambda,\frac{1}{1-\lambda},\frac{\lambda}{\lambda-1},\frac{\lambda-1}{\lambda}\}$, $\omega=\frac{1+\sqrt{-3}}{2}$

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