

ハンドル体の写像類群のホモロジー的アナロジーについて

Susumu Hirose
廣瀬 進 (佐賀大学理工学部)

Department of Mathematics
Faculty of Science and Engineering
Saga University

1. INTRODUCTION

A 3-dimensional handlebody H_g is an orientable 3-manifold constructed from a 3-ball by attaching g 1-handles. We denote the boundary of H_g by Σ_g , which is an orientable closed surface of genus g . Let $\mathcal{M}_g$ be the mapping class group of Σ_g and $\mathcal{H}_g$ be the mapping class group of H_g , for short, we call this group the *handlebody group*. For elements a, b and c of a group, we write $\bar{c} = c^{-1}$, and $a * b = ab\bar{a}$. Let P_g be a planar surface constructed from a 2-disk by removing g copies of disjoint 2-disks. As indicated in Figure 1, we denote the boundary components of P_g by $\gamma_0, \gamma_2, \dots, \gamma_{2g}$, and denote some properly embedded arcs of P_g by $\gamma_1, \gamma_3, \dots, \gamma_{2g+1}$, $\beta_2, \beta_4, \dots, \beta_{2g-2}$ and $\beta'_2, \beta'_4, \dots, \beta'_{2g-2}$. The 3-manifold $P_g \times [-1, 1]$ is homeomorphic to H_g . On $\partial(P_g \times [-1, 1]) = \Sigma_g$, we define $c_{2i-1} = \partial(\gamma_{2i-1} \times [-1, 1])$ ($1 \leq i \leq g+1$), $b_{2j} = \partial(\beta_{2j} \times [-1, 1])$, $b'_{2j} = \partial(\beta'_{2j} \times [-1, 1])$ ($2 \leq j \leq g-1$), and $c_{2k} = \gamma_{2k} \times \{0\}$ ($1 \leq k \leq g$). In Figures 2 and 3, these circles are illustrated and oriented. For simple close curve a on Σ_g , we define the Dehn twist T_a about a as indicated in Figure 4.

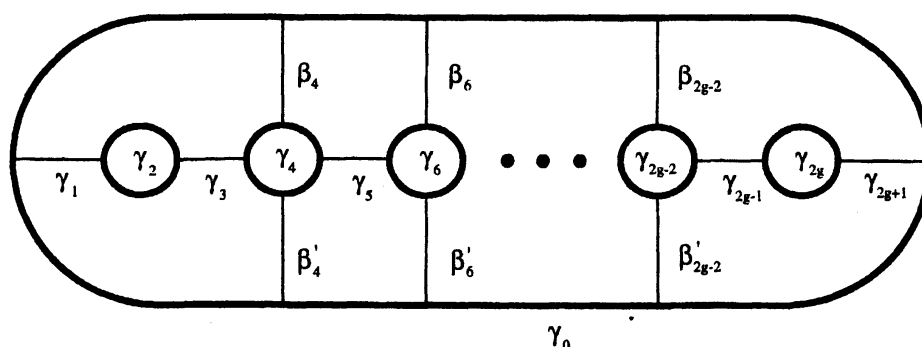


FIGURE 1

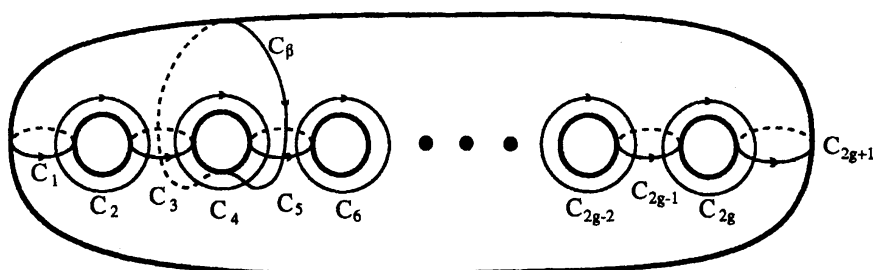


FIGURE 2

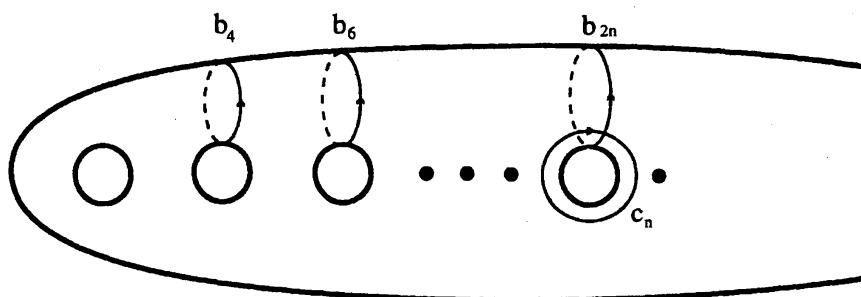


FIGURE 3

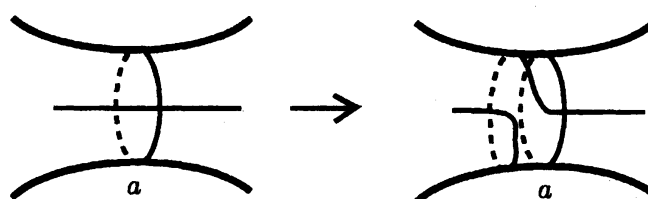


FIGURE 4

For short, we denote T_{c_i} by C_i , and $T_{b_{2i}}$ by B_{2i} . As elements of $H_1(\Sigma_g, \mathbb{Z})$, we take

$$\begin{aligned} x_1 &= -c_1, & y_1 &= -c_2 \\ x_i &= b_{2i}, & y_i &= -c_{2i}, \text{ where } 2 \leq i \leq g-1, \\ x_g &= -c_{2g}, & y_g &= -c_{2g+1}. \end{aligned}$$

Then, $\{x_1, y_1, \dots, x_g, y_g\}$ is a basis of $H_1(\Sigma_g, \mathbb{Z})$, and satisfy $(x_i, y_j) = \delta_{i,j}$, $(x_i, x_j) = (y_i, y_j) = 0$ for the intersection form $(,)$. Let E_g be a identity $g \times g$ matrix, and

$$J = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

We define $\text{Sp}(2g) = \{M \in GL(2g, \mathbb{Z}) \mid MJM' = J\}$, where M' means a transpose of M . Let p be a point on Σ_g . We can characterize the handlebody group $\mathcal{H}_g$ by the actions of each elements on the fundamental group $\pi_1(\Sigma_g, p)$. Let l_1 be an arc on Σ_g which begins from p and ends on c_1 , l_i ($2 \leq i \leq g-1$) be an arc Σ_g which begins from p and ends on b_{2i} , and l_g be an arc on Σ_g which begins from p and ends on c_{2g} . We denote $\mathcal{N}$ the normal closure of $\{l_1 c_1 \bar{l}_1, l_2 b_4 \bar{l}_2, \dots, l_{g-1} b_{2g-2} \bar{l}_{g-1}, l_g c_{2g} \bar{l}_g\}$, then $\mathcal{H}_g = \{\phi \in \mathcal{M}_g \mid \phi(\mathcal{N}) = \mathcal{N}\}$. We define a homological analogue of $\mathcal{H}_g$. Let N be the $\mathbb{Z}$ -submodule of $H_1(\Sigma_g, \mathbb{Z})$ generated by $\{x_1, \dots, x_g\}$, and $\mathcal{HH}_g$ be a subgroup of $\mathcal{M}_g$ defined by $\mathcal{HH}_g = \{\phi \in \mathcal{M}_g \mid \phi_*(N) = N\}$. We call $\mathcal{HH}_g$ the *homological handlebody group* of genus g . For each element ϕ of $\mathcal{M}_g$, we define a $2g \times 2g$ matrix M_ϕ by

$$(\phi(x_1), \phi(x_2), \dots, \phi(x_g), \phi(y_1), \phi(y_2), \dots, \phi(y_g)) = (x_1, x_2, \dots, x_g, y_1, y_2, \dots, y_g) M_\phi.$$

Then, M_ϕ is an element of $\text{Sp}(2g)$, and the map μ from $\mathcal{M}_g$ to $\text{Sp}(2g)$ defined by mapping ϕ to M_ϕ is a surjection. On the other hand, $\mu|_{\mathcal{H}_g}$ is not a surjection. We define a subgroup $ur\text{Sp}(2g)$ of $\text{Sp}(2g)$ by

$$ur\text{Sp}(2g) = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \text{Sp}(2g) \right\},$$

where A , B , and D are $g \times g$ matrices, and 0 is a $g \times g$ zero matrix. We show the following theorem

Theorem 1.1. $\mu(\mathcal{H}_g) = ur\text{Sp}(2g)$.

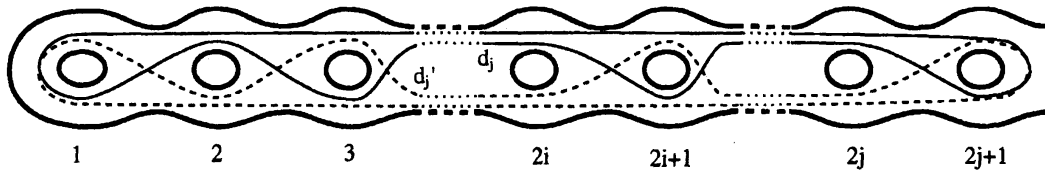


FIGURE 5

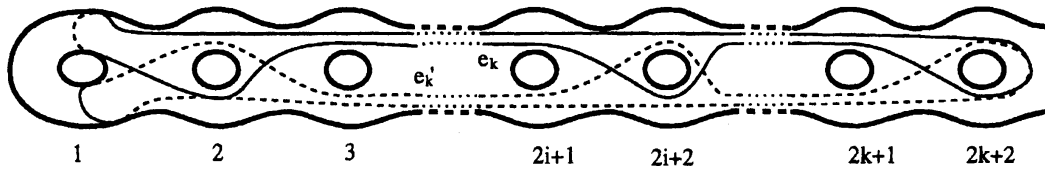


FIGURE 6

By definition, $\mathcal{HH}_g = \mu^{-1}(urSp(2g))$. Let $[a]$ be the largest integer n which satisfies $n \leq a$, and d_j, d'_j, e_k, e'_k are indicated in Figures 5 and 6. We show

Theorem 1.2. *If $g \geq 3$, $\mathcal{HH}_g$ is generated by $C_1, C_2C_1^2C_2, C_2C_1C_3C_2, C_{2i}C_{2i-1}B_{2i}C_{2i}, C_{2i}C_{2i+1}B_{2i}C_{2i}$ ($2 \leq i \leq g-1$), $C_{2g}C_{2g-1}C_{2g+1}C_{2g}, T_{d_j}\overline{T_{d'_j}}$ ($1 \leq j \leq [\frac{g-1}{2}]$), and $T_{e_k}\overline{T_{e'_k}}$ ($1 \leq k \leq [\frac{g-2}{2}]$).*

The author does not know whether $\mathcal{HH}_2$ is finitely generated or not. This note is a survey of a paper [1].

2. PROOF OF THEOREM 1.1

It is easy to see that $\mu(\mathcal{HH}_g) \subset \text{urSp}(2g)$. We show that $\text{urSp}(2g) \subset \mu(\mathcal{HH}_g)$. Let S_0 be a $g \times g$ symmetric matrix, and U_1, U_2, U_3 be $g \times g$ unimodular matrices given by

$$S_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, U_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, U_3 = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

By applying the argument by Hua and Reiner [2], we show

Lemma 2.1. *The group $\text{urSp}(2g)$ is generated by*

$$\left\{ \begin{pmatrix} E_g & S_0 \\ 0 & E_g \end{pmatrix}, \begin{pmatrix} U_i & 0 \\ 0 & (U'_i)^{-1} \end{pmatrix}, \text{ where } i = 1, 2, 3 \right\}.$$

□

Suzuki [5] introduced elements ρ (cyclic translation of handles), ω_1 (twisting a knob), ρ_{12} (interchanging two knobs), and θ_{12} (sliding) of $\mathcal{H}_g$. In [5], their actions on the fundamental group of Σ_g were listed. With using this list, we show

$$\mu(C_1) = \begin{pmatrix} E_g & S_0 \\ 0 & E_g \end{pmatrix}, \mu(\rho) = \begin{pmatrix} U_1 & 0 \\ 0 & (U'_1)^{-1} \end{pmatrix},$$

$$\mu(\rho_{12}\theta_{12}\rho_{12}^{-1}) = \begin{pmatrix} U_2 & 0 \\ 0 & (U'_2)^{-1} \end{pmatrix}, \mu(\omega_1) = \begin{pmatrix} U_3 & 0 \\ 0 & (U'_3)^{-1} \end{pmatrix}.$$

The above observation shows that $\text{urSp}(2g) \subset \mu(\mathcal{H}_g)$.

3. PROOF OF THEOREM 1.2

We denote the kernel of μ by $\mathcal{I}_g$ and call this the *Torelli group*. By Theorem 1.1, we can show that $\mathcal{HH}_g$ is generated by $\mathcal{H}_g \cup \mathcal{I}_g$. For $g \geq 3$, we find finite subsets $\mathcal{S}$ of $\mathcal{I}_g$ such that $\mathcal{H}_g \cup \mathcal{S}$ generates $\mathcal{HH}_g$. Johnson [3] showed that, when g is larger

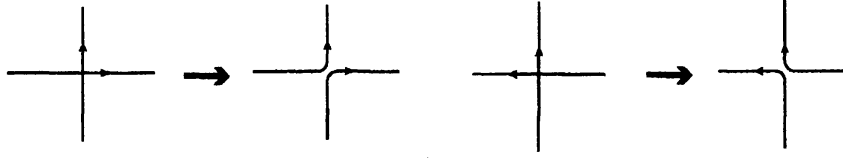


FIGURE 7

than or equal to 3, $\mathcal{I}_g$ is finitely generated. We review his result. We orient and call simple closed curves as is indicated in Figure 2, and call $(c_1, c_2, \dots, c_{2g+1})$ and $(c_\beta, c_5, \dots, c_{2g+1})$ as *chains*. For oriented simple closed curves d and e which mutually intersect in one point, we construct an oriented simple closed curve $d + e$ from $d \cup e$ as follows: choose a disk neighborhood of the intersection point and in it make a replacement as indicated in Figure 7. For a consecutive subset $\{c_i, c_{i+1}, \dots, c_j\}$ of a chain, let $c_i + \dots + c_j$ be the oriented simple closed curve constructed by repeated applications of the above operations. Let $(i_1, \dots, i_{r+1})$ be a subsequence of $(1, 2, \dots, 2g+1)$ (Resp. $(\beta, 5, \dots, 2g+1)$). We construct the union of circles $\mathcal{C} = c_{i_1} + \dots + c_{i_2-1} \cup c_{i_2} + \dots + c_{i_3-1} \cup \dots \cup c_{i_r} + \dots + c_{i_{r+1}-1}$. If r is odd, the regular neighborhood of $\mathcal{C}$ is an oriented compact surface with 2 boundary components. Let ϕ be the element of $\mathcal{M}_g$ defined as the composition of the positive Dehn twist along the boundary curve to the left of $\mathcal{C}$ and the negative Dehn twist along the boundary curve to the right of $\mathcal{C}$. Then, ϕ is an element of $\mathcal{I}_g$. We denote ϕ by $[i_1, \dots, i_{r+1}]$, and call this *the odd subchain map* of $(c_1, c_2, \dots, c_{2g+1})$ (Resp. $(c_\beta, c_5, \dots, c_{2g+1})$). Johnson [3] showed the following theorem:

Theorem 3.1. [3, Main Theorem] *For $g \geq 3$, the odd subchain maps of the two chains $(c_1, c_2, \dots, c_{2g+1})$ and $(c_\beta, c_5, \dots, c_{2g+1})$ generate $\mathcal{I}_g$. $\square$*

By taking conjugations of odd subchain maps by elements of $\mathcal{H}_g$ and applying the following theorem by Takahashi [6], we show Theorem 1.2.

Theorem 3.2. [6] *$\mathcal{H}_g$ is generated by $C_1, C_2C_1^2C_2, C_2C_1C_3C_2, C_{2i}C_{2i-1}B_{2i}C_{2i}, C_{2i}C_{2i+1}B_{2i}C_{2i}$ ($2 \leq i \leq g-1$), $C_{2g}C_{2g-1}C_{2g+1}C_{2g}$. $\square$*

ACKNOWLEDGMENTS

A part of this work was done while the author stayed at Michigan State University as a visiting scholar sponsored by the Japanese Ministry of Education, Culture, Sports, Science and Technology. He is grateful to the Department of Mathematics, Michigan State University, especially to Professor Nikolai V. Ivanov, for their hospitality.

REFERENCES

1. S. Hirose, The action of the handlebody group on the first homology group of a surface, preprint
2. L.K. Hua and I. Reiner, On the generators of the symplectic modular group, *Trans. Amer. Math. Soc.* 65(1949), 415–426.
3. D. Johnson, The structure of the Torelli Group I: A finite set of generators for $\mathcal{I}$, *Annals of Math.* 118(1983), 423–442.
4. D. McCullough and A. Miller, The genus 2 Torelli group is not finitely generated, *Topology Appl.* 22 (1986), 43–49.
5. S. Suzuki, On homeomorphisms of a 3-dimensional handlebody, *Canad. J. Math.* 29 (1977), 111–124.
6. M. Takahashi, On the generators of the mapping class group of a 3-dimensional handlebody, *Proc. Japan Acad. Ser. A Math. Sci.* 71 (1995), 213–214.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING, SAGA UNIVERSITY,
SAGA, 840 JAPAN

E-mail address: hirose@ms.saga-u.ac.jp