

Analytic torsion for irreducible holomorphic
symplectic fourfolds with involution

Dai Imai

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Chapter 0

Introduction

In this thesis, we study equivariant analytic torsion for irreducible holomorphic symplectic manifolds of $K3^{[2]}$ -type with antisymplectic involution. We introduce a holomorphic torsion invariant for these manifolds and study the invariant as a function on the orthogonal modular variety associated to the periods of these manifolds. To explain our results, let us recall equivariant analytic torsion and equivariant Quillen metrics.

0.1 Analytic torsion and Quillen metrics

The notion of analytic torsion for complex manifolds was introduced by Ray-Singer [72]. It is the exponential of the derivative at zero of the weighted alternating sum of various spectral zeta functions. In [71], Quillen defined a metric on the determinant of the cohomology by using analytic torsion. He introduced the product of the L^2 -metric on the determinant of the cohomology and the analytic torsion, and this metric is called the Quillen metric. One of the remarkable properties of the Quillen metric is that, for a holomorphic hermitian vector bundle over a family of compact Kähler manifolds, the Quillen metric is a C^∞ hermitian metric on the determinant of the cohomology even if the dimension of the cohomology of the fibers is not a locally constant function on the base space.

In [13], [14], [15], Bismut-Gillet-Soulé calculated the curvature of the determinant of the cohomology endowed with the Quillen metric for locally Kähler proper holomorphic submersions. In particular, they established the Grothendieck-Riemann-Roch theorem at the level of differential forms at bidegree $(1, 1)$. In [17], Bismut-Lebeau established the immersion formula for Quillen metrics, which is a comparison formula for Quillen metrics for closed immersions. Using the Bismut-Lebeau theorem, Gillet-Soulé [33] proved the arithmetic Riemann-Roch theorem in Arakelov geometry, where the determinant of the cohomology is endowed with the Quillen metric.

In [11], Bismut studied Quillen metrics in the equivariant setting and obtained the immersion formula and the anomaly formula for equivariant Quillen metrics. In [53], Ma proved the curvature formula for equivariant Quillen metrics. Making use of these

results, Köhler-Rössler [49] proved the fixed point formula of Lefschetz type in Arakelov geometry, which is an equivariant analog of the arithmetic Riemann-Roch theorem in [33].

On the other hand, to extend the Kronecker limit formula in higher dimensions, Yoshikawa [82] constructed an invariant of 2-elementary K3 surfaces by using equivariant analytic torsion. He proved that for each deformation type, his invariant is expressed as the Petersson norm of a certain automorphic form obtained as the tensor product of a Borcherds product and a Siegel modular form.

One of the goals of this thesis is to generalize the result of Yoshikawa [82] to a class of higher dimensional hyperkähler manifolds. In this thesis, we construct an invariant of $K3^{[2]}$ -type manifolds with antisymplectic involution by using the equivariant analytic torsion and we show that, in some special cases, it coincides with Yoshikawa's invariant. Hence, in some cases, our invariant is expressed as the Petersson norm of a certain explicit automorphic form. Moreover, we compare our invariant with the BCOV invariant of a certain class of Calabi-Yau 4-folds. Let us recall the BCOV torsion and the BCOV invariant.

0.2 BCOV invariants

In their study of mirror symmetry at genus one [8], Bershadsky-Cecotti-Ooguri-Vafa introduced the following combination of Ray-Singer analytic torsions:

$$\mathcal{T}_{BCOV}(Z, \omega_Z) = \prod_{p \geq 0} \tau(\overline{\Omega}_Z^p)^{(-1)^p p},$$

where Z is a compact Ricci-flat Kähler manifold and $\tau(\overline{\Omega}_Z^p)$ is the analytic torsion of Ω_Z^p . This combination of analytic torsions is called the BCOV torsion. The conjecture of Bershadsky-Cecotti-Ooguri-Vafa claims the equivalence of the following two functions via the mirror map; one is the generating series of the genus one Gromov-Witten invariants of a Calabi-Yau 3-fold, and the other is the BCOV torsion viewed as a function on the base space of its mirror family of Calabi-Yau 3-folds.

After Bershadsky-Cecotti-Ooguri-Vafa, in mathematics, the BCOV invariant for Calabi-Yau manifolds was introduced by Fang-Lu-Yoshikawa [27] in dimension 3, and by Eriksson-Freixas i Montplet-Mourougane [25], [26] in arbitrary dimension. Besides the BCOV torsion, they introduced certain correction terms consisting of a Bott-Chern secondary class and various covolumes of cohomology lattices to get a genuine invariant of Calabi-Yau manifolds (See §4.2). Then the BCOV invariant of the mirror family of a Calabi-Yau hypersurface in projective space is determined by Fang-Lu-Yoshikawa [27] in dimension 3, and by Eriksson-Freixas i Montplet-Mourougane [26] in arbitrary dimension. Moreover, Eriksson-Freixas i Montplet-Mourougane extend the mirror symmetry conjecture at genus one to arbitrary dimension [26]. On the other hand, the genus one Gromov-Witten invariants of a Calabi-Yau hypersurface of a projective space was determined by Zinger [89], [90]. By comparing the formula for the BCOV invariant

and the generating function of the genus one Gromov-Witten invariants, the mirror symmetry at genus one between a Calabi-Yau hypersurface of $\mathbb{P}^n$ and its mirror family is established by Zinger [89], [90] and Fang-Lu-Yoshikawa [27] for $n = 4$, and by Zinger [89], [90] and Eriksson-Freixas i Montplet-Mourougane [25], [26] for $n \geq 5$.

As for the BCOV invariant of complex tori of higher dimension and hyperkähler manifolds, its triviality is proved by Eriksson-Freixas i Montplet-Mourougane [25]. However, for Ricci-flat compact Kähler manifolds with torsion canonical bundle, the BCOV torsion is not necessarily trivial when it is viewed as a function on the moduli space. In [82], Yoshikawa studied the analytic torsion (in fact BCOV torsion) of Ricci-flat Enriques surfaces and proved that the corresponding function on the moduli space is expressed as the Petersson norm of the Borcherds Φ -function, a remarkable automorphic form on the period domain for Enriques surfaces discovered by Borcherds [20]. On the other hand, very recently, Oberdieck [65] proves the explicit formula for the Gromov-Witten invariants of an Enriques surface conjectured by Klemm-Mariño [46] and he shows that the generating series of the genus one Gromov-Witten invariants of an Enriques surface is derived from a certain power of the Borcherds Φ -function. In this way, for Enriques surfaces, the equivalence of the BCOV torsion and the genus one Gromov-Witten invariants is established by Oberdieck [65] and Yoshikawa [82].

In this thesis, we study the BCOV invariant of the Calabi-Yau 4-folds constructed by Camere-Garbagnati-Mongardi [24], who obtained their Calabi-Yau 4-folds as a crepant resolution of the quotient of a $K3^{[2]}$ -type manifold by an antisymplectic involution. Under a certain assumption on the deformation type, we show that the BCOV invariant of a Calabi-Yau 4-fold of Camere-Garbagnati-Mongardi coincides with our invariant of the corresponding $K3^{[2]}$ -type manifold with involution, up to a universal constant. As a result, we show that the BCOV invariant of the universal covering of the Hilbert scheme of 2-points on an Enriques surface is expressed as the Petersson norm of the Borcherds Φ -function. Furthermore, we show that the BCOV invariant of the Calabi-Yau 4-fold of Camere-Garbagnati-Mongardi obtained from a 2-elementary K3 surface with no fixed curves of positive genus, is expressed as the Petersson norm of the Borcherds product of Yoshikawa [84] and Gritsenko [35]. Let us explain our results in more detail.

0.3 The invariant of $K3^{[2]}$ -type manifolds with involution

Let X be an irreducible holomorphic symplectic manifold. Namely, X is a simply-connected compact Kähler manifold admitting an everywhere non-degenerate holomorphic 2-form η such that $H^0(X, \Omega_X^2)$ is generated by η . It is known that there exist a unique primitive integral quadric form q_X on $H^2(X, \mathbb{Z})$ of signature $(3, b_2(X) - 3)$ and a positive rational number $c_X \in \mathbb{Q}$ such that

$$q_X(\alpha)^n = c_X \int_X \alpha^{2n}$$

for any $\alpha \in H^2(X, \mathbb{Z})$. If $b_2(X) = 6$, we also require that $q_X(\omega) > 0$ for any Kähler class ω . The quadric form q_X is called the Beauville-Bogomolov-Fujiki form ([5]). Let $(\cdot, \cdot) : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}$ be the integral bilinear form corresponding to q_X .

An irreducible holomorphic symplectic manifold X is a manifold of $K3^{[2]}$ -type if X is deformation equivariant to the Hilbert scheme of length 2 zero-dimensional subschemes of a K3 surface. Then $H^2(X, \mathbb{Z})$ is isomorphic to

$$L_2 = L_{K3} \oplus \mathbb{Z}e,$$

where L_{K3} is the K3 lattice, i.e. an even unimodular lattice of signature $(3, 19)$, and e is an element with $e^2 = -2$ and $(e, L_{K3}) = 0$. In this thesis, we study $K3^{[2]}$ -type manifolds. We assume that X is a $K3^{[2]}$ -type manifold.

Let $\iota : X \rightarrow X$ be a holomorphic involution on X . A holomorphic involution ι on X satisfying

$$\iota^* \eta = -\eta$$

is called antisymplectic. In this thesis, *involutions on an irreducible holomorphic symplectic manifold are always assumed to be antisymplectic.*

In [18], Boissière-Camere-Sarti classified the hyperbolic sublattices of L_2 obtained as the invariant subspace $H^2(X, \mathbb{Z})^\iota$ for a $K3^{[2]}$ -type manifold (X, ι) with involution. In [45], these sublattices of L_2 are called admissible sublattices. There are 150 isomorphism classes of the pairs consisting of an admissible sublattice of L_2 and the corresponding involution of L_2 .

For an admissible sublattice M , we set

$$\Delta(M) = \{\delta \in M; \delta^2 = -2, \text{ or } \delta^2 = -10, (\delta, L_2) = 2\mathbb{Z}\} \text{ and } \tilde{\mathcal{C}}_M = \{x \in M \otimes \mathbb{R}; x^2 > 0\}.$$

For $\delta \in \Delta(M)$, we define $\delta^\perp = \{x \in \tilde{\mathcal{C}}_M : (x, \delta) = 0\}$. Then $\bigcup_{\delta \in \Delta(M)} \delta^\perp$ is a closed subset of $\tilde{\mathcal{C}}_M$ and a connected component of $\tilde{\mathcal{C}}_M \setminus \bigcup_{\delta \in \Delta(M)} \delta^\perp$ is called a Kähler-type chamber of M . The set of all Kähler-type chambers of M is denoted by $\text{KT}(M)$.

A pair (X, α) is called a marked manifold of $K3^{[2]}$ -type if X is a manifold of $K3^{[2]}$ -type and $\alpha : H^2(X, \mathbb{Z}) \rightarrow L_2$ is an isometry of lattices. By [39], there exists a moduli space $\mathfrak{M}_{L_2}$ of marked manifolds of $K3^{[2]}$ -type. We fix a connected component $\mathfrak{M}_{L_2}^\circ$ of $\mathfrak{M}_{L_2}$.

Let (M, ι_M) be a pair consisting of an admissible sublattice of L_2 and the corresponding involution of L_2 . Let $\mathcal{K} \in \text{KT}(M)$. For the $K3^{[2]}$ -type manifold (X, ι) with involution, a marking $\alpha : H^2(X, \mathbb{Z}) \rightarrow L_2$ of (X, ι) is said to be admissible for $(M, \mathcal{K})$ if

$$(X, \alpha) \in \mathfrak{M}_{L_2}^\circ, \quad \alpha(H^2(X, \mathbb{Z})^\iota) = M \quad \text{and} \quad \alpha(\mathcal{K}_X^\iota) \subset \mathcal{K},$$

where $\mathcal{K}_X^\iota \subset H^2(X, \mathbb{R})^\iota$ is the set of ι -invariant Kähler classes of X . Moreover ι is called an involution of type $(M, \mathcal{K})$ if there exists an admissible marking for $(M, \mathcal{K})$.

In [45], Joumaah proved that any two manifolds of $K3^{[2]}$ -type with involution of type $(M, \mathcal{K})$ are deformation equivalent. Let $M^\perp$ be the orthogonal complement of M in L_2 . He defined the connected component $\Omega_{M^\perp}^+$ of the period domain

$$\Omega_{M^\perp} = \{[\eta] \in \mathbb{P}(M_{\mathbb{C}}^\perp); (\eta, \eta) = 0, (\eta, \bar{\eta}) > 0\}$$

corresponding to $(M, \mathcal{K})$ and introduced an orthogonal modular variety $\mathcal{M}_{M, \mathcal{K}}$, which is an arithmetic quotient of the bounded symmetric domain $\Omega_{M^\perp}^+$, to define the period of a manifold of $K3^{[2]}$ -type with involution of type $(M, \mathcal{K})$ (See Section 2.2.3). Let $\tilde{\mathcal{M}}_{M, \mathcal{K}}$ be the set of all isomorphism classes of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$. The period of $(X, \iota) \in \tilde{\mathcal{M}}_{M, \mathcal{K}}$ is defined by

$$P_{M, \mathcal{K}}(X, \iota) = [\alpha(H^0(X, \Omega_X^2))] \in \mathcal{M}_{M, \mathcal{K}},$$

where α is an admissible marking for $(M, \mathcal{K})$, and the period map is denoted by $P_{M, \mathcal{K}} : \tilde{\mathcal{M}}_{M, \mathcal{K}} \rightarrow \mathcal{M}_{M, \mathcal{K}}$. Joumaah constructed a reduced divisor $\bar{\mathcal{D}}_{M^\perp}$ on $\mathcal{M}_{M, \mathcal{K}}$ and proved that the image of $P_{M, \mathcal{K}}$ is given by the Zariski open subset

$$\mathcal{M}_{M, \mathcal{K}}^\circ = \mathcal{M}_{M, \mathcal{K}} \setminus \bar{\mathcal{D}}_{M^\perp}$$

and that the period map $P_{M, \mathcal{K}} : \tilde{\mathcal{M}}_{M, \mathcal{K}} \rightarrow \mathcal{M}_{M, \mathcal{K}}^\circ = \mathcal{M}_{M, \mathcal{K}} \setminus \bar{\mathcal{D}}_{M^\perp}$ is generically injective.

Now, let us explain the construction of an invariant of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$. For this, let us recall the notion of equivariant analytic torsion. Let V be a compact Kähler manifold with a holomorphic involution $\theta : V \rightarrow V$. Consider the μ_2 -action on V induced from θ . Let $\bar{E} = (E, h_E)$ be a μ_2 -equivariant holomorphic vector bundle on V endowed with a μ_2 -invariant hermitian metric. For $g \in \mu_2$, the equivariant analytic torsion of $\bar{E}$ is defined by

$$\tau_g(\bar{E}) = \exp \left\{ - \sum_{q=0}^{\dim V} (-1)^q q \zeta'_{0, q, g}(0) \right\},$$

where $\zeta_{0, q, g}(s)$ is the equivariant spectral zeta function of the Laplacian acting on the E -valued smooth $(0, q)$ -forms on V . If $g = 1$, we write $\tau(\bar{E})$ instead of $\tau_1(\bar{E})$ and call $\tau(\bar{E})$ the (usual) analytic torsion of $\bar{E}$.

Let (X, ι) be a manifold of $K3^{[2]}$ -type with involution of type $(M, \mathcal{K})$. Let ω_X be an ι -invariant Kähler form on X . Then ω_X induces the corresponding hermitian metric h_X on the holomorphic cotangent bundle Ω_X^1 , and we set $\bar{\Omega}_X^1 = (\Omega_X^1, h_X)$. The equivariant analytic torsion of $\bar{\Omega}_X^1$ with respect to the μ_2 -action induced from ι is denoted by $\tau_\iota(\bar{\Omega}_X^1)$.

We denote the fixed locus of $\iota : X \rightarrow X$ by X^ι . By [6], X^ι is a possibly disconnected smooth complex surface. We denote by $\bar{\mathcal{O}}_{X^\iota}$ the trivial line bundle $\mathcal{O}_{X^\iota}$ equipped with the trivial metric. The analytic torsion of $\bar{\mathcal{O}}_{X^\iota}$ is denoted by $\tau(\bar{\mathcal{O}}_{X^\iota})$. If $X^\iota = \sqcup_i Z_i$ is the decomposition of X^ι into the connected components, then

$$\tau(\bar{\mathcal{O}}_{X^\iota}) = \prod_i \tau(\bar{\mathcal{O}}_{Z_i}).$$

The volume of (X, ω_X) is defined by $\text{Vol}(X, \omega_X) = \int_X \frac{\omega_X^4}{4!}$. As above, if $X^\iota = \sqcup_i Z_i$ is the decomposition of X^ι into the connected components, then the volume of $(X^\iota, \omega_X|_{X^\iota})$ is defined by

$$\text{Vol}(X^\iota, \omega_X|_{X^\iota}) = \prod_i \int_{Z_i} \frac{\omega_X|_{Z_i}^2}{2!}.$$

We denote the covolume of the lattice $\text{Im}(H^1(X^\iota, \mathbb{Z}) \rightarrow H^1(X^\iota, \mathbb{R}))$ in the cohomology $H^1(X^\iota, \mathbb{R})$ by $\text{Vol}_{L^2}(H^1(X^\iota, \mathbb{Z}), \omega_X|_{X^\iota})$, where $H^1(X^\iota, \mathbb{R})$ is endowed with the L^2 -inner product induced from the Kähler form $\omega_X|_{X^\iota}$.

We define the real-valued smooth function φ on X^ι by

$$\varphi = \frac{\omega_X^4/4!}{\eta^2 \wedge \bar{\eta}^2 \text{Vol}(X, \omega_X)} \Big|_{X^\iota}.$$

Obviously φ is independent of the choice of η . We define a positive real number $A(X, \iota, \omega_X)$ by

$$A(X, \iota, \omega_X) = \exp \left[\int_{X^\iota} (\log \varphi) \Omega \right],$$

where Ω is the differential form on X^ι defined by

$$\Omega = c_1(TX^\iota, h_{X^\iota})^2 - 8c_2(TX^\iota, h_{X^\iota}) - c_1(TX, h_X)|_{X^\iota}^2 + 3c_2(TX, h_X)|_{X^\iota}.$$

Set

$$t = \text{Tr}(\iota_M) + 2.$$

We define a real number $\tau_{M, \mathcal{K}}(X, \iota)$ by

$$\begin{aligned} \tau_{M, \mathcal{K}}(X, \iota) &= \tau_\iota(\bar{\Omega}_X^1) \text{Vol}(X, \omega_X)^{\frac{(t-1)(t-7)}{16}} A(X, \iota, h_X) \\ &\quad \times \tau(\bar{\mathcal{O}}_{X^\iota})^{-2} \text{Vol}(X^\iota, \omega_{X^\iota})^{-2} \text{Vol}_{L^2}(H^1(X^\iota, \mathbb{Z}), \omega_{X^\iota}). \end{aligned}$$

As an application of the curvature formula for Quillen metrics [13], [53], we have the following.

Theorem 0.3.1 (Theorem 5.3.13). *The real number $\tau_{M, \mathcal{K}}(X, \iota)$ is independent of the choice of an ι -invariant Kähler form. In particular, $\tau_{M, \mathcal{K}}(X, \iota)$ is an invariant of (X, ι) .*

By Theorem 0.3.1 and Joumaah's theorem [45], $\tau_{M, \mathcal{K}}$ is viewed as a real-valued function on $\mathcal{M}_{M, \mathcal{K}}^\circ$. Namely,

$$\tau_{M, \mathcal{K}}(p) = \tau_{M, \mathcal{K}}(X, \iota) \quad ((X, \iota) \in P_{M, \mathcal{K}}^{-1}(p))$$

is independent of the choice of $(X, \iota) \in P_{M, \mathcal{K}}^{-1}(p)$.

In this thesis, we study the function $\tau_{M, \mathcal{K}}$ on $\mathcal{M}_{M, \mathcal{K}}^\circ$. More precisely, we prove the curvature formula for $\tau_{M, \mathcal{K}}$ and the algebraicity of the singularity of $\tau_{M, \mathcal{K}}$, and for some cases, we show the coincidence of $\tau_{M, \mathcal{K}}$ and Yoshikawa's invariant of 2-elementary K3 surfaces, as well as the coincidence of $\tau_{M, \mathcal{K}}$ and the BCOV invariant of Calabi-Yau 4-folds of Camere-Garbagnati-Mongardi.

0.4 The curvature formula for $\tau_{M,\mathcal{K}}$

Let $\omega_{\mathcal{M}_{M,\mathcal{K}}}$ be the orbifold Kähler form on $\mathcal{M}_{M,\mathcal{K}}$ induced from the Kähler form of the Bergman metric on the period domain $\Omega_{M^\perp}$. In Lemma 5.3.15, we will prove the existence of a smooth $(1,1)$ -form $\sigma_{M,\mathcal{K}}$ on $\mathcal{M}_{M,\mathcal{K}}^\circ$ such that for any $(X, \iota) \in \tilde{\mathcal{M}}_{M,\mathcal{K}}$ we have

$$\mathcal{P}_{M,\mathcal{K}}^* \sigma_{M,\mathcal{K}} = c_1(\pi_* \Omega_{\mathcal{X}^\iota / \text{Def}(X,\iota)}^1, h_{L^2}) - c_1(R^1 \pi_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) - 2c_1(\pi_* K_{\mathcal{X}^\iota / \text{Def}(X,\iota)}, h_{L^2}),$$

where $\mathcal{P}_{M,\mathcal{K}} : \text{Def}(X, \iota) \rightarrow \mathcal{M}_{M,\mathcal{K}}$ is the period map of the Kuranishi family $\pi : (\mathcal{X}, \iota) \rightarrow \text{Def}(X, \iota)$ of (X, ι) . Recall that $d^c = \frac{\partial - \bar{\partial}}{4\pi i}$.

Theorem 0.4.1 (Theorem 5.3.16). *The function $\tau_{M,\mathcal{K}}$ on $\mathcal{M}_{M,\mathcal{K}}^\circ$ is smooth. Moreover, the following equation of differential forms on $\mathcal{M}_{M,\mathcal{K}}^\circ$ holds:*

$$-dd^c \log \tau_{M,\mathcal{K}} = \frac{(t+1)(t+7)}{8} \omega_{\mathcal{M}_{M,\mathcal{K}}} + \sigma_{M,\mathcal{K}}.$$

There is an application of Theorem 0.4.1 to families of $K3^{[2]}$ -type manifolds with involution.

Theorem 0.4.2 (Theorem 5.3.19). *Suppose that $t \neq -7, -1$ and $q(X^\iota) = p_g(X^\iota) = 0$ for each $(X, \iota) \in \tilde{\mathcal{M}}_{M,\mathcal{K}}$. Then there exists no irreducible projective curve on $\mathcal{M}_{M,\mathcal{K}}^\circ$. In particular, if $f : (\mathcal{X}, \iota) \rightarrow S$ is a family of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$, and if S is compact, then f is isotrivial. Namely any two fibers of f are isomorphic to each other.*

0.5 The algebraicity of the singularity of $\tau_{M,\mathcal{K}}$

Let $\mathcal{X}$ be a smooth projective manifold of dimension $n+1$ and let $\iota : \mathcal{X} \rightarrow \mathcal{X}$ be a holomorphic involution. We consider the μ_2 -action on $\mathcal{X}$ induced from ι . Let C be a smooth projective curve with trivial μ_2 -action, and let $f : \mathcal{X} \rightarrow C$ be a μ_2 -equivariant holomorphic surjection. Then the μ_2 -action preserves the fibers of f .

Let $\Sigma_f = \{x \in \mathcal{X} ; df(x) = 0\}$ be the critical locus of f and let $\Delta_f = f(\Sigma_f)$ be the discriminant locus of f . The fiber of f at $s \in C$ is denoted by X_s . We set $C^\circ = C \setminus \Delta_f$, $\mathcal{X}^\circ = f^{-1}(C^\circ)$, and $f^\circ = f|_{\mathcal{X}^\circ}$. Then $f^\circ : \mathcal{X}^\circ \rightarrow C^\circ$ is a family of projective manifolds with μ_2 -action.

Let $h_{\mathcal{X}}$ be a μ_2 -invariant Kähler metric on $\mathcal{X}$. The relative tangent bundle $T\mathcal{X}/C$ is the μ_2 -equivariant subbundle of $T\mathcal{X}|_{\mathcal{X} \setminus \Sigma_f}$ defined by

$$T\mathcal{X}/C = \text{Ker}(f_*|_{\mathcal{X} \setminus \Sigma_f} : T\mathcal{X}|_{\mathcal{X} \setminus \Sigma_f} \rightarrow f^*TC|_{\mathcal{X} \setminus \Sigma_f}).$$

We set $h_{\mathcal{X}/C} = h_{\mathcal{X}}|_{T\mathcal{X}/C}$. For $s \in C^\circ$, the restriction $h_s := h_{\mathcal{X}/C}|_{X_s}$ is a μ_2 -invariant Kähler metric on X_s .

Fix $0 \in \Delta_f$ and let (Δ, s) be a local coordinate on C centered at 0 such that $\Delta \cap \Delta_f = \{0\}$ and that $s(\Delta)$ is the unit disk in $\mathbb{C}$. We set $X = f^{-1}(\Delta)$ and $\Delta^* = \Delta \setminus \{0\}$. We assume that the singular fiber X_0 is a simple normal crossing divisor.

We define a function $\tau_\iota(\bar{\Omega}_{\mathcal{X}/S}^1)$ on Δ^* by

$$\tau_\iota(\bar{\Omega}_{\mathcal{X}/S}^1)(s) = \tau_\iota(\bar{\Omega}_{X_s}^1) \quad (s \in \Delta^*),$$

where $\tau_\iota(\bar{\Omega}_{X_s}^1)$ is the equivariant analytic torsion of the holomorphic cotangent bundle $\Omega_{X_s}^1$ endowed with the μ_2 -invariant hermitian metric h_s induced from $h_{\mathcal{X}/C}$. Since $h^q(\Omega_{X_s}^1)(q \geq 0)$ is independent of $s \in \Delta^*$, $\tau_\iota(\bar{\Omega}_{\mathcal{X}/S}^1)$ is a smooth function on Δ^* . As an equivariant analog of the results [12], [25], [27], [83], we prove the algebraicity of the singularity of $\tau_\iota(\bar{\Omega}_{\mathcal{X}/S}^1)$.

Theorem 0.5.1 (Theorem 6.1.14). *There exists a rational number $c(X_0, \Omega_{\mathcal{X}/C}^1)$ such that:*

$$\log \tau_\iota(\bar{\Omega}_{\mathcal{X}/S}^1)(s) = c(X_0, \Omega_{\mathcal{X}/C}^1) \log |s|^2 + O(\log \log |s|^{-1}) \quad (s \rightarrow 0).$$

Note that $\Omega_{\mathcal{X}/S}^1$ is not a locally free sheaf on $\mathcal{X}$ since $\Sigma_f \neq \emptyset$.

Let M be an admissible sublattice of L_2 and let $\mathcal{K} \in \text{KT}(M)$ be a Kähler-type chamber. Let $C \subset \mathcal{M}_{M,\mathcal{K}}^*$ be an irreducible projective curve, where $\mathcal{M}_{M,\mathcal{K}}^*$ is the Baily-Borel compactification of $\mathcal{M}_{M,\mathcal{K}}$. We assume that C is not contained in $\bar{\mathcal{D}}_{M^\perp} \cup (\mathcal{M}_{M,\mathcal{K}}^* \setminus \mathcal{M}_{M,\mathcal{K}})$. Let $p \in C \cap \bar{\mathcal{D}}_{M^\perp}$ be a smooth point of C . Choose a coordinate (Δ, s) on C centered at p such that $\Delta \cap \bar{\mathcal{D}}_{M^\perp} = \{p\}$ and $s(\Delta)$ is the unit disk. As an application of Theorem 0.5.1, we prove that the invariant $\tau_{M,\mathcal{K}}$ has an algebraic singularity along the divisor $\bar{\mathcal{D}}_{M^\perp}$ in the following sense.

Theorem 0.5.2 (Theorem 6.2.7). *There exists a rational number $a \in \mathbb{Q}$ such that:*

$$\log \tau_{M,\mathcal{K}}(s) = a \log |s|^2 + O(\log \log |s|^{-1}).$$

0.6 A comparison with Yoshikawa's invariant

In order to explain the applications of Theorems 0.4.1 and 0.5.2, let us recall the notion of 2-elementary K3 surfaces. Let M_0 be a primitive hyperbolic 2-elementary sublattice of the K3 lattice L_{K3} . A K3 surface (Y, σ) with antisymplectic involution is a 2-elementary K3 surface of type M_0 if there exists a marking $\alpha : H^2(X, \mathbb{Z}) \rightarrow L_{K3}$ such that $\alpha(H^2(X, \mathbb{Z})^\sigma) = M_0$. The period domain for 2-elementary K3 surfaces of type M_0

$$\Omega_{M_0^\perp} = \{[\eta] \in \mathbb{P}(M_{0,\mathbb{C}}^\perp); (\eta, \eta) = 0, (\eta, \bar{\eta}) > 0\}$$

consists of two connected components, each of which is a bounded symmetric domain of type IV of dimension $20 - \text{rk}(M_0)$. We fix one of the connected components and denote it by $\Omega_{M_0^\perp}^+$. Let $O(M_0^\perp)$ be the group of isometries of $M_0^\perp$, which acts projectively on

$\Omega_{M_0^\perp}$. Let $O^+(M_0^\perp)$ be the subgroup of $O(M_0^\perp)$ of index 2 preserving the connected components of $\Omega_{M_0^\perp}$. The group $O^+(M_0^\perp)$ acts on $\Omega_{M_0^\perp}^+$ properly discontinuously, so that we have the orthogonal modular variety

$$\mathcal{M}_{M_0} = \Omega_{M_0^\perp}^+ / O^+(M_0^\perp).$$

Set $\Delta(M_0^\perp) = \{d \in M_0^\perp; d^2 = -2\}$ and $\mathcal{D}_{M_0^\perp} = \bigcup_{d \in \Delta(M_0^\perp)} d^\perp$. Then $\mathcal{D}_{M_0^\perp}$ is a reduced divisor on $\Omega_{M_0^\perp}^+$. Since the $O^+(M_0^\perp)$ -action on $\Omega_{M_0^\perp}^+$ induces the $O^+(M_0^\perp)$ -action on $\mathcal{D}_{M_0^\perp}$, the quotient $\bar{\mathcal{D}}_{M_0^\perp}$ is a reduced divisor on $\mathcal{M}_{M_0}$. By [82], its complement $\mathcal{M}_{M_0}^\circ = \mathcal{M}_{M_0} \setminus \bar{\mathcal{D}}_{M_0^\perp}$ is a coarse moduli space of 2-elementary K3 surfaces of type M_0 .

Let (Y, σ) be a 2-elementary K3 surface of type M_0 . Making use of equivariant analytic torsion of (Y, σ) and the analytic torsion of Y^σ , Yoshikawa introduced an invariant $\tau_{M_0}(Y, \sigma)$ of (Y, σ) and determined its structure as an automorphic function on the moduli space. Namely, τ_{M_0} is expressed as the Petersson norm of an automorphic form on $\Omega_{M_0^\perp}^+$ obtained as the tensor product of an explicit Borchers product and an explicit Siegel modular form ([52], [82], [86], [87]).

We set

$$\widetilde{M}_0 = M_0 \oplus \mathbb{Z}e.$$

Then $\widetilde{M}_0$ is an admissible sublattice of L_2 . Let $Y^{[2]}$ be the Hilbert scheme of 2-points on Y and let $\sigma^{[2]} : Y^{[2]} \rightarrow Y^{[2]}$ be the involution induced from σ . Then $\sigma^{[2]} : Y^{[2]} \rightarrow Y^{[2]}$ is antisymplectic and the $\sigma^{[2]}$ -invariant subspace of $H^2(Y^{[2]}, \mathbb{Z})$ is isomorphic to $\widetilde{M}_0$. Moreover, the invariant Kähler cone of $(Y^{[2]}, \sigma^{[2]})$ corresponds to a Kähler-type chamber having the face $M_{0, \mathbb{R}} = e^\perp$. Such a Kähler-type chamber is called natural in this thesis. We consider the case where $r = \text{rk}(M_0) \leq 17$ and the divisor $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible. There exist 23 isometry classes of such M_0 .

Theorem 0.6.1 (Theorem 6.4.12). *Suppose that $\text{rk}(M_0) \leq 17$ and the divisor $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible. Let $\mathcal{K} \in \text{KT}(\widetilde{M}_0)$ be natural. Then there exists a positive constant $C_{M_0} > 0$ depending only on M_0 such that, for any 2-elementary K3 surface (Y, σ) of type M_0 , the following identity holds:*

$$\tau_{\widetilde{M}_0, \mathcal{K}}(Y^{[2]}, \sigma^{[2]}) = C_{M_0} \tau_{M_0}(Y, \sigma)^{-2(r-9)}.$$

In this case, the function $\tau_{\widetilde{M}_0, \mathcal{K}}$ on $\mathcal{M}_{M_0, \mathcal{K}}^\circ$ is expressed as the Petersson norm of an explicit automorphic form on $\Omega_{M_0^\perp}^+$.

Let us consider the case $M_0 = \langle 2 \rangle$. Here $M_0 = \langle 2 \rangle$ is the rank 1 lattice generated by an element h with $h^2 = +2$. Since $\bar{\mathcal{D}}_{M_0^\perp}$ is not irreducible, this case is not covered by Theorem 0.6.1. By Joumaah [45], the admissible sublattice $\widetilde{M}_0 = M_0 \oplus \mathbb{Z}e$ admits two deformation types, one natural and the other non-natural. Let $\mathcal{K}$ be a natural Kähler-type chamber and let $\mathcal{K}'$ be a non-natural Kähler-type chamber. A 2-elementary K3 surface (Y, σ) of type $\langle 2 \rangle$ is given by the pair consisting of the double cover $Y \rightarrow \mathbb{P}^2$ branched over a smooth sextic C and its covering involution $\sigma : Y \rightarrow Y$. Then

$(Y^{[2]}, \sigma^{[2]})$ is a manifold of $K3^{[2]}$ -type with involution of type $(\widetilde{M}_0, \mathcal{K})$. The fixed locus of $\sigma^{[2]} : Y^{[2]} \rightarrow Y^{[2]}$ is isomorphic to $C^{(2)} \sqcup Y/\sigma$. Since $Y/\sigma = \mathbb{P}^2$, we have the Mukai flop $f : Y^{[2]} \dashrightarrow \text{elm}_{Y/\sigma}(Y^{[2]})$, where $\text{elm}_{Y/\sigma}(Y^{[2]})$ is a manifold of $K3^{[2]}$ -type ([36]). We define $\text{elm}_{Y/\sigma}(\sigma^{[2]}) = f \circ \sigma \circ f^{-1}$. Then $\text{elm}_{Y/\sigma}(\sigma^{[2]})$ is a biregular involution. Moreover, $(\text{elm}_{Y/\sigma}(Y^{[2]}), \text{elm}_{Y/\sigma}(\sigma^{[2]}))$ is an irreducible holomorphic symplectic manifold of $K3^{[2]}$ -type with involution of type $(\widetilde{M}_0, \mathcal{K}')$.

Theorem 0.6.2 (Theorem 6.4.15). *There exist positive constants $C_1, C_2 > 0$ such that for any 2-elementary $K3$ surface (Y, σ) of type $M_0 = \langle 2 \rangle$, the following identities hold:*

$$\begin{aligned}\tau_{\widetilde{M}_0, \mathcal{K}}(Y^{[2]}, \sigma^{[2]}) &= C_1 \tau_{M_0}(Y, \sigma)^{16}, \\ \tau_{\widetilde{M}_0, \mathcal{K}'}(\text{elm}_{Y/\sigma}(Y^{[2]}), \text{elm}_{Y/\sigma}(\sigma^{[2]})) &= C_2 \tau_{M_0}(Y, \sigma)^{16}.\end{aligned}$$

In this case, $\tau_{\widetilde{M}_0, \mathcal{K}}$ and $\tau_{\widetilde{M}_0, \mathcal{K}'}$ are again expressed as the Petersson norm of an explicit automorphic form on $\Omega_{M_0^\perp}^+$.

0.7 The BCOV invariant of the Calabi-Yau 4-folds of Camere-Garbagnati-Mongardi

Let $M \subset L_2$ be an admissible sublattice and let $\mathcal{K} \in \text{KT}(M)$ be a Kähler-type chamber. Let (X, ι) be a manifold of $K3^{[2]}$ -type with involution of type $(M, \mathcal{K})$. The blowup of X along the fixed locus X^ι is denoted by $\widetilde{X}$ and the involution of $\widetilde{X}$ induced from ι is denoted by $\tilde{\iota}$. Camere-Garbagnati-Mongardi [24] proved that its quotient $Z = \widetilde{X}/\tilde{\iota}$ is a Calabi-Yau 4-fold and the canonical map $Z \rightarrow X/\iota$ is a crepant resolution.

Let M_0 be a primitive hyperbolic 2-elementary sublattice of L_{K3} . As before, set

$$\widetilde{M}_0 = M_0 \oplus \mathbb{Z}e.$$

Let $\mathcal{K} \in \text{KT}(\widetilde{M}_0)$ be a natural Kähler-type chamber. Namely, the hyperplane $M_{0, \mathbb{R}}$ in $\widetilde{M}_{0, \mathbb{R}}$ is a face of $\mathcal{K}$.

If there is a 2-elementary $K3$ surface (Y, σ) of type M_0 such that $(X, \iota) = (Y^{[2]}, \sigma^{[2]})$, then we denote X, ι and Z by X_Y, ι_Y and Z_Y , respectively.

Let $\tau_{\text{BCOV}}(Z_Y)$ be the BCOV invariant of Z_Y constructed by Eriksson-Freixas i Montplet-Mourougane [25], [26]. We compare the two invariants $\tau_{\widetilde{M}_0, \mathcal{K}}(X_Y, \iota_Y)$ and $\tau_{\text{BCOV}}(Z_Y)$.

Theorem 0.7.1 (Theorem 7.1.14). *If $r(M_0) \leq 17$ and $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible, then there exists a constant C_{M_0} depending only on M_0 such that*

$$\tau_{\text{BCOV}}(Z_Y) = C_{M_0} \tau_{\widetilde{M}_0, \mathcal{K}}(X_Y, \iota_Y)$$

for any 2-elementary $K3$ surface (Y, σ) of type M_0 .

Now, let us give some applications of Theorem 0.7.1. Assume that $M_0 = U(2) \oplus E_8(2)$, where U is the even unimodular lattice of signature $(1, 1)$ and E_8 is the negative definite even unimodular lattice associated with the Dynkin diagram E_8 . Here, for a lattice $\Lambda = (\mathbb{Z}^r, (\cdot, \cdot)_\Lambda)$, we define $\Lambda(k) = (\mathbb{Z}^r, k(\cdot, \cdot)_\Lambda)$. Then $\text{rk}(M_0) = 10 \leq 17$ and $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible by [61]. For a 2-elementary K3 surface (Y, σ) of type M_0 , its fixed locus Y^σ is empty and the quotient Y/σ is an Enriques surface. Moreover, $\mathcal{M}_{M_0}^\circ$ can be regarded as a coarse moduli space of Enriques surfaces. Oguiso-Schröer [66] showed that Z_Y is the universal cover of the Hilbert scheme of 2-points on the Enriques surface Y/σ .

In [82], Yoshikawa proved that there exists a constant C depending only on $M_0 = U(2) \oplus E_8(2)$ such that

$$\tau_{M_0}(Y, \sigma) = C \|\Phi([Y/\sigma])\|^{-\frac{1}{2}} \quad (0.7.1)$$

for any 2-elementary K3 surface (Y, σ) of type M_0 , where $\|\Phi([Y/\sigma])\|$ is the Petersson norm of the Borcherds Φ -function [20], [21], evaluated at the period of the Enriques surface Y/σ .

Theorem 0.7.2 (Theorem 7.2.2). *There exists a constant C depending only on $M_0 = U(2) \oplus E_8(2)$ such that*

$$\tau_{BCOV}(\widetilde{S^{[2]}}) = C \|\Phi([S])\|$$

for any Enriques surface S , where $\widetilde{S^{[2]}}$ is the universal cover of the Hilbert scheme of 2-points on S .

Let $k = 1, \dots, 6$ and assume that $M_0 = \Lambda_k(2)^\perp$, where Λ_k is an odd unimodular lattice of signature $(2, 10 - k)$ and the orthogonal complement is considered in L_{K3} . Then the fixed locus Y^σ of a 2-elementary K3 surface of type $M_0 = \Lambda_k(2)^\perp$ is the disjoint union of k -rational curves and $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible by [28] and [87].

In [84], [87], Yoshikawa proved that there exists a constant C_k depending only on k such that

$$\tau_{M_0}(Y, \sigma) = C_k \|\Phi_k([Y])\|^{-\frac{1}{2}} \quad (0.7.2)$$

for any 2-elementary K3 surface (Y, σ) of type M_0 , where $\|\Phi_k([Y])\|$ is the Petersson norm of the automorphic form Φ_k of weight $k + 4$ constructed by Yoshikawa [84] and Gritsenko [35], evaluated at the period of (Y, σ) .

Theorem 0.7.3 (Theorem 7.2.4). *There exists a constant C'_k depending only on k such that*

$$\tau_{BCOV}(Z_Y) = C'_k \|\Phi_k([Y])\|^{k+1}$$

for any 2-elementary K3 surface (Y, σ) of type M_0 .

After the work of Eriksson-Freixas i Montplet-Mourougane [26], to our knowledge, Theorems 0.7.2 and 0.7.3 are the first to give an explicit formula for the BCOV invariant for locally universal family of Calabi-Yau 4-folds, whose moduli space has dimension strictly greater than 1.

This thesis is organized as follows. Chapters 1-4 are mainly preliminaries. In Chapter 1, we recall some basic results concerning equivariant analytic torsion and equivariant Quillen metrics. In Chapter 2, we review some standard facts about irreducible holomorphic symplectic manifolds as well as the work of Joumaah concerning irreducible holomorphic symplectic manifolds with involution. In Chapter 3, we recall Borchers products and Siegel modular forms. In Chapter 4, we recall two types of holomorphic torsion invariants. First, we recall the invariant of 2-elementary K3 surfaces of Yoshikawa and his explicit formula in terms of automorphic forms. Second, we recall some basic results concerning the BCOV invariant obtained by Eriksson-Freixas i Montplet-Mourougane.

Chapters 5, 6 and 7 are based on the author's papers [42], [43] and [44], respectively. In Chapter 5, we construct the invariant $\tau_{M,\mathcal{K}}$ of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$ and prove its curvature formula (Theorems 5.3.13 and 5.3.16). In Chapter 6, we show the algebraicity of the singularity of the invariant $\tau_{M,\mathcal{K}}$ (Theorem 6.2.7). As an application, we prove that in some cases, the invariant $\tau_{M,\mathcal{K}}$ coincides with the invariant of 2-elementary K3 surfaces of Yoshikawa (Theorems 6.4.12 and 6.4.15). In Chapter 7, we prove the coincidence of the curvature formula for the invariant $\tau_{M,\mathcal{K}}$ and that for the BCOV invariant of the corresponding Calabi-Yau 4-folds of Camere-Garbagnati-Mongardi (Theorem 7.1.2), and we compare these invariants under certain assumption of $(M, \mathcal{K})$ (Theorem 7.1.14). In some cases, we give an explicit formula for the BCOV invariant of these manifolds (Theorems 7.2.2 and 7.2.4).

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Chapter 1

Equivariant analytic torsion and equivariant Quillen metrics

1.1 Equivariant analytic torsion

1.1.1 Spectral decomposition of the Laplacian

Let M be a compact Kähler manifold of dimension m . Let h_M be a Kähler metric on M and denote its Kähler form by ω_M . The space of smooth (p, q) -forms on M is denoted by $A^{p,q}(M)$.

Let E be a holomorphic vector bundle on M and let h_E be a hermitian metric on E . In this thesis, $c_i(E, h_E)$ denotes the i -th Chern form of (E, h_E) with respect to the Chern connection. The space of E -valued smooth (p, q) -forms on M is denoted by $A^{p,q}(M, E)$ or $A^{p,q}(E)$.

Let $T^{*(0,1)}M$ be the C^∞ complex vector bundle on M whose fiber is the space of cotangent vectors of type $(0, 1)$. The Kähler metric h_M induces a hermitian metric on $T^{*(0,1)}M$ and this hermitian metric is also denoted by h_M . Then the metrics h_M and h_E induce a hermitian metric h on the complex vector bundle $E \otimes \wedge^q T^{*(0,1)}M$. The L^2 -metric on $A^{0,q}(M, E)$ is defined by

$$\langle \alpha, \beta \rangle_{L^2} = \int_M h(\alpha, \beta) \frac{\omega_M^m}{m!} \quad (\alpha, \beta \in A^{0,q}(M, E)). \quad (1.1.1)$$

The Dolbeault operator of E is denoted by $\bar{\partial}_E : A^{0,q}(M, E) \rightarrow A^{0,q+1}(M, E)$, and its formal adjoint is denoted by $\bar{\partial}_E^* : A^{0,q}(M, E) \rightarrow A^{0,q-1}(M, E)$. We define the Hodge-Kodaira Laplacian $D_{0,q}^2$ acting on $A^{0,q}(M, E)$ by

$$D_{0,q}^2 = (\bar{\partial}_E + \bar{\partial}_E^*)^2 : A^{0,q}(M, E) \rightarrow A^{0,q}(M, E).$$

Let F be a C^∞ complex vector bundle on M . Let $x \in M$ and let U be an open neighborhood of x equipped with a system of real local coordinates $x_1, \dots, x_{2m}$ and a trivialization of F . Then U can be identified with a domain of $\mathbb{R}^{2m}$. Let $N =$

$\text{rk}_{\mathbb{R}}(\wedge^q T^{*(0,1)} M \otimes F)$ be the rank of $\wedge^q T^{*(0,1)} M \otimes F$ viewed as a real vector bundle. Then an F -valued smooth $(0, q)$ -form on U can be regarded as an N -tuple of smooth functions on U .

Let $P : A^{0,q}(F) \rightarrow A^{0,q}(F)$ be a partial differential operator of order d . On U , P can be written as

$$P = P(x, D) = (P_1, \dots, P_N), \quad P_i = \sum_{j=1}^N \sum_{|\alpha| \leq d} a_{\alpha, i, j}(x) D_x^\alpha,$$

where $\alpha = \{\alpha_1, \dots, \alpha_{2m}\}$ is a multi-index, D_x^α is a differential operator defined by

$$D_x^\alpha = D_1^{\alpha_1} \dots D_{2m}^{\alpha_{2m}}, \quad D_i = \frac{\partial}{\partial x_i}$$

and $a_{\alpha, i, j}(x)$ is a smooth function on U .

Definition 1.1.1. *The principal symbol $\sigma(P)(x, \xi)$ is a N -tuple of homogeneous polynomials of degree d in ξ defined by*

$$\sigma(P)(x, \xi) = (\sigma_1(P)(x, \xi), \dots, \sigma_N(P)(x, \xi)), \quad \sigma_i(P)(x, \xi) = \sum_{j=1}^N \sum_{|\alpha|=d} a_{\alpha, i, j}(x) \xi^\alpha,$$

where $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_{2m}^{\alpha_{2m}}$.

The principal symbol $\sigma(P)(x, \xi)$ can be regarded as a linear operator on $\wedge^q T_x^{*(0,1)} M \otimes F_x \cong \mathbb{R}^N$ given by the matrix

$$\sigma(P)(x, \xi) = \sum_{|\alpha|=d} \xi^\alpha A_\alpha,$$

where $A_\alpha = (a_{\alpha, i, j}(x))$ is the $N \times N$ -matrix.

Definition 1.1.2. *A partial differential operator $P : A^{0,q}(F) \rightarrow A^{0,q}(F)$ of order d is called elliptic if for each $x \in M$ and each $\xi \neq 0$, the principal symbol $\sigma(P)(x, \xi) \in \text{End}(F_x)$ is an isomorphism.*

Proposition 1.1.3. *The principal symbol of the Hodge-Kodaira Laplacian $D_{0,q}^2$ is given by*

$$\sigma(D_{0,q}^2)(x, \xi) = -\frac{1}{2} \|\xi\|^2 1_{F_x}.$$

Here ξ is regarded as a cotangent vector $\xi = \xi_1 dx_1 + \dots + \xi_{2m} dx_{2m} \in T_x^* M$ and $\|\xi\|^2$ is the norm of ξ with respect to the Kähler metric h_M . In particular, $D_{0,q}^2$ is a second-order self-adjoint elliptic operator.

Proof. See [48, Theorem 3.16] or [7, Theorem 3.52 and Proposition 3.67]. $\square$

For $\lambda \in \mathbb{R}$, we set $E(\lambda) = \{\alpha \in A^{0,q}(M, E); D_{0,q}^2 \alpha = \lambda \alpha\}$. The completion of the pre-Hilbert space $A^{0,q}(M, E)$ with respect to the L^2 metric $\langle \cdot, \cdot \rangle_{L^2}$ is denoted by $L^{0,q}(M, E)$. Since $\langle D_{0,q}^2 \alpha, \alpha \rangle_{L^2} \geq 0$ for all $\alpha \in A^{0,q}(M, E)$, we have $\lambda \geq 0$ if $E(\lambda) \neq 0$.

Theorem 1.1.4. *There exists a sequence $\{\lambda_n\}_{n \in \mathbb{Z}_{\geq 0}} \subset \mathbb{R}_{\geq 0}$ with $\lambda_n \rightarrow +\infty$ such that the following hold:*

- (1) $0 < \dim E(\lambda_n) < +\infty$.
- (2) If $n \neq m$, then $E(\lambda_n)$ is orthogonal to $E(\lambda_m)$.
- (3) $L^{0,q}(M, E) = \widehat{\bigoplus_{n \in \mathbb{Z}_{\geq 0}} E(\lambda_n)}$, where the right hand side of the equality is the completion of the algebraic direct sum $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} E(\lambda_n)$.

Proof. This theorem is proved via the spectral decomposition theorem of the self-adjoint compact operator $(D_{0,q}^2 + 1)^{-1}$. See [48, Appendix Theorem 7.6] and [31, Lemma 1.6.3]. $\square$

Let $\text{rk}(E \otimes \wedge^q T^{*(0,1)} M)$ be the rank of $E \otimes \wedge^q T^{*(0,1)} M$. The volume of (M, ω_M) is denoted by

$$\text{Vol}(M, \omega_M) = \int_M \frac{\omega_M^m}{m!}.$$

Let $\Gamma(z)$ be the Gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (\text{Re } z > 0).$$

Then $\Gamma(z)$ extends to a meromorphic function on $\mathbb{C}$ with simple poles at $0, -1, -2, \dots$.

Theorem 1.1.5. *Let $\{\lambda_n\}_{n \in \mathbb{Z}_{\geq 0}}$ be the spectrum of the Laplacian $D_{0,q}^2$ and let $a_n = \dim E(\lambda_n)$ be the multiplicity of λ_n . We define a function $N(x)$ on $\mathbb{R}_{\geq 0}$ by*

$$N(x) = \sum_n a_n,$$

where the summation runs over all n with $\lambda_n \leq x$. Then the following formula holds:

$$N(x) = \frac{\text{rk}(E \otimes \wedge^q T^{*(0,1)} M) \text{Vol}(M, \omega_M)}{(4\pi)^{\frac{m}{2}} \Gamma(\frac{m}{2} + 1)} x^{\frac{m}{2}} + o(x^{\frac{m}{2}}) \quad (x \rightarrow \infty).$$

Proof. See [7, Corollary 2.43]. $\square$

Theorem 1.1.6. *Let*

$$0 \leq \mu_1 \leq \mu_2 \leq \dots$$

be the eigenvalues of the Laplacian $D_{0,q}^2$ counted with multiplicity. Then there exists a constant C such that the following asymptotic formula holds:

$$\mu_k = C k^{\frac{2}{m}} + o(k^{\frac{2}{m}}) \quad (k \rightarrow +\infty).$$

Proof. By Theorem 1.1.5, the result follows from [76, Proposition 13.1]. $\square$

1.1.2 Heat kernels and their asymptotic expansion

Let $pr_2 : \mathbb{R}_{>0} \times M \times M \rightarrow M$ and $pr_3 : \mathbb{R}_{>0} \times M \times M \rightarrow M$ be the projections to the second and third factor, respectively.

Definition 1.1.7. A smooth section $K^{0,q}(\cdot, \cdot, \cdot)$ of the smooth complex vector bundle

$$pr_2^*(E \otimes \wedge^q T^{*(0,1)} M) \otimes pr_3^*(E^* \otimes \wedge^q T^{*(0,1)} M)$$

on $\mathbb{R}_{>0} \times M \times M$ is called a heat kernel of $D_{0,q}^2$ if the following conditions hold:

(1) For every $y_0 \in M$, the following equation holds:

$$(\partial_t + D_{0,q}^2)K^{0,q}(t, x, y_0) = 0,$$

where $D_{0,q}^2$ acts on the x -variable.

(2) For all $s \in A^{0,q}(E)$, the following holds:

$$\lim_{t \rightarrow 0} \int_M K^{0,q}(t, x, y) s(y) dv(y) = s(x).$$

Theorem 1.1.8. There exists a heat kernel for the operator $D_{0,q}^2$. Moreover, the heat kernel for $D_{0,q}^2$ is unique.

Proof. See [7, Proposition 2.17, Theorem 2.20 and Theorem 2.23]. □

We define an operator $e^{-tD_{0,q}^2}$ on $L^{0,q}(M, E)$ by

$$(e^{-tD_{0,q}^2} s)(x) = \int_M K^{0,q}(t, x, y) s(y) dv(y) \quad (s \in L^{0,q}(M, E)).$$

By Definition 1.1.7 (1), (2), $e^{-tD_{0,q}^2} s$ is a smooth section of the vector bundle $pr_2^*(E \otimes \wedge^q T^{*(0,1)} M)$ on $\mathbb{R}_{>0} \times M$ and it satisfies the following heat equation:

$$(\partial_t + D_{0,q}^2)e^{-tD_{0,q}^2} s = 0, \quad \lim_{t \rightarrow 0} e^{-tD_{0,q}^2} s = s.$$

Let

$$\Delta_M = \{(x, x) \in M \times M; x \in M\}$$

be the diagonal set. Let $d(x, y)$ be the distance function on M with respect to h_M .

Fix a real coordinate system $\{(U_i, x_1, \dots, x_{4m})\}_{i \in I}$ on $M \times M$. Since M is compact, we can assume that I is a finite set. For each i , U_i can be identified with a relatively compact domain of $\mathbb{R}^{4m}$. Then the space of C^∞ functions on U_i is endowed with the C^l -norm $\|\cdot\|_l$. The C^l -norm $\|f\|_l$ of a C^∞ function f on $M \times M$ is defined by the supremum of the C^l -norms $\sup_{i \in I} \|f|_{U_i}\|_l$.

Let $pr'_1 : M \times M \rightarrow M$ and $pr'_2 : M \times M \rightarrow M$ be the projections to the first and second factor, respectively.

Theorem 1.1.9. *Fix $N \in \mathbb{Z}_{\geq m+1}$. Then there exist an open neighborhood U of Δ_M , smooth sections $\Phi_i(x, y)$ ($i \geq 0$) of the C^∞ complex vector bundle on U*

$$\text{Hom}(pr_2'^*(E \otimes \wedge^q T^{*(0,1)} M), pr_1'^*(E \otimes \wedge^q T^{*(0,1)} M))$$

with compact support in U , and a smooth section $R_{N+1}(t, x, y)$ of the C^∞ complex vector bundle on $\mathbb{R}_{\geq 0} \times M \times M$

$$\text{Hom}(pr_3^*(E \otimes \wedge^q T^{*(0,1)} M), pr_2^*(E \otimes \wedge^q T^{*(0,1)} M))$$

satisfying the following:

(1) *On $\mathbb{R}_{>0} \times M \times M$,*

$$K^{0,q}(t, x, y) = \frac{1}{(4\pi t)^m} e^{-\frac{d(x,y)^2}{4t}} \sum_{i=0}^N t^i \Phi_i(x, y) + R_{N+1}(t, x, y).$$

(2) $\phi_0(x, x) = 1_{E \otimes \wedge^q T^{*(0,1)} M}$.

(3) *For all $k, l \geq 0$, there exists a constant $C_{N,l,k} > 0$ such that*

$$\|\partial_t^k R_{N+1}(t, x, y)\|_l \leq C_{N,l,k} t^{N+1-m-l-2k} \quad (t \in (0, 1]).$$

Proof. See [7, Theorem 2.30]. □

1.1.3 Equivariant spectral zeta function and equivariant analytic torsion

Let $\iota : M \rightarrow M$ be a holomorphic involution of M . In what follows, we consider the μ_2 -action on M induced from ι . Assume that h_M is μ_2 -invariant.

We also assume that E is a μ_2 -equivariant holomorphic vector bundle on M and that h_E is μ_2 -invariant.

We define a μ_2 -action on $A^{0,q}(M, E)$ by

$$(g \cdot \alpha)(x) = g(\alpha(g^{-1}(x))) \quad (\alpha \in A^{0,q}(M, E), g \in \mu_2, x \in M).$$

The L^2 -inner product defined in (1.1.1) is a μ_2 -invariant hermitian inner product.

We denote the spectrum of $D_{0,q}^2$ by $\sigma(D_{0,q}^2)$ and the eigenspace of $D_{0,q}^2$ associated with an eigenvalue $\lambda \in \sigma(D_{0,q}^2)$ by $E_{0,q}(\lambda)$. By Theorem 1.1.4, $\sigma(D_{0,q}^2)$ is a discrete subset in $\mathbb{R}_{\geq 0}$ and $E_{0,q}(\lambda)$ is finite dimensional.

Definition 1.1.10. *Let $g \in \mu_2$. The spectral zeta function of $D_{0,q}^2$ is defined by*

$$\zeta_{0,q,g}(s) = \sum_{\lambda \in \sigma(D_{0,q}^2) \setminus \{0\}} \lambda^{-s} \text{Tr}(g|_{E_{0,q}(\lambda)}) \quad (s \in \mathbb{C}, \text{Re } s > m).$$

By Theorem 1.1.6, $\zeta_{0,q,g}(s)$ converges uniformly on compact subsets of $\{s \in \mathbb{C}, \operatorname{Re} s > m\}$.

Definition 1.1.11. Let H be a Hilbert space and let $\{e_i\}$ be a complete orthonormal system of H .

(1) Let A be an bounded operator on H . Set

$$\|A\|_2 = \left[\sum_i \|Ae_i\|^2 \right]^{\frac{1}{2}}.$$

Then A is called a Hilbert-Schmidt operator if $\|A\|_2 < \infty$.

(2) A bounded operator P on H is called a trace class operator if there exist Hilbert-Schmidt operators A, B such that $P = AB$.

(3) Let P be a trace class operator on H . The trace of P is defined by

$$\operatorname{Tr}(P) = \sum_i (Pe_i, e_i).$$

For a trace class operator P , $\operatorname{Tr}(P)$ is independent of the choice of $\{e_i\}$ and $|\operatorname{Tr}(P)| < \infty$.

Proposition 1.1.12. For $t > 0$ and $g \in \mu_2$, the operator $ge^{-tD_{0,q}^2}$ is a trace class operator on $L^{0,q}(M, E)$ and its trace is given by

$$\operatorname{Tr}(ge^{-tD_{0,q}^2}) = \int_M \operatorname{tr}(g_x \cdot K^{0,q}(t, g^{-1}x, x)) dx.$$

Proof. Note that $ge^{-tD_{0,q}^2}$ is the integral operator with smooth kernel $g_x \cdot K^{0,q}(t, g^{-1}x, y)$. The result follows from [54, Lemma D.1.5]. $\square$

Let $m' = \dim M^g$ be the dimension of the fixed locus M^g of $g : M \rightarrow M$.

Theorem 1.1.13. There exist a sequence $\{a_i(g)\}_{i \geq 0}$ of real numbers such that $\operatorname{Tr}(ge^{-tD_{0,q}^2})$ has the following asymptotic expansion as $t \searrow 0$.

$$\operatorname{Tr}(ge^{-tD_{0,q}^2}) \sim (4\pi t)^{-m'} \sum_{i=0}^{\infty} a_i(g) t^i.$$

Namely, for any $N \in \mathbb{Z}_{>0}$, there exists a positive constant $C_N > 0$ such that for any $0 < t \leq 1$,

$$\left| \operatorname{Tr}(ge^{-tD_{0,q}^2}) - (4\pi t)^{-m'} \sum_{i=0}^N a_i(g) t^i \right| \leq C_N t^{N+1-m'}.$$

Proof. See [7, Theorem 6.11]. $\square$

Since

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda t} dt$$

for $\lambda > 0$, the spectral zeta function $\zeta_{0,q,g}(s)$ can be expressed as the Mellin transform

$$\zeta_{0,q,g}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left\{ \text{Tr}(g e^{-tD_{0,q}^2}) - \text{tr}(g|_{\text{Ker}(D_{0,q}^2)}) \right\} dt \quad (\text{Re } s > m').$$

Theorem 1.1.14. *The series $\zeta_{0,q,g}(s)$ convergents uniformly on compact subsets of $\{s \in \mathbb{C}; \text{Re } s > m'\}$. Moreover, $\zeta_{0,q,g}(s)$ extends to a meromorphic function on $\mathbb{C}$, which has possible poles of order at most 1 at $s = m' - k$ ($k = 0, 1, \dots, m' - 1$) and is holomorphic at $s = 0$.*

Proof. The result follows from Theorem 1.1.13 and [7, Lemma 9.34]. $\square$

We set $\overline{E} = (E, h_E)$.

Definition 1.1.15. *Let $g \in \mu_2$. The equivariant analytic torsion of $\overline{E} = (E, h_E)$ on (M, ω_M) is defined by*

$$\tau_g(\overline{E}) = \exp \left\{ - \sum_{q=0}^n (-1)^q q \zeta'_{0,q,g}(0) \right\}.$$

If $g = 1$, it is the (usual) analytic torsion of $\overline{E}$ and is denoted by $\tau(\overline{E})$ instead of $\tau_1(\overline{E})$.

1.2 Equivariant Quillen metrics

1.2.1 Harmonic forms and cohomologies

We keep the notation and assumption in Section 1.1. Namely, M is a compact Kähler manifold, $\iota : M \rightarrow M$ is a holomorphic involution of M , h_M is an μ_2 -invariant Kähler metric on M with the Kähler form ω_M , E is a μ_2 -equivariant holomorphic vector bundle on M and h_E is a μ_2 -invariant hermitian metric on E .

We define the space of E -valued harmonic $(0, q)$ -forms on M by

$$K^q(M, E) = \text{Ker } D_{0,q}^2 \cap A^{0,q}(M, E).$$

Note that $K^q(M, E)$ depends on the Kähler metric h_M and the hermitian metric h_E . By Hodge theory, there is a μ_2 -equivariant canonical isomorphism of complex vector spaces

$$K^q(M, E) \cong H^q(M, E), \quad \alpha \mapsto [\alpha]. \quad (1.2.1)$$

We denote by h_{L^2} the μ_2 -invariant hermitian inner product on the cohomology $H^q(M, E)$ induced from the L^2 -inner product on $A^{0,q}(M, E)$ via the isomorphism (1.2.1). We denote by $H^q(M, E)_\pm$ the (± 1) -eigenspace of $\iota^* : H^q(M, E) \rightarrow H^q(M, E)$. Then the decomposition $H^q(M, E) = H^q(M, E)_+ \oplus H^q(M, E)_-$ is orthogonal with respect to h_{L^2} .

1.2.2 Definition of equivariant Quillen metrics

We define the determinant of the cohomology of E by

$$\lambda(E) = \bigotimes_{q \geq 0} (\det H^q(M, E))^{(-1)^q}.$$

The hermitian inner product h_{L^2} on $H^q(M, E)$ induces the hermitian metric $\|\cdot\|_{\lambda(E), L^2}$ on $\lambda(E)$.

Definition 1.2.1. *We define the Quillen metric on $\lambda(E)$ by*

$$\|\alpha\|_{\lambda(E), Q}^2 = \tau(\overline{E}) \|\alpha\|_{\lambda(E), L^2}^2 \quad (\alpha \in \lambda(E)).$$

We set

$$\lambda_\pm(E) = \bigotimes_{q \geq 0} (\det H^q(M, E)_\pm)^{(-1)^q}.$$

We define the equivariant determinant of the cohomology of E by

$$\lambda_{\mu_2}(E) = \lambda_+(E) \oplus \lambda_-(E).$$

The μ_2 -invariant hermitian inner product h_{L^2} on $H^q(M, E)$ induces the hermitian metric $\|\cdot\|_{\lambda_\pm(E), L^2}$ on $\lambda_\pm(E)$. We define the equivariant metric on $\lambda_{\mu_2}(E)$ by

$$\|\alpha\|_{\lambda_{\mu_2}(E), L^2}(\iota) = \|\alpha_+\|_{\lambda_+(E), L^2} \cdot \|\alpha_-\|_{\lambda_-(E), L^2}^{-1} \quad (\alpha = (\alpha_+, \alpha_-) \in \lambda_{\mu_2}(E), \alpha_\pm \neq 0),$$

and call it the equivariant L^2 -metric on $\lambda_{\mu_2}(E)$.

Definition 1.2.2. *We define the equivariant Quillen metric on $\lambda_{\mu_2}(E)$ by*

$$\|\alpha\|_{\lambda_{\mu_2}(E), Q}^2(\iota) = \tau_g(\overline{E}) \|\alpha\|_{\lambda_{\mu_2}(E), L^2}^2(\iota) \quad (\alpha \in \lambda_{\mu_2}(E)).$$

1.2.3 The curvature formula for the equivariant Quillen metrics

Let $\mathcal{M}$ and B be complex manifolds of dimension $m + n$ and n , respectively. Let $\iota : \mathcal{M} \rightarrow \mathcal{M}$ be a holomorphic involution. Then ι induces a μ_2 -action on $\mathcal{M}$. We consider the trivial μ_2 -action on B . The fixed locus of $\iota : \mathcal{M} \rightarrow \mathcal{M}$ is denoted by $\mathcal{M}^\iota$.

Let $\pi : (\mathcal{M}, \iota) \rightarrow B$ be a proper surjective μ_2 -equivariant holomorphic submersion. Suppose that π is locally Kähler. Namely, for each point $b \in B$ there is an open

neighborhood U of b such that $\pi^{-1}(U)$ is Kähler. The fiber of π is denoted by M_b ($b \in B$) or simply M . Since π is μ_2 -equivariant, the involution ι induces a holomorphic involution on each fiber M_b , which is denoted by ι_b or simply ι .

Let $h_{\mathcal{M}/B}$ be an μ_2 -invariant hermitian metric on the relative tangent bundle $T\mathcal{M}/B = \text{Ker}(\pi_* : T\mathcal{M} \rightarrow \pi^*TB)$ which is fiberwise Kähler. Set $h_b = h_{\mathcal{M}/B}|_{M_b}$ ($b \in B$). This is an μ_2 -invariant Kähler metric on M_b . Its Kähler form is denoted by ω_b and we set $\omega_{\mathcal{M}/B} = \{\omega_b\}_{b \in B}$.

Let $\bar{\mathcal{E}} = (\mathcal{E}, h_{\mathcal{E}})$ be a μ_2 -equivariant holomorphic hermitian vector bundle on $\mathcal{M}$. We assume that $\dim H^q(M_b, \mathcal{E}|_{M_b})$ is independent of $b \in B$ for all $q \geq 0$. Then $R^q\pi_*\mathcal{E}$ is a locally free sheaf for all $q \geq 0$ and we may regard it as a holomorphic vector bundle on B . The fiber of $R^q\pi_*\mathcal{E}$ at $b \in B$ is $H^q(M_b, \mathcal{E}|_{M_b})$ and $R^q\pi_*\mathcal{E}$ is equipped with the μ_2 -invariant hermitian metric whose restriction to the fiber $H^q(M_b, \mathcal{E}|_{M_b})$ coincides with the L^2 -metric. This metric is called the L^2 -metric and denoted by h_{L^2} .

The equivariant determinant of the cohomology of $\mathcal{E}$ is defined by

$$\lambda_{\mu_2}(\mathcal{E}) = \lambda_+(\mathcal{E}) \oplus \lambda_-(\mathcal{E}), \quad \lambda_{\pm}(\mathcal{E}) = \det(R^q\pi_*\mathcal{E})_{\pm},$$

where $(R^q\pi_*\mathcal{E})_{\pm}$ is the (± 1) -eigenbundle of $R^q\pi_*\mathcal{E}$. For an open subset $U \subset B$, a holomorphic section $\sigma = (\sigma_+, \sigma_-)$ of $\lambda_{\mu_2}(\mathcal{E})|_U$ is called an admissible section if both σ_+ and σ_- are nowhere vanishing on U .

Let $g \in \mu_2$. We define a real-valued function on B by

$$\tau_g(\bar{\mathcal{E}})(b) = \tau_g(\bar{\mathcal{E}}|_{M_b}) \quad (b \in B).$$

Let $\mathcal{E}_{\pm}$ be the (± 1) -eigenbundle of the μ_2 -action on $\mathcal{E}|_{\mathcal{M}^{\iota}}$, and the restriction of $h_{\mathcal{E}}$ to $\mathcal{E}_{\pm}$ is denoted by $h_{\pm}$. The curvature form of $(\mathcal{E}_{\pm}, h_{\pm})$ with respect to the Chern connection is denoted by $R_{\pm}$. Recall that the equivariant Todd form and the equivariant Chern character form are differential forms on $\mathcal{M}^{\iota}$ defined by

$$Td_{\iota}(\mathcal{E}, h_{\mathcal{E}}) = Td\left(-\frac{R_+}{2\pi i}\right) \det\left(\frac{I}{I + \exp(+\frac{R_-}{2\pi i})}\right), \quad (1.2.2)$$

and

$$ch_{\iota}(\mathcal{E}, h_{\mathcal{E}}) = ch\left(-\frac{R_+}{2\pi i}\right) - ch\left(-\frac{R_-}{2\pi i}\right), \quad (1.2.3)$$

respectively. Here, for a square matrix A , we set

$$Td(A) = \det\left(\frac{A}{I - e^{-A}}\right)$$

and

$$ch(A) = \text{Tr}(e^A).$$

If α is a differential form, then $[\alpha]^{(p,q)}$ denotes the component of α of bidegree (p, q) .

Since $\pi : \mathcal{M} \rightarrow B$ is a μ_2 -equivariant holomorphic submersion with trivial μ_2 -action on B , its restriction $\pi : \mathcal{M}^\iota \rightarrow B$ is also a holomorphic submersion.

Let $\pi_* : A^{p,q}(\mathcal{M}^\iota) \rightarrow A^{p-m,q-m}(B)$ be the integration along the fibers of π . Namely, for $\alpha \in A^{p,q}(\mathcal{M}^\iota)$, we define $\pi_*(\alpha)$ by the formula:

$$\int_B \pi_*(\alpha) \wedge \beta = \int_{\mathcal{M}^\iota} \alpha \wedge \pi^* \beta$$

for all $\beta \in A^{m+n-p,m+n-q}(B)$ with compact support.

Set $d^c = \frac{\partial - \bar{\partial}}{4\pi i}$. Then we have $dd^c = -\frac{\partial \bar{\partial}}{2\pi i}$.

Theorem 1.2.3. *The function $\tau_\iota(\bar{\mathcal{E}})$ is smooth on B . Moreover, it satisfies the following differential equation on B :*

$$-dd^c \log \tau_\iota(\bar{\mathcal{E}}) + \sum_{q \geq 0} (-1)^q [ch_\iota(R^q \pi_* \mathcal{E}, h_{L^2})]^{(1,1)} = [\pi_* Td_\iota(T\mathcal{M}/B, h_{\mathcal{M}/B}) ch_\iota(\mathcal{E}, h_{\mathcal{E}})]^{(1,1)}.$$

Proof. See [13, Theorem 0.1] and [53, Theorem 2.12.]. □

1.2.4 Bott-Chern secondary classes

Recall that M is a compact Kähler manifold of dimension m and $\iota : M \rightarrow M$ is a holomorphic involution. Set

$$\tilde{A}(M^\iota) = \bigoplus_{p=0}^m A^{p,p}(M^\iota) / (\text{Im } \partial + \text{Im } \bar{\partial}).$$

Theorem 1.2.4. *There exists a map which attaches to a datum $(M, \iota, \mathcal{E}, h_{E'}, h_E, h_{E''})$, where M is a compact Kähler manifold, ι is a holomorphic involution of M , $\mathcal{E}$ is a short exact sequence of μ_2 -equivariant holomorphic vector bundles on M*

$$\mathcal{E} : 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

equipped with μ_2 -invariant hermitian metrics $h_{E'}$, h_E and $h_{E''}$ on E' , E and E'' , an element $\widetilde{Td}_\iota(\mathcal{E}; h_{E'}, h_E, h_{E''}) \in \tilde{A}(M^\iota)$ such that the following hold:

(1) *For each $(M, \iota, \mathcal{E}, h_{E'}, h_E, h_{E''})$, the following identity holds:*

$$-dd^c \widetilde{Td}_\iota(\mathcal{E}; h_{E'}, h_E, h_{E''}) = Td_\iota(E', h_{E'}) Td_\iota(E'', h_{E''}) - Td_\iota(E, h_E).$$

(2) *For each μ_2 -equivariant holomorphic map $f : M' \rightarrow M$, the following identity holds:*

$$\widetilde{Td}_\iota(f^* \mathcal{E}; f^* h_{E'}, f^* h_E, f^* h_{E''}) = f^* \widetilde{Td}_\iota(\mathcal{E}; h_{E'}, h_E, h_{E''})$$

(3) *If $\mathcal{E}$ splits and if $(E, h_E) = (E' \oplus E'', h_{E'} \oplus h_{E''})$, then $\widetilde{Td}_\iota(\mathcal{E}; h_{E'}, h_E, h_{E''}) = 0$.*

Proof. See [49, Theorem 3.4] and [13, e), f)]. $\square$

Theorem 1.2.5. *There exists a map which attaches to a datum $(M, \iota, \mathcal{E}, h_{E'}, h_E, h_{E''})$, where M is a compact Kähler manifold, ι is a holomorphic involution of M , $\mathcal{E}$ is a short exact sequence of μ_2 -equivariant holomorphic vector bundles on M*

$$\mathcal{E} : 0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

equipped with μ_2 -invariant hermitian metrics $h_{E'}$, h_E and $h_{E''}$ on E' , E and E'' , an element $\tilde{ch}_l(\mathcal{E}; h_{E'}, h_E, h_{E''}) \in \hat{A}(M^\iota)$ such that the following hold:

(1) *For each $(M, \iota, \mathcal{E}, h_{E'}, h_E, h_{E''})$, the following identity holds:*

$$-dd^c \tilde{ch}_l(\mathcal{E}; h_{E'}, h_E, h_{E''}) = ch_l(E', h_{E'}) + ch_l(E'', h_{E''}) - ch_l(E, h_E).$$

(2) *For each μ_2 -equivariant holomorphic map $f : M' \rightarrow M$, the following identity holds:*

$$\tilde{ch}_l(f^* \mathcal{E}; f^* h_{E'}, f^* h_E, f^* h_{E''}) = f^* \tilde{ch}_l(\mathcal{E}; h_{E'}, h_E, h_{E''})$$

(3) *If $\mathcal{E}$ splits and if $(E, h_E) = (E' \oplus E'', h_{E'} \oplus h_{E''})$, then $\tilde{ch}_l(\mathcal{E}; h_{E'}, h_E, h_{E''}) = 0$.*

Proof. See [49, Theorem 3.4] and [13, e), f)]. $\square$

1.2.5 The embedding formula for equivariant Quillen metrics

Let W be a compact Kähler manifold and let $\iota : W \rightarrow W$ be an involution. Consider the μ_2 -action on W induced from ι . Let h_W be a μ_2 -invariant Kähler metric on W and denote the associated Kähler form by ω_W . The fixed locus of $\iota : W \rightarrow W$ is denoted by W^ι . Let $i : H \hookrightarrow W$ be the inclusion.

Let H be a smooth hypersurface of W which intersects W^ι transversally. Assume that H is preserved by the μ_2 -action on W . The fixed locus of $\iota|_H : H \rightarrow H$ is denoted by H^ι . Set $h_H := h_W|_H$. Then h_H is a μ_2 -invariant Kähler metric on H .

Let F be a μ_2 -equivariant holomorphic vector bundle on W and let h_F be a μ_2 -invariant hermitian metric on F .

Let $[H]$ be the μ_2 -equivariant holomorphic line bundle on W associated with the divisor H and let $s_H \in H^0(W, [H])$ be a holomorphic section with $\text{div}(s_H) = H$. Let $h_{[H]}$ be a μ_2 -invariant hermitian metric on $[H]$ and let $h_{[H]}^{-1}$ be the μ_2 -invariant hermitian metric on $[H]^{-1}$ induced from $h_{[H]}$. The μ_2 -action on $[H]$ is given by

$$g(s_H(x)) = s_H(g(x)) \quad (g \in \mu_2, x \in W).$$

We remark that the μ_2 -action on $[H]$ defined on $W \setminus H$ as above extends to a μ_2 -action on $[H]$ defined on W . Since H intersects W^ι transversally, s_H is a μ_2 -invariant holomorphic section of $[H]$. The holomorphic section s_H induces the following short exact sequence of μ_2 -equivariant coherent sheaves on W .

$$0 \rightarrow \mathcal{O}_W([H]^{-1} \otimes F) \xrightarrow{\otimes s_H} \mathcal{O}_W(F) \rightarrow i_* \mathcal{O}_H(F|_H) \rightarrow 0. \quad (1.2.4)$$

The normal bundle of H in W is denoted by $N_{H/W}$. Let $ds_H \in H^0(H, N_{H/W}^* \otimes [H]|_H)$ be the nowhere vanishing holomorphic section with the following condition: for an open covering $\{U_\alpha\}_\alpha$ of a neighborhood of H , $ds_H|_{U_\alpha} \in H^0(H \cap U_\alpha, N_{H/W}^* \otimes [H]|_H)$ is defined by

$$ds_H|_{U_\alpha} = df_\alpha \otimes \sigma_\alpha,$$

where $\sigma_\alpha \in H^0(U_\alpha, [H])$ is a nowhere-vanishing holomorphic section of $[H]$ and $f_\alpha \in \mathcal{O}(U_\alpha)$ is such that $s_H|_{U_\alpha} = f_\alpha \sigma_\alpha$. Therefore, $ds_H \in H^0(H, N_{H/W}^* \otimes [H]|_H)$ induces an isomorphism $N_{H/W} \cong [H]|_H$ of μ_2 -equivariant holomorphic line bundles on H . Moreover, we have the isomorphism

$$N_{H/W} \otimes F \cong ([H] \otimes F)|_H$$

of μ_2 -equivariant holomorphic vector bundles on H . Let a_N be a μ_2 -invariant hermitian metric on $N_{H/W}$.

Definition 1.2.6. *The metrics $h_{[H]}^{-1} \otimes h_F$ and h_F verify assumption (A) with respect to a_N and $h_F|_H$ if $ds_H \in H^0(H, N_{H/W}^* \otimes [H]|_H)$ induces the following isometry of holomorphic hermitian vector bundles on H*

$$(N_{H/W} \otimes F, a_N \otimes h_F) \cong ([H] \otimes F)|_H, (h_{[H]} \otimes h_F)|_H.$$

For the definition of the assumption (A) in the general case, we refer to [10, Definition 1.5].

There exists a short exact sequence of μ_2 -equivariant holomorphic vector bundles on H

$$\mathcal{E} : 0 \rightarrow TH \rightarrow TW|_H \rightarrow N_{H/W} \rightarrow 0.$$

Let $\widetilde{Td}_l(\mathcal{E}, (h_H, h_W|_H, a_N))$ be the Bott-Chern class (Theorem 1.2.4) such that

$$-dd^c \widetilde{Td}_l(\mathcal{E}, (h_H, h_W|_H, a_N)) = Td_l(TW, h_W) - Td_l(TH, h_H)Td_l(N_{H/W}, a_N).$$

Let $\lambda_{\mu_2}(F|_H)$ (resp. $\lambda_{\mu_2}(F)$ and $\lambda_{\mu_2}([H]^{-1} \otimes F)$) be the equivariant determinant of the cohomology of $F|_H$ (resp. F and $[H]^{-1} \otimes F$). By (1.2.4),

$$\lambda_\pm(F|_H) \otimes \lambda_\pm(F)^{-1} \otimes \lambda_\pm([H]^{-1} \otimes F)$$

is canonically isomorphic to $\mathbb{C}$. We define

$$\lambda_\pm = \lambda_\pm(F|_H) \otimes \lambda_\pm(F)^{-1} \otimes \lambda_\pm([H]^{-1} \otimes F), \quad \lambda_{\mu_2} = \lambda_+ \oplus \lambda_-.$$

The canonical element of λ_{μ_2} is denoted by $\sigma = (\sigma_+, \sigma_-)$.

The Riemann zeta function is denoted by $\zeta(s)$. We define

$$\zeta(\pi, s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.$$

Define the formal power series $R(x)$, $R(\pi, x)$ by

$$R(x) = \sum_{n \geq 1, n \text{ odd}} \left(\sum_{j=1}^n \frac{1}{j} \zeta(-n) + 2\zeta'(-n) \right) \frac{x^n}{n!},$$

and

$$R(\pi, x) = \sum_{n \geq 1, n \text{ odd}} \left(\sum_{j=1}^n \frac{1}{j} \zeta(\pi, -n) + 2\zeta'(\pi, -n) \right) \frac{x^n}{n!}.$$

For a complex vector bundle E on a manifold M , let $R(E), R(\pi, E) \in H^\bullet(M, \mathbb{R})$ be the additive genera corresponding to the formal power series $R(x)$ and $R(\pi, x)$, respectively.

Recall that H intersects W^ι transversally.

Theorem 1.2.7. *The following identity holds :*

$$\begin{aligned} \log \|\sigma\|_{Q, \lambda_{\mu_2}}^2(\iota) = & - \int_{W^\iota} \frac{Td_\iota(TW, h_W) ch_\iota(F, h_F)}{Td([H], h_{[H]})} \log h_{[H]}(s_H, s_H) \\ & - \int_{H^\iota} \frac{\widetilde{Td}_\iota(\mathcal{E}, (h_H, h_W|_H, a_N)) ch_\iota(F, h_F)}{Td(N_{H/W}, a_N)} \\ & - \int_{W^\iota} Td_\iota(TW) R(TW^\iota) R(\pi, N_{W^\iota/W}) ch_\iota(F, h_F) \\ & + \int_{H^\iota} Td_\iota(TH) R(TH^\iota) R(\pi, N_{H^\iota/H}) ch_\iota(F|_H, h_{F|_H}). \end{aligned}$$

Proof. See [11, Theorem 0.1]. By the assumption that H intersects W^ι transversally, the μ_2 -action on $[H]|_{W^\iota}$ is trivial. Then we used the explicit formula for the Bott-Chern current [16, Remark 3.5, especially (3.23), Theorem 3.15 and Theorem 3.17] to get the first term in the right hand side, and we used the triviality of the μ_2 -action on $[H]|_{W^\iota}$ to get the second term of the right hand side. $\square$

1.2.6 The anomaly formula for the Quillen metrics

Recall that M is a compact Kähler manifold and $\iota : M \rightarrow M$ is a holomorphic involution. Let

$$E_\bullet : 0 \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_m \rightarrow 0$$

be an exact sequence of μ_2 -equivariant holomorphic vector bundles on M . Set

$$\lambda_\pm = \otimes_i \lambda_\pm(E_i)^{(-1)^i} \quad \lambda_{\mu_2} = \lambda_+ \oplus \lambda_-. \quad (1.2.5)$$

Then $\lambda_\pm$ is canonically isomorphic to $\mathbb{C}$ and the canonical element is denoted by $\sigma_\pm$. Set $\sigma_{KM} = (\sigma_+, \sigma_-) \in \lambda_{\mu_2}$.

Let $\tilde{ch}_\iota(\overline{E}_\bullet)$ be the Bott-Chern secondary class (Theorem 1.2.5) such that

$$-dd^c \tilde{ch}_\iota(\overline{E}_\bullet) = \sum_i (-1)^i ch_\iota(E_i, h_{E_i}).$$

Let h_i be a μ_2 -invariant hermitian metric on E_i and let $\|\cdot\|_{Q, \lambda_{\mu_2}(E_i)}(\iota)$ be the equivariant Quillen metric on $\lambda_{\mu_2}(E_i)$ with respect to h_M and h_i . Let $\|\cdot\|_{Q, \lambda_{\mu_2}}(\iota)$ be the equivariant Quillen metric on λ_{μ_2} given by the metric $\otimes_i \|\cdot\|_{Q, \lambda_{\mu_2}(E_i)}(\iota)$ via (1.2.5).

Theorem 1.2.8. *The following identity holds:*

$$\log \|\sigma_{KM}\|_{Q, \lambda_{\mu_2}}^2(\iota) = \int_{M^\iota} Td_\iota(TM, h_M) \tilde{ch}_\iota(\overline{E}_\bullet).$$

Proof. This theorem follows from the immersion formula for equivariant Quillen metrics applied to the immersion $\emptyset \hookrightarrow M^\iota$. See [11, Theorem 0.1] and [49, Theorem 3.7]. $\square$

1.2.7 The singularity of Quillen metrics

Let $\mathcal{M}$ be a projective manifold and let C be a smooth projective curve. Let $\pi : \mathcal{M} \rightarrow C$ be a surjective holomorphic map.

Let $\Sigma_\pi = \{x \in \mathcal{X}; d\pi(x) = 0\}$ be the critical locus of π and let $\Delta_\pi = \pi(\Sigma_\pi)$ be the discriminant locus of π . The fiber of π at $s \in C$ is denoted by M_s .

We set $C^\circ = C \setminus \Delta_\pi$, $\mathcal{M}^\circ = \pi^{-1}(C^\circ)$, and $\pi^\circ = \pi|_{\mathcal{M}^\circ}$. Then $\pi^\circ : \mathcal{M}^\circ \rightarrow C^\circ$ is a family of projective manifolds.

Let $h_{\mathcal{M}}$ be a Kähler metric on $\mathcal{M}$. The relative tangent bundle $T\mathcal{M}/C$ is the subbundle of $T\mathcal{M}|_{\mathcal{M} \setminus \Sigma_\pi}$ defined by

$$T\mathcal{M}/C = \text{Ker}(\pi_*|_{\mathcal{M} \setminus \Sigma_\pi} : T\mathcal{M}|_{\mathcal{M} \setminus \Sigma_\pi} \rightarrow \pi^*TC|_{\mathcal{M} \setminus \Sigma_\pi}).$$

We set $h_{\mathcal{M}/C} = h_{\mathcal{M}}|_{T\mathcal{M}/C}$.

Fix $0 \in \Delta_\pi$ and let (Δ, s) be a local coordinate on C centered at 0 such that $\Delta \cap \Delta_\pi = \{0\}$ and that $s(\Delta)$ is the unit disk in $\mathbb{C}$. We set $M = \pi^{-1}(\Delta)$ and $\Delta^* = \Delta \setminus \{0\}$.

Let (F, h_F) be a holomorphic hermitian vector bundle on $\mathcal{M}$. Let $\lambda(F)$ be the determinant of the cohomology of F . Then $\lambda(F)$ is equipped with the Quillen metric $\|\cdot\|_{Q, \lambda(F)}$ with respect to $h_{\mathcal{M}/C}$ and h_F . Let σ be a nowhere-vanishing holomorphic section of $\lambda(F)|_\Delta$.

Let $\Pi : \mathbb{P}(T\mathcal{M})^\vee \rightarrow \mathcal{M}$ be the projective space bundle on $\mathcal{M}$. The Gauss map $\mu : \mathcal{M} \setminus \Sigma_\pi \rightarrow \mathbb{P}(T\mathcal{M})^\vee$ is defined by $\mu(x) = [T_x M_{\pi(x)}]$. Then μ extends to a meromorphic map $\mu : \mathcal{M} \rightarrow \mathbb{P}(T\mathcal{M})^\vee$. There exists a resolution $q : (\widetilde{\mathcal{M}}, E) \rightarrow (\mathcal{M}, \Sigma_\pi)$ of the indeterminacy of μ such that $\tilde{\mu} = \mu \circ q$ extends to a holomorphic map $\widetilde{\mathcal{M}} \rightarrow \mathbb{P}(T\mathcal{M})^\vee$ and such that E is a normal crossing divisor of $\widetilde{\mathcal{M}}$.

Let $\mathcal{U}$ be the universal hyperplane bundle on $\mathbb{P}(T\mathcal{M})^\vee$. Then we have the following exact sequence of holomorphic vector bundles on $\mathbb{P}(T\mathcal{M})^\vee$:

$$0 \rightarrow \mathcal{U} \rightarrow \Pi^*T\mathcal{M} \rightarrow \mathcal{H} \rightarrow 0,$$

where $\mathcal{H} = \Pi^*T\mathcal{M}/\mathcal{U}$.

We define a topological invariant of the germ $(\pi : (\mathcal{M}, M_0) \rightarrow (C, 0), F)$ by

$$\alpha(X_0, F) = \int_{E \cap q^{-1}(M_0)} \tilde{\mu} \left\{ Td(\mathcal{U}) \frac{Td(\mathcal{H}) - 1}{c_1(\mathcal{H})} \right\} q^*ch(F).$$

For functions f, g on $\Delta \setminus \{0\}$, we write $f \equiv g$ if $f - g \in C^0(\Delta)$.

Theorem 1.2.9. *The following identity of functions on $\Delta \setminus \{0\}$ holds:*

$$\log \|\sigma\|_{Q, \lambda(F)}^2 \equiv \alpha(X_0, F) \log |s|^2.$$

Proof. See [83, Theorem 1.1]. □

Chapter 2

Hyperkähler manifolds with involution

2.1 Hyperkähler manifolds

2.1.1 Lattices

Definition 2.1.1. A lattice L is a finitely generated free abelian group equipped with an integral non-degenerate symmetric bilinear form $(\cdot, \cdot) : L \times L \rightarrow \mathbb{Z}$.

The rank of a lattice L is denoted by $\text{rk}(L)$. For $K = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$, we set $L_K = L \otimes_{\mathbb{Z}} K$. The signature of L is denoted by $\text{sign}(L)$. The dual lattice is denoted by $L^\vee = \text{Hom}(L, \mathbb{Z})$. It can be identified with the subgroup of $L_{\mathbb{Q}}$ defined by $\{x \in L_{\mathbb{Q}}; (x, y) \in \mathbb{Z} \text{ for all } y \in L\}$. Since the bilinear form $(\cdot, \cdot)$ is non-degenerate, there is an injection $L \rightarrow L^\vee$. The quotient $A_L = L^\vee/L$ is a finite abelian group called the discriminant group of L .

A lattice L is called unimodular if $A_L = \{0\}$. For each $l \in L$, set $l^2 = (l, l) \in \mathbb{Z}$. A lattice L is called even if l^2 is an even number for any $l \in L$. We denote by U the even unimodular lattice of rank 2 of signature $(1, 1)$ with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and denote by E_8 the *negative* definite even unimodular lattice of rank 8 associated with the Dynkin diagram E_8 . We set

$$L_{K3} = E_8^{\oplus 2} \oplus U^{\oplus 3},$$

and

$$L_2 = L_{K3} \oplus \mathbb{Z}e,$$

where $e^2 = -2$ and $(e, L_{K3}) = 0$. Let $k \in \mathbb{Z}$ and let $(\cdot, \cdot)$ be the bilinear form of a lattice L . We denote by $L(k)$ the lattice which is the free abelian group L equipped with the bilinear form $k(\cdot, \cdot)$.

The isometry group of L is denoted by $O(L)$. A sublattice $S \subset L$ is primitive if the quotient L/S is a free abelian group. A lattice L is hyperbolic if $\text{sign}(L) = (1, \text{rk}(L) - 1)$.

A lattice L is 2-elementary if there exists a non-negative integer $l(L) \in \mathbb{Z}_{\geq 0}$ such that A_L is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{l(L)}$.

For a lattice L and $l \in L$, the reflection $s_l : L_{\mathbb{R}} \rightarrow L_{\mathbb{R}}$ is defined by

$$s_l(x) = x - \frac{2(x, l)}{(l, l)}l.$$

By [68, Theorem 43.3], any isometry $g \in O(L_{\mathbb{R}})$ can be expressed as the product of reflections

$$g = s_{v_1} \dots s_{v_m},$$

where $v_1, \dots, v_m$ are elements of $L_{\mathbb{R}}$. We define the real spinor norm $sn_{\mathbb{R}}(g)$ by

$$sn_{\mathbb{R}}(g) = \begin{cases} +1 & \text{if } -(v_1)^2 \dots -(v_m)^2 > 0, \\ -1 & \text{if } -(v_1)^2 \dots -(v_m)^2 < 0. \end{cases}$$

This is independent of the expression $g = s_{v_1} \dots s_{v_m}$ (cf. [68, Proposition 54.6]). We define a subgroup $O^+(L)$ of $O(L)$ by

$$O^+(L) = \{g \in O(L); sn_{\mathbb{R}}(g) = +1\}.$$

Let N be a lattice of signature $(2, n)$. We set

$$\Omega_N = \{[\eta] \in \mathbb{P}(N_{\mathbb{C}}); (\eta, \eta) = 0, (\eta, \bar{\eta}) > 0\}.$$

Then Ω_N is a complex manifold consisting of two connected components Ω_1 and Ω_2 , both of which are isomorphic to a bounded symmetric domain of type IV of dimension n . For $v \in N$ with $(v, v) \neq 0$, we have

$$s_v(\Omega_1) = \begin{cases} \Omega_1 & \text{if } (v, v) < 0 \\ \Omega_2 & \text{if } (v, v) > 0 \end{cases}$$

and we have

$$O^+(N) = \{g \in O(N); g \text{ preserves } \Omega_1\}. \quad (2.1.1)$$

Let Γ be a finite index subgroup of $O^+(N)$. For each $i = 1, 2$, Γ acts on Ω_i projectively. We define an orthogonal modular variety $\mathcal{M}_i$ by

$$\mathcal{M}_i = \Omega_i / \Gamma.$$

By [2, Theorems 10.4 and 10.11], $\mathcal{M}_i$ has a compactification $\mathcal{M}_i^*$, called the Baily-Borel compactification, such that $\mathcal{M}_i^*$ is an irreducible normal projective variety of dimension n and such that the boundary $\mathcal{M}_i^* \setminus \mathcal{M}_i$ is of codimension ≥ 2 if $n \geq 3$.

Let Λ be a lattice of signature $(3, n)$. By [57, Lemma 4.1], an isometry $g \in O(\Lambda)$ is of real spinor norm $+1$ if and only if it acts on $H^2(\tilde{\mathcal{C}}_{\Lambda}, \mathbb{Z}) \cong \mathbb{Z}$ by $+1$, where $\tilde{\mathcal{C}}_{\Lambda} = \{x \in \Lambda_{\mathbb{R}}; x^2 > 0\}$. A generator of $H^2(\tilde{\mathcal{C}}_{\Lambda}, \mathbb{Z}) \cong \mathbb{Z}$ is called an orientation class of $\tilde{\mathcal{C}}_{\Lambda}$. Let $h \in \Lambda$ be an element with $(h, h) > 0$. Since the signature of the lattice $h^{\perp} \subset \Lambda$ is $(2, n)$, $\Omega_{h^{\perp}} = \Omega_{\Lambda} \cap h^{\perp}$ consists of two connected components. If an isometry $g \in O(\Lambda)$ is of real spinor norm $+1$ and $g(h) = h$, it preserves the connected components of $\Omega_{h^{\perp}}$.

2.1.2 Definition and fundamental properties

Definition 2.1.2. *A simply-connected compact Kähler manifold X is an irreducible holomorphic symplectic manifold if there exists an everywhere non-degenerate holomorphic 2-form η such that $H^0(X, \Omega_X^2) = \mathbb{C}\eta$.*

Definition 2.1.3. *A compact Riemannian manifold (M, g) of dimension $4n$ is a hyperkähler manifold if its holonomy group is equal to $Sp(n)$.*

If (M, g) is hyperkähler, then there exist three complex structures I, J , and K on M with $IJ = -JI = K$ such that (M, g, I) , (M, g, J) , (M, g, K) are all irreducible holomorphic symplectic manifolds [36, Proposition 23.3].

On the other hand, let X be an irreducible holomorphic symplectic manifold, and let $\alpha \in H^2(X, \mathbb{R})$ be a Kähler class of X . By Yau [81], there exists a unique Ricci-flat Kähler form $\omega_{X,0}$ such that $[\omega_{X,0}] = \alpha$ in the cohomology $H^2(X, \mathbb{R})$. If M is the real manifold underlying X and g is the Ricci-flat Riemannian metric corresponding to $\omega_{X,0}$, then (M, g) is a hyperkähler manifold [36, Proposition 5.11].

Definition 2.1.4. *A 2-dimensional irreducible holomorphic symplectic manifold is called a K3 surface.*

Theorem 2.1.5. *The Hilbert scheme $Y^{[n]}$ of length n zero-dimensional subschemes of a K3 surface Y is an irreducible holomorphic symplectic manifold of dimension $2n$.*

Proof. See [5, Théorème 3]. □

Definition 2.1.6. *An irreducible holomorphic symplectic manifold X is of K3^[n]-type if X is deformation equivalent to the Hilbert scheme of n -points of a K3 surface.*

Note that any small deformation of an irreducible holomorphic symplectic manifold is also an irreducible holomorphic symplectic manifold (cf. [5, Proposition 9]).

2.1.3 The Beauville-Bogomolov-Fujiki form

Let X be an irreducible holomorphic symplectic manifold of dimension $2n$ and let η be a holomorphic symplectic 2-form on X with $\int_X (\eta\bar{\eta})^n = 1$. We define a quadratic form $q_{X,\eta} : H^2(X, \mathbb{C}) \rightarrow \mathbb{C}$ by

$$q_{X,\eta}(\alpha) = \frac{n}{2} \int_X (\eta\bar{\eta})^{n-1} \alpha^2 + (1-n) \left(\int_X \eta^n \bar{\eta}^{n-1} \alpha \right) \left(\int_X \eta^{n-1} \bar{\eta}^n \alpha \right).$$

The k -th Betti number of X is denoted by $b_k(X)$. The quadratic form $q_{X,\eta}$ induces a quadratic form $q_{X,\eta} : H^2(X, \mathbb{R}) \rightarrow \mathbb{R}$ on the real cohomology group $H^2(X, \mathbb{R})$. Let ω_X be a Kähler form on X .

Lemma 2.1.7. *The quadratic form $q_{X,\eta} : H^2(X, \mathbb{R}) \rightarrow \mathbb{R}$ is non-degenerate and has signature $(3, b_2(X) - 3)$. More precisely, $q_{X,\eta}$ is positive definite on $\mathbb{R}[\omega_X] \oplus \mathbb{R}[\operatorname{Re} \eta] \oplus \mathbb{R}[\operatorname{Im} \eta]$ and negative definite on the primitive part of $H^{1,1}(X)$ with respect to the Kähler class $[\omega_X]$.*

Proof. See [36, Corollary 23.11]. □

Theorem 2.1.8. *There exists a positive number $a > 0$ such that*

$$q_X := a q_{X,\eta} : H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}$$

is a primitive integral quadratic form on $H^2(X, \mathbb{Z})$. Moreover, there exists a positive rational number $c_X \in \mathbb{Q}_{>0}$ such that

$$q_X(\alpha)^n = c_X \int_X \alpha^{2n} \quad (\alpha \in H^2(X, \mathbb{Z})).$$

Proof. See [36, Proposition 23.14]. □

In this paper, the quadric form q_X is called the Beauville-Bogomolov-Fujiki form. Let $(\cdot, \cdot) : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}$ be the integral bilinear form corresponding to q_X . In what follows, the cohomology group $H^2(X, \mathbb{Z})$ of an irreducible holomorphic symplectic manifold X is endowed with the lattice structure $(\cdot, \cdot)$ induced from the Beauville-Bogomolov-Fujiki form q_X .

Example 2.1.9. If X is a K3 surface, then the bilinear form $(\cdot, \cdot)$ on $H^2(X, \mathbb{Z})$ induced from q_X is given by the cup product pairing and isometric to L_{K3} .

Example 2.1.10. If the n -th symmetric product of a K3 surface Y is denoted by $Y^{(n)}$, the Hilbert scheme $Y^{[n]}$ is obtained as a crepant resolution of $Y^{(n)}$. Such a crepant resolution is unique by [29, Theorem (2.2)]. Let E be its exceptional divisor and the Poincaré dual of E is denoted by $[E] \in H^2(Y^{[n]}, \mathbb{Z})$. Let $\tau : Y^{[n]} \rightarrow Y^{(n)}$ be the crepant resolution, $\pi : Y^n \rightarrow Y^{(n)}$ be the projection, and $p_k : Y^n \rightarrow Y$ be the projection to the k -th factor. We define an injection $i : H^2(Y, \mathbb{Z}) \rightarrow H^2(Y^{[n]}, \mathbb{Z})$ by $i(\alpha) = \tau^*(\beta)$, where $\beta \in H^2(Y^{(n)}, \mathbb{Z})$ is determined by

$$\pi^*(\beta) = \sum_k p_k^*(\alpha).$$

By [5, Proposition 6], $i : H^2(Y, \mathbb{Z}) \rightarrow H^2(Y^{[n]}, \mathbb{Z})$ preserves the Beauville-Bogomolov-Fujiki forms and

$$H^2(Y^{[n]}, \mathbb{Z}) \cong i(H^2(Y, \mathbb{Z})) \oplus \mathbb{Z}\varepsilon, \tag{2.1.2}$$

where $\varepsilon \in H^2(Y^{[n]}, \mathbb{Z})$ is the cohomology class such that $2\varepsilon = [E]$.

2.1.4 Periods of irreducible holomorphic symplectic manifolds

Fix a lattice Λ . We consider an irreducible holomorphic symplectic manifold X such that $H^2(X, \mathbb{Z}) \cong \Lambda$. Let Ω_Λ be the period domain defined by

$$\Omega_\Lambda = \{[\eta] \in \mathbb{P}(\Lambda_{\mathbb{C}}); (\eta, \eta) = 0, (\eta, \bar{\eta}) > 0\}.$$

Let $\alpha : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ be a (Λ) -marking. Namely, α is an isometry of lattices. By the Hodge-Riemann bilinear relation, we have $\alpha(H^0(X, \Omega_X^2)) \in \Omega_\Lambda$.

Definition 2.1.11. *The point $\alpha(H^0(X, \Omega_X^2))$ of Ω_Λ is called the period of (X, α) .*

Consider the Kuranishi family $\pi : \mathcal{X} \rightarrow \text{Def}(X)$ of X . The fiber of π at $t \in \text{Def}(X)$ is denoted by X_t . Since X admits a symplectic form on TX , we have

$$H^0(X, TX) \cong H^0(X, \Omega_X^1) = 0.$$

Hence $\pi : \mathcal{X} \rightarrow \text{Def}(X)$ is a universal deformation for any of its fiber. Moreover, it follows from the Bogomolov-Tian-Todorov theorem (cf. [36, Theorem 14.10]) that the Kuranishi space $\text{Def}(X)$ is smooth of dimension $\dim H^1(X, TX)$.

If $\text{Def}(X)$ is sufficiently small, the marking $\alpha : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ extends to a morphism $\alpha : R^2\pi_*\mathbb{Z} \rightarrow \Lambda_{\text{Def}(X)}$ of sheaves on $\text{Def}(X)$, where $\Lambda_{\text{Def}(X)}$ is the constant sheaf of stalk Λ .

We define the period map $P : \text{Def}(X) \rightarrow \Omega_\Lambda$ by

$$P(t) = \alpha(H^0(X_t, \Omega_{X_t}^2)).$$

Since $\pi_*\Omega_{\mathcal{X}/\text{Def}(X)}^2$ is a holomorphic subbundle of $R^2\pi_*\mathbb{C} \otimes \mathcal{O}_{\text{Def}(X)}$, $P : \text{Def}(X) \rightarrow \Omega_\Lambda$ is holomorphic (cf. [36, p.177]).

Theorem 2.1.12. *The period map $P : \text{Def}(X) \rightarrow \Omega_\Lambda$ for the Kuranishi family $\pi : \mathcal{X} \rightarrow \text{Def}(X)$ is an open embedding.*

Proof. See [36, Proposition 22.11]. □

Theorem 2.1.13. *There exists a moduli space $\mathfrak{M}_\Lambda$ of Λ -marked irreducible holomorphic symplectic manifolds.*

Proof. See [39, 1.18]. □

We define the period map $P : \mathfrak{M}_\Lambda \rightarrow \Omega_\Lambda$ by

$$P(X, \alpha) = \alpha(H^0(X, \Omega_X^2)).$$

Fix a connected component $\mathfrak{M}_\Lambda^\circ$ of $\mathfrak{M}_\Lambda$. The restriction of $P : \mathfrak{M}_\Lambda \rightarrow \Omega_\Lambda$ to $\mathfrak{M}_\Lambda^\circ$ is denoted by $P_\circ : \mathfrak{M}_\Lambda^\circ \rightarrow \Omega_\Lambda$. Then the surjectivity of the period map is known.

Theorem 2.1.14. *The map $P_\circ : \mathfrak{M}_\Lambda^\circ \rightarrow \Omega_\Lambda$ is surjective.*

Proof. See [36, Proposition 25.12]. □

The fiber of $P_\circ$ is characterized as follows.

Theorem 2.1.15. *For any $p \in \Omega_\Lambda$, the fiber $P_\circ^{-1}(p)$ consists of pairwise inseparable points. Here a pair of points $x, y \in P_\circ^{-1}(p)$ is called inseparable if $U \cap V \neq \emptyset$ for any open neighborhoods U and V of x and y , respectively.*

Proof. See [79, Theorem 1.16] or [3, Theorem 1.1]. □

Definition 2.1.16. Let X, X_1, X_2 be irreducible holomorphic symplectic manifolds.

- (1) A parallel-transport operator $f : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$ is an isomorphism of lattices such that there exist a family $p : \mathcal{X} \rightarrow B$ of irreducible holomorphic symplectic manifolds over a possibly reducible analytic base B , two points $b_1, b_2 \in B$, and a continuous path $\gamma : [0, 1] \rightarrow B$ with $\gamma(0) = b_1, \gamma(1) = b_2$ such that $p^{-1}(b_i) \cong X_i$ ($i = 1, 2$) and such that the parallel-transport in the local system $R^2p_*\mathbb{Z}$ induces $f : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$.
- (2) A parallel-transport operator $g : H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$ is called a monodromy operator. The subgroup $\text{Mon}^2(X)$ of $O(H^2(X, \mathbb{Z}))$ consisting of all monodromy operators of X is called the monodromy group.

Recall that the second cohomology group $H^2(X, \mathbb{Z})$ of a compact Kähler manifold X is endowed with the structure of integral Hodge structure of weight 2. Namely, there exists a $\mathbb{C}$ -vector subspaces $\{H^{p,q}\}_{p+q=2}$ with $H^{p,q} \cong H^q(X, \Omega_X^p)$ such that

$$H^2(X, \mathbb{C}) = \bigoplus_{p+q=2} H^{p,q}, \quad \overline{H^{p,q}} = H^{q,p}.$$

Theorem 2.1.17. Let X and Y be irreducible holomorphic symplectic manifolds. Then the following hold:

- (1) X and Y are bimeromorphic if and only if there exists a parallel-transport operator $f : H^2(X, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z})$ inducing an isomorphism of integral Hodge structures.
- (2) Let $f : H^2(X, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z})$ be a parallel-transport operator inducing an isomorphism of integral Hodge structures. Then there exists an isomorphism $\phi : Y \rightarrow X$ with $\phi^* = f$ if and only if there exists a Kähler class α on X such that $f(\alpha)$ is a Kähler class on Y .

Proof. See [57, Theorem 1.3]. □

By Theorem 2.1.17, the integral second cohomology lattices of bimeromorphic irreducible holomorphic symplectic manifolds are isometric to each other.

Let X be an irreducible holomorphic symplectic manifold. The Kähler cone of X is denoted by $\mathcal{K}_X$. Let $\mathcal{C}_X$ be the connected component of the positive cone of $H^{1,1}(X, \mathbb{R})$

$$\tilde{\mathcal{C}}_X = \{x \in H^{1,1}(X, \mathbb{R}); x^2 > 0\}$$

which contains $\mathcal{K}_X$. For $w \in H^{1,1}(X, \mathbb{R})$, define the hyperplane $w^\perp$ of $H^{1,1}(X, \mathbb{R})$ by

$$w^\perp = \{x \in H^{1,1}(X, \mathbb{R}); (x, w) = 0\}.$$

Definition 2.1.18. A Kähler-type chamber of X is defined to be the subset of $\mathcal{C}_X$ of the form $g(f^*\mathcal{K}_{\tilde{X}})$, where $g \in \text{Mon}^2(X)$ is an isomorphism of integral Hodge structures, $\tilde{X}$ is another irreducible holomorphic symplectic manifold and $f : X \dashrightarrow \tilde{X}$ is a bimeromorphic map.

Definition 2.1.19. Let $z \in H^{1,1}(X, \mathbb{Q})$ be a rational cohomology class with $z^2 < 0$. Then z is called a *monodromy birationally minimal* if there exist an isomorphism of integral Hodge structures $g \in \text{Mon}^2(X)$, an irreducible holomorphic symplectic manifold $\tilde{X}$ and a bimeromorphic map $f : X \dashrightarrow \tilde{X}$ such that the hyperplane $g(z)^\perp$ in $H^{1,1}(X, \mathbb{R})$ contains a face of $f^*\mathcal{K}_{\tilde{X}}$. Namely, the intersection $g(z)^\perp \cap \partial f^*\mathcal{K}_{\tilde{X}}$ contains an open subset of $g(z)^\perp$.

Theorem 2.1.20. The connected components of $\mathcal{C}_X \setminus \bigcup_z z^\perp$ are the Kähler-type chambers, where the union is taken over all monodromy birationally minimal classes on X .

Proof. See [1, Theorem 6.2]. $\square$

Let $\Delta(X) \subset H^{1,1}(X, \mathbb{Z})$ be the set of all primitive monodromy birationally minimal classes. An element $D \in \Delta(X)$ is called a *wall divisor*.

Theorem 2.1.21. Assume that X is a manifold of $K3^{[2]}$ -type. Then $D \in H^{1,1}(X, \mathbb{Z})$ is a wall divisor if and only if one of the following holds:

- (1) $(D, D) = -2$,
- (2) $(D, D) = -10$ and $(D, H^2(X, \mathbb{Z})) = 2\mathbb{Z}$.

Proof. See [38, Theorem 23], [58, Theorem 1.11] and [59, Proposition 2.12]. $\square$

2.2 Involutions of $K3^{[2]}$ -type manifolds

2.2.1 Manifolds of $K3^{[2]}$ -type and antisymplectic involution

Let X be an irreducible holomorphic symplectic manifold and η a holomorphic symplectic 2-form on X . For any holomorphic involution $\iota : X \rightarrow X$, we have $\iota^*\eta = \eta$ or $-\eta$. A holomorphic involution $\iota : X \rightarrow X$ which satisfies $\iota^*\eta = -\eta$ is called *antisymplectic*.

Example 2.2.1. Let Y be a K3 surface and let $\sigma : Y \rightarrow Y$ be an antisymplectic involution of Y . The involution σ induces a holomorphic involution $\sigma^{[2]} : Y^{[2]} \rightarrow Y^{[2]}$, and $\sigma^{[2]}$ is an antisymplectic involution. The involution $\sigma^{[2]}$ on $Y^{[2]}$ is called the *natural involution*.

Example 2.2.2. We recall the example of antisymplectic involution in [67, proof of Corollary 2.11] and [45, Example 9.12]. Let $C \subset \mathbb{P}^2$ be a smooth sextic curve. The double covering of $\mathbb{P}^2$ branched over C is denoted by $\pi : Y \rightarrow \mathbb{P}^2$, and the covering involution is denoted by $\sigma : Y \rightarrow Y$. Then Y is a K3 surface and σ is antisymplectic. By Example 2.2.1, (Y, σ) induces a manifold of $K3^{[2]}$ -type $(Y^{[2]}, \sigma^{[2]})$ with antisymplectic involution. Its fixed locus is isomorphic to $C^{(2)} \sqcup (Y/\sigma) = C^{(2)} \sqcup \mathbb{P}^2$. Let $f : Y^{[2]} \dashrightarrow \text{elm}_{Y/\sigma}(Y^{[2]})$ be the Mukai flop of $Y^{[2]}$ along $Y/\sigma = \mathbb{P}^2$. (See [36, Example 21.7].) Set $\text{elm}_{Y/\sigma}(\sigma^{[2]}) = f \circ \sigma^{[2]} \circ f^{-1}$. Then $\text{elm}_{Y/\sigma}(Y^{[2]})$ is a manifold of $K3^{[2]}$ -type. By [67, proof of Corollary 2.11], $\text{elm}_{Y/\sigma}(\sigma^{[2]}) : \text{elm}_{Y/\sigma}(Y^{[2]}) \rightarrow \text{elm}_{Y/\sigma}(Y^{[2]})$ is biregular and is an antisymplectic involution.

The invariant subspace of $H^2(X, \mathbb{Z})$ is defined by

$$H^2(X, \mathbb{Z})^\iota = \{\alpha \in H^2(X, \mathbb{Z}); \iota^* \alpha = \alpha\}.$$

Proposition 2.2.3. *If ι is antisymplectic, then the following hold:*

- (1) $(\eta, H^2(X, \mathbb{Z})^\iota) = 0$.
- (2) $H^2(X, \mathbb{Z})^\iota \subset H^{1,1}(X, \mathbb{Z})$.
- (3) $H^2(X, \mathbb{Z})^\iota$ is hyperbolic. Namely, $\text{sign}(q_X|_{H^2(X, \mathbb{Z})^\iota}) = (1, \text{rk } H^2(X, \mathbb{Z})^\iota - 1)$.
- (4) X is projective.

Proof. See [45, Proposition 4.4]. □

In what follows, *involutions on a $K3^{[2]}$ -type manifold always imply antisymplectic ones.*

Definition 2.2.4. *Let $\mathcal{X}$ and S be complex manifolds, let $f : \mathcal{X} \rightarrow S$ be a surjective proper holomorphic submersion, and let $\iota : \mathcal{X} \rightarrow \mathcal{X}$ be a holomorphic involution. Then $f : (\mathcal{X}, \iota) \rightarrow S$ is called a family of $K3^{[2]}$ -type manifolds with involution if it satisfies the following three conditions:*

- (1) *For each $s \in S$, $X_s = f^{-1}(s)$ is a manifold of $K3^{[2]}$ -type.*
- (2) *$f \circ \iota = f$.*
- (3) *$\iota : \mathcal{X} \rightarrow \mathcal{X}$ induces an involution $\iota_s : X_s \rightarrow X_s$ for all $s \in S$.*

Let (X, ι) be a manifold of $K3^{[2]}$ -type with involution, and let $\pi : (\mathcal{X}, X) \rightarrow (\text{Def}(X), 0)$ be the Kuranishi family of X with $\pi^{-1}(0) = X$. Since $\pi : \mathcal{X} \rightarrow \text{Def}(X)$ is a universal family ([36, §22.1]), there exist holomorphic involutions $I : \mathcal{X} \rightarrow \mathcal{X}$ and $J : (\text{Def}(X), 0) \rightarrow (\text{Def}(X), 0)$ such that $I|_X = \iota$ and the following diagram commutes

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{I} & \mathcal{X} \\ \pi \downarrow & & \downarrow \pi \\ (\text{Def}(X), 0) & \xrightarrow{J} & (\text{Def}(X), 0) \end{array}$$

The fixed locus of J is called the local deformation space $\text{Def}(X, \iota)$ of (X, ι) .

Theorem 2.2.5. *Let (X, ι) be a manifold of $K3^{[2]}$ -type with involution, and let η be a holomorphic symplectic 2-form on X . Set*

$$t = \text{Tr}(\iota^*|_{H^{1,1}(X)}).$$

Then the following hold.

- (1) *The fixed locus X^ι is a smooth Lagrangian submanifold of X . Namely, X^ι is a smooth complex surface such that $\eta|_{X^\iota} = 0$.*
- (2) $\int_{X^\iota} c_1(X^\iota)^2 = t^2 - 1$, $\chi(\mathcal{O}_{X^\iota}) = \frac{t^2 + 7}{8}$, and $\int_{X^\iota} c_2(X^\iota) = \frac{t^2 + 23}{2}$.
- (3) *The local deformation space $\text{Def}(X, \iota)$ is smooth of dimension $\frac{21 - t}{2}$.*
- (4) *t is an odd integer with $-19 \leq t \leq 21$.*

Proof. See [6, Theorems 1 and 2]. □

Example 2.2.6. By the construction of $i : H^2(Y, \mathbb{Z}) \rightarrow H^2(Y^{[2]}, \mathbb{Z})$ and E in Example 2.1.10, the involution $\sigma^{[2]} : Y^{[2]} \rightarrow Y^{[2]}$ satisfies $(\sigma^{[2]})^* \circ i = i \circ \sigma^*$ and $(\sigma^{[2]})^*[E] = [E]$. Therefore, we have

$$H^2(Y^{[2]}, \mathbb{Z})^{\sigma^{[2]}} = i(H^2(Y, \mathbb{Z})^\sigma) \oplus \mathbb{Z}\varepsilon.$$

By [45, Example 5.1], we have $\text{Def}(Y, \sigma) = \text{Def}(Y^{[2]}, \sigma^{[2]})$. Hence any deformation of $(Y^{[2]}, \sigma^{[2]})$ is induced from a deformation of (Y, σ) .

2.2.2 Deformation types of $K3^{[2]}$ -type manifolds with involution

The moduli space, and the period map for $K3^{[2]}$ -type manifolds with antisymplectic involution are studied by Joumaah [45]. Following Joumaah [45], we recall his results.

Let X be a manifold of $K3^{[2]}$ -type and $\alpha : H^2(X, \mathbb{Z}) \rightarrow L_2$ be a marking. The pair (X, α) is called a marked manifold of $K3^{[2]}$ -type. Let $\text{Mon}^2(L_2)$ be the subgroup of the isometry group $O(L_2)$ defined by

$$\text{Mon}^2(L_2) = \alpha \circ \text{Mon}^2(X) \circ \alpha^{-1}.$$

By [57, Theorem 9.1], the group $\text{Mon}^2(L_2)$ is a normal subgroup of $O(L_2)$ and is independent of the choice of (X, α) . By [57, Lemma 9.2], we have

$$\text{Mon}^2(L_2) = O^+(L_2).$$

Similarly, we can define the monodromy group $\text{Mon}^2(L_{K3})$. By [19, Theorem A], we have

$$\text{Mon}^2(L_{K3}) = O^+(L_{K3}).$$

Definition 2.2.7. *Let M be a sublattice of L_2 and $\iota_M \in \text{Mon}^2(L_2)$ an involution. The pair (M, ι_M) is called an admissible sublattice of L_2 if M is hyperbolic and the invariant sublattice $(L_2)^{\iota_M}$ of ι_M is equal to M .*

Let $\mathfrak{M}_{L_2}$ be the moduli space of marked manifolds of $K3^{[2]}$ -type (cf. Theorem 2.1.13). We fix a connected component $\mathfrak{M}_{L_2}^\circ$ of $\mathfrak{M}_{L_2}$.

Definition 2.2.8. Let (M, ι_M) be an admissible sublattice of L_2 and let (X, ι) be a manifold of $K3^{[2]}$ -type with involution. An isometry $\alpha : H^2(X, \mathbb{Z}) \rightarrow L_2$ is called an $(M-)$ admissible marking of (X, ι) if

$$(X, \alpha) \in \mathfrak{M}_{L_2}^\circ \quad \text{and} \quad \alpha \circ \iota^* = \iota_M \circ \alpha.$$

Moreover ι is of type M if there exists an M -admissible marking of (X, ι) .

Let (M, ι_M) be an admissible sublattice of L_2 .

Definition 2.2.9. Define a set $\Delta(M)$ by

$$\Delta(M) = \{\delta \in M; \delta^2 = -2, \text{ or } \delta^2 = -10, (\delta, L_2) = 2\mathbb{Z}\}.$$

We set $\tilde{\mathcal{C}}_M = \{x \in M_{\mathbb{R}}; x^2 > 0\}$. For $\delta \in \Delta(M)$, we define

$$\delta^\perp = \{x \in \tilde{\mathcal{C}}_M : (x, \delta) = 0\}.$$

Definition 2.2.10. A connected component of $\tilde{\mathcal{C}}_M \setminus \bigcup_{\delta \in \Delta(M)} \delta^\perp$ is called a Kähler-type chamber of M . The set of all Kähler-type chambers of M is denoted by $\text{KT}(M)$.

We set

$$\Gamma(M) = \{\sigma \in \text{Mon}^2(L_2); \sigma \circ \iota_M = \iota_M \circ \sigma\},$$

and we define a subgroup Γ_M of $O(M)$ by

$$\Gamma_M = \{\sigma|_M \in O(M); \sigma \in \Gamma(M)\}.$$

Since $O(M)$ preserves $\Delta(M)$, the group Γ_M acts on $\text{KT}(M)$.

Let (X, ι) be a manifold of $K3^{[2]}$ -type with involution of type M , and let $\alpha : H^2(X, \mathbb{Z}) \rightarrow L_2$ be an M -admissible marking of (X, ι) . The ι -invariant Kähler cone of X is defined by

$$\mathcal{K}_X^\iota = \{\omega \in H^{1,1}(X, \mathbb{Z}); \omega \text{ is a Kähler class, } \iota^*\omega = \omega\}.$$

We denote by $\rho(X, \iota, \alpha)$ the Kähler-type chamber containing $\alpha(\mathcal{K}_X^\iota)$. Let $\alpha, \beta : H^2(X, \mathbb{Z}) \rightarrow L_2$ be two M -admissible markings of (X, ι) . Since $(X, \alpha), (X, \beta) \in \mathfrak{M}_{L_2}^\circ$, we have $\alpha \circ \beta^{-1} \in \text{Mon}^2(L_2)$. Moreover, $\alpha \circ \beta^{-1} \in \Gamma(M)$ and hence

$$[\rho(X, \iota, \alpha)] = [\rho(X, \iota, \beta)]$$

in $\text{KT}(M)/\Gamma_M$. Therefore the class $[\rho(X, \iota, \alpha)] \in \text{KT}(M)/\Gamma_M$ is independent of the choice of α , which we denote $\rho(X, \iota)$.

Theorem 2.2.11. The map ρ induces a bijection from the set of deformation types of $K3^{[2]}$ -type manifolds with antisymplectic involution of type M to $\text{KT}(M)/\Gamma_M$.

Proof. See [45, Theorem 9.11]. $\square$

Let $\mathcal{K} \in \text{KT}(M)$ and fix $h \in \mathcal{K} \cap M$. Then $h^\perp = h^\perp \cap L_2$ is a lattice of signature $(2, 20)$ containing $M^\perp$, and $\Omega_{h^\perp}$ consists of two connected components. By [57, (4.1)], the connected component $\mathfrak{M}_{L_2}^\circ$ determines the connected component $\Omega_{h^\perp}^+$ of $\Omega_{h^\perp}$. Namely, for any $(X, \alpha) \in \mathfrak{M}_{L_2}^\circ$ and $p = [\sigma] \in \Omega_{h^\perp}^+$, the orientation of $\tilde{\mathcal{C}}_X = \{x \in H^2(X, \mathbb{R}); q_X(x) > 0\}$ determined by the real and the imaginary parts of the holomorphic 2-form on X and a Kähler class of X is compatible with the orientation of $\tilde{\mathcal{C}}_{L_2} = \{x \in L_{2, \mathbb{R}}; x^2 > 0\}$ determined by the real 3-dimensional vector space $W := \text{Span}_{\mathbb{R}}\{\text{Re } \sigma, \text{Im } \sigma, h\}$ associated to the basis $\{\text{Re } \sigma, \text{Im } \sigma, h\}$ via the isomorphism $\tilde{\mathcal{C}}_X \cong \tilde{\mathcal{C}}_{L_2}$ induced from the marking α . Let $\Omega_{M^\perp}^+$ be the connected component of $\Omega_{M^\perp}$ with $\Omega_{M^\perp}^+ \subset \Omega_{h^\perp}^+$.

We denote by $\tilde{\mathcal{M}}_{M, \mathcal{K}}$ the set of isomorphism classes of $K3^{[2]}$ -type manifolds (X, ι) with involution of type M such that $\rho(X, \iota) = [\mathcal{K}]$ in $\text{KT}(M)/\Gamma_M$.

Definition 2.2.12. Let $(X, \iota) \in \tilde{\mathcal{M}}_{M, \mathcal{K}}$. An isometry $\alpha : H^2(X, \mathbb{Z}) \rightarrow L_2$ is admissible for $(M, \mathcal{K})$ if α is an M -admissible marking with $\alpha(\mathcal{K}_X^\iota) \subset \mathcal{K}$, where $\mathcal{K}_X^\iota$ is the ι -invariant Kähler cone of X i.e. $\mathcal{K}_X^\iota = \{\omega \in \mathcal{K}_X; \iota^* \omega = \omega\}$. Moreover, the involution ι is of type $(M, \mathcal{K})$ if there exists an admissible marking of (X, ι) .

2.2.3 Periods of $K3^{[2]}$ -type manifolds with involution

For each $(X, \iota) \in \tilde{\mathcal{M}}_{M, \mathcal{K}}$, there exists an admissible marking for $(M, \mathcal{K})$. Moreover, if $\alpha, \beta : H^2(X, \mathbb{Z}) \rightarrow L_2$ are two admissible markings for $(M, \mathcal{K})$, then we have $\alpha \circ \beta^{-1} \in \Gamma(\mathcal{K})$, where

$$\Gamma(\mathcal{K}) = \{\sigma \in \text{Mon}^2(L_2); \sigma \circ \iota_M = \iota_M \circ \sigma \text{ and } \sigma(\mathcal{K}) = \mathcal{K}\} = \{\sigma \in \Gamma(M); \sigma(\mathcal{K}) = \mathcal{K}\}.$$

Let

$$\Gamma_{M^\perp, \mathcal{K}} = \{\sigma|_{M^\perp} \in O(M^\perp); \sigma \in \Gamma(\mathcal{K})\}.$$

For $g \in \Gamma_{M^\perp, \mathcal{K}}$, there exists $\sigma \in \Gamma(\mathcal{K})$ such that $\sigma|_{M^\perp} = g$. Since $\Gamma(\mathcal{K}) \subset \text{Mon}^2(L_2) = O^+(L_2)$, we have $sn_{\mathbb{R}}(\sigma) = +1$. On the other hand, since $(\sigma|_M)(\mathcal{K}) = \mathcal{K}$, we have $sn_{\mathbb{R}}(\sigma|_M) = +1$. Therefore, $sn_{\mathbb{R}}(g) = sn_{\mathbb{R}}(\sigma|_{M^\perp}) = +1$ and $g \in O^+(M^\perp)$. Thus, $\Gamma_{M^\perp, \mathcal{K}}$ is contained in $O^+(M^\perp)$. By [45, Proposition 10.2], $\Gamma_{M^\perp, \mathcal{K}}$ is a finite index subgroup of $O^+(M^\perp)$. By (2.1.1), $O^+(M^\perp)$ preserves $\Omega_{M^\perp}^+$. Therefore we obtain an orthogonal modular variety

$$\mathcal{M}_{M, \mathcal{K}} = \Omega_{M^\perp}^+ / \Gamma_{M^\perp, \mathcal{K}}.$$

Definition 2.2.13. We define the period map $P_{M, \mathcal{K}} : \tilde{\mathcal{M}}_{M, \mathcal{K}} \rightarrow \mathcal{M}_{M, \mathcal{K}}$ by

$$P_{M, \mathcal{K}}(X, \iota) = [\alpha(H^{2,0}(X))],$$

where $\alpha : H^2(X, \mathbb{Z}) \rightarrow L_2$ is an admissible marking for $(M, \mathcal{K})$.

As before, let

$$\Delta(M^\perp) = \{\delta \in M^\perp; \delta^2 = -2, \text{ or } \delta^2 = -10, (\delta, L_2) = 2\mathbb{Z}\},$$

and set

$$\mathcal{D}_{M^\perp} = \bigcup_{\delta \in \Delta(M^\perp)} H_\delta \subset \Omega_{M^\perp},$$

where

$$H_\delta = \{x \in \Omega_{M^\perp}; (x, \delta) = 0\}. \quad (2.2.1)$$

By [45, Lemma 7.7], $\mathcal{D}_{M^\perp}$ is locally finite in $\Omega_{M^\perp}$ and is viewed as a reduced divisor on $\Omega_{M^\perp}$. Set

$$\bar{\mathcal{D}}_{M^\perp} = \mathcal{D}_{M^\perp} / \Gamma_{M^\perp, \mathcal{K}}.$$

Then $\bar{\mathcal{D}}_{M^\perp}$ is a reduced divisor on $\mathcal{M}_{M, \mathcal{K}}$. We set

$$\mathcal{M}_{M, \mathcal{K}}^\circ = \mathcal{M}_{M, \mathcal{K}} \setminus \bar{\mathcal{D}}_{M^\perp} \quad \text{and} \quad \Omega_{M^\perp}^\circ = \Omega_{M^\perp}^+ \setminus \mathcal{D}_{M^\perp}.$$

Proposition 2.2.14. *The image of the period map $P_{M, \mathcal{K}}$ is $\mathcal{M}_{M, \mathcal{K}}^\circ$.*

Proof. See [45, Lemma 9.5 and Proposition 9.9]. $\square$

The period map $P_{M, \mathcal{K}} : \tilde{\mathcal{M}}_{M, \mathcal{K}} \rightarrow \mathcal{M}_{M, \mathcal{K}}^\circ$ is not injective but generically injective.

Theorem 2.2.15. *There exists a $\Gamma_{M^\perp, \mathcal{K}}$ -invariant effective reduced divisor $\mathcal{D}_\mathcal{K}$ such that*

$$P_{M, \mathcal{K}} : P_{M, \mathcal{K}}^{-1}(\mathcal{M}_{M, \mathcal{K}}^\circ \setminus \bar{\mathcal{D}}_\mathcal{K}) \rightarrow \mathcal{M}_{M, \mathcal{K}}^\circ \setminus \bar{\mathcal{D}}_\mathcal{K}$$

is bijective, where $\bar{\mathcal{D}}_\mathcal{K} = \mathcal{D}_\mathcal{K} / \Gamma_{M^\perp, \mathcal{K}}$.

Proof. See [45, Theorem 10.5]. $\square$

Definition 2.2.16. *Let $f : (\mathcal{X}, \iota) \rightarrow S$ be a family of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$. We define the period map $P_{M, \mathcal{K}} : S \rightarrow \mathcal{M}_{M, \mathcal{K}}$ for $f : (\mathcal{X}, \iota) \rightarrow S$ by*

$$P_{M, \mathcal{K}}(s) = P_{M, \mathcal{K}}(X_s, \iota_s), \quad (s \in S).$$

Since $f_* \Omega_{\mathcal{X}/S}^2$ is a holomorphic vector subbundle of the flat bundle $R^2 f_* \mathbb{C} \otimes \mathcal{O}_S$, $P_{M, \mathcal{K}}$ is holomorphic ([36, 22.3]).

Definition 2.2.17. *Let $(X, \iota), (X', \iota') \in \tilde{\mathcal{M}}_{M, \mathcal{K}}$. We call (X, ι) and (X', ι') inseparable if their universal deformations $\pi : (\mathcal{X}, \iota) \rightarrow \text{Def}(X, \iota)$ and $\pi' : (\mathcal{X}', \iota') \rightarrow \text{Def}(X', \iota')$ contain isomorphic fibers, where $\text{Def}(X, \iota)$ and $\text{Def}(X', \iota')$ are viewed as germs.*

Lemma 2.2.18. *If $(X, \iota), (X', \iota') \in \tilde{\mathcal{M}}_{M, \mathcal{K}}$ satisfy $P_{M, \mathcal{K}}(X, \iota) = P_{M, \mathcal{K}}(X', \iota')$, then (X, ι) and (X', ι') are inseparable.*

Proof. See [45, Proposition 10.7]. $\square$

Chapter 3

Automorphic forms

In this section, we briefly recall two types of automorphic forms, Borcherds products and Siegel modular forms. These automorphic forms will be used to express the holomorphic torsion invariant of $K3^{[2]}$ -type manifolds with involution in the later chapters.

3.1 Borcherds products

3.1.1 The metaplectic group, Weil representations and elliptic modular forms

Let $\mathbb{H}$ be the complex upper half plane defined by

$$\mathbb{H} = \{\tau \in \mathbb{C}; \operatorname{Im} \tau > 0\}.$$

Let $Mp_2(\mathbb{Z})$ be the metaplectic group defined by

$$Mp_2(\mathbb{Z}) = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d} \right); \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \sqrt{c\tau + d} \in \mathcal{O}(\mathbb{H}) \right\},$$

where $\sqrt{c\tau + d}$ is a holomorphic function on the upper half plane $\mathbb{H}$ with its square $c\tau + d$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, write $j(\gamma, \tau) = c\tau + d$.

Let $(\gamma, \sqrt{j(\gamma, \tau)}), (\gamma', \sqrt{j(\gamma', \tau)}) \in Mp_2(\mathbb{Z})$. Since

$$j(\gamma, \gamma'\tau)j(\gamma', \tau) = j(\gamma\gamma', \tau),$$

$\sqrt{j(\gamma, \gamma'\tau)}\sqrt{j(\gamma', \tau)}$ is a square root of $j(\gamma\gamma', \tau)$. Therefore, we can define the multiplication of $Mp_2(\mathbb{Z})$ by

$$(\gamma, \sqrt{j(\gamma, \tau)}) \cdot (\gamma', \sqrt{j(\gamma', \tau)}) = (\gamma\gamma', \sqrt{j(\gamma, \gamma'\tau)}\sqrt{j(\gamma', \tau)}).$$

Set

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad Z = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right) \in Mp_2(\mathbb{Z})$$

Then $Mp_2(\mathbb{Z})$ is generated by S, T (cf. [74, Chapter VII, §1, Theorem 2]). In particular,

$$Mp_2(\mathbb{Z}) = \langle S, T; S^2 = (ST)^3 = Z, Z^4 = 1 \rangle.$$

Let Λ be an even lattice of signature (b^+, b^-) and let $\Lambda^\vee$ be its dual lattice. Set $A_\Lambda = \Lambda^\vee / \Lambda$. Let $\{\mathbf{e}_\gamma\}_{\gamma \in A_\Lambda}$ be the canonical basis of the group ring $\mathbb{C}[A_\Lambda]$. We define the Weil representation $\rho_\Lambda : Mp_2(\mathbb{Z}) \rightarrow GL(\mathbb{C}[A_\Lambda])$ by

$$\rho_\Lambda(T)\mathbf{e}_\gamma = e^{\pi i \gamma^2} \mathbf{e}_\gamma, \quad \rho_\Lambda(S)\mathbf{e}_\gamma = i^{-\frac{b^+ - b^-}{2}} \frac{1}{\sqrt{|A_\Lambda|}} \sum_{\delta \in A_\Lambda} e^{-2\pi i(\gamma, \delta)} \mathbf{e}_\delta$$

Since Λ is an even lattice, $e^{\pi i \gamma^2}$ and $e^{-2\pi i(\gamma, \delta)}$ are independent of the choice of a representative of $\gamma, \delta \in A_\Lambda = \Lambda^\vee / \Lambda$.

Definition 3.1.1. A $\mathbb{C}[A_\Lambda]$ -valued holomorphic function $F : \mathbb{H} \rightarrow \mathbb{C}[A_\Lambda]$ on $\mathbb{H}$ is called a $\mathbb{C}[A_\Lambda]$ -valued elliptic modular form of weight $1 - b^-/2$ and of type ρ_Λ if

- (1) for $(\gamma, \sqrt{j(\gamma, \tau)}) \in Mp_2(\mathbb{Z})$, the following identity of holomorphic functions on $\mathbb{H}$ holds:

$$F(\gamma \cdot \tau) = \sqrt{j(\gamma, \tau)}^{2-b^-} \rho(\gamma, \sqrt{j(\gamma, \tau)}) F(\tau).$$

- (2) $F(\tau)$ is meromorphic at $\tau = +i\infty$ and the Fourier series expansion of $F(\tau)$ has integral coefficients.

We write $F(\tau) = \sum_{\gamma \in A_\Lambda} f_\gamma(\tau) \mathbf{e}_\gamma$. By (1), we have

$$F(\tau + 1) = \rho(T)F(\tau),$$

which implies that

$$f_\gamma(\tau + 1) = e^{\pi i \gamma^2} f_\gamma(\tau).$$

Therefore, for each $\gamma \in A_\Lambda$, $e^{-\pi i \gamma^2 \tau} f_\gamma(\tau)$ is a $\mathbb{Z}$ -invariant holomorphic function on $\mathbb{H}$. If we set $q = e^{2\pi i \tau}$, then $e^{-\pi i \gamma^2 \tau} f_\gamma(\tau)$ is regarded as a holomorphic function on the punctured disk $\Delta^* = \{z \in \mathbb{C}; 0 < |z| < 1\}$. Therefore, $F(\tau)$ admits the following Fourier series expansion:

$$F(\tau) = \sum_{\gamma \in A_\Lambda} \mathbf{e}_\gamma \sum_{k \in \mathbb{Z} + \gamma^2/2} c_\gamma(k) q^k, \quad q = e^{2\pi i \tau}.$$

By (2) the Fourier coefficients $c_\gamma(k)$ satisfy

- (1) $c_\gamma(k) \in \mathbb{Z}$,
- (2) $c_\gamma(k) = 0$ if $k \neq \mathbb{Z} + \gamma^2/2$ or $k \ll 0$,
- (3) $c_\gamma(k) = c_{-\gamma}(k)$.

3.1.2 Automorphic forms on a bounded symmetric domain of type IV

We assume that the lattice Λ is an even lattice of signature $(2, b^-)$ and that there exist an integer $N \in \mathbb{Z}$ and a hyperbolic lattice L of signature $(1, b^- - 1)$ such that

$$\Lambda = U(N) \oplus L.$$

An element of Λ is denoted by (m, n, l) , where $m, n \in \mathbb{Z}$, $l \in L$.

Recall that

$$\Omega_\Lambda = \{[\eta] \in \mathbb{P}(\Lambda_\mathbb{C}); (\eta, \eta) = 0, (\eta, \bar{\eta}) > 0\},$$

and that Ω_Λ is the disjoint union of two bounded symmetric domains of type IV of dimension b^- . Set

$$\mathcal{C}_L = \{x \in L_\mathbb{R}; x^2 > 0\}.$$

We identify Ω_Λ with the tube domain $L_\mathbb{R} + i\mathcal{C}_L \subset \Lambda_\mathbb{C}$ via the following isomorphism

$$L_\mathbb{R} + i\mathcal{C}_L \rightarrow \Omega_\Lambda, \quad y \mapsto \left[\left(-\frac{y^2}{2}, \frac{1}{N}, y \right) \right]. \quad (3.1.1)$$

Let $\mathcal{C}_\Lambda = \mathcal{C}_\Lambda^+ \sqcup \mathcal{C}_\Lambda^-$ be the decomposition into connected components. Each connected component of Ω_Λ corresponds to the connected components $L_\mathbb{R} + i\mathcal{C}_L^\pm$ of the tube domain $L_\mathbb{R} + i\mathcal{C}_L$. The connected component of Ω_Λ corresponding to $L_\mathbb{R} + i\mathcal{C}_L^+$ is denoted by Ω_Λ^+ .

Fix a vector $l \in \Lambda_\mathbb{R}$ with $l^2 \geq 0$. Set

$$j_\Lambda(\gamma, [z]) = \frac{(\gamma(z), l)}{(z, l)}, \quad ([z] \in \Omega_\Lambda^+, \gamma \in O^+(\Lambda)).$$

Since $(z, l) \neq 0$ for any $[z] \in \Omega_\Lambda^+$, $j_\Lambda(\gamma, [z])$ is a nowhere-vanishing holomorphic function on Ω_Λ^+ .

Definition 3.1.2. Let $\Gamma \subset O^+(\Lambda)$ be a subgroup of finite index and let $\chi : \Gamma \rightarrow \mathbb{C}^*$ be a character. A holomorphic function F on Ω_Λ^+ is an automorphic form for Γ of weight k and of type χ if the following identity holds:

$$F(\gamma \cdot [z]) = \chi(\gamma) j_\Lambda(\gamma, [z])^k F([z]), \quad ([z] \in \Omega_\Lambda^+, \gamma \in \Gamma).$$

Define the Petersson norm $\|F([z])\|$ by

$$\|F([z])\|^2 = \left(\frac{(z, \bar{z})}{|(l, z)|^2} \right)^k |F([z])|^2.$$

If the character $\chi : \Gamma \rightarrow \mathbb{C}^*$ is unitary, $\|F\|^2$ is a Γ -invariant C^∞ function on Ω_Λ^+ .

Remark 3.1.3. Assume that $\text{rk } \Lambda \geq 5$. By [56, Chapter IX, Remark 1.6 (iv), Proposition 5.3, Theorem 5.4], $\Gamma/[\Gamma, \Gamma]$ is a finite group. Hence every character $\chi : \Gamma \rightarrow \mathbb{C}^*$ is unitary and $\|F\|^2$ is a Γ -invariant C^∞ function on Ω_Λ^+ .

We may regard an automorphic form F on Ω_Λ^+ as a holomorphic function on $L_\mathbb{R} + i\mathcal{C}_L^+$ via the isomorphism (3.1.1).

3.1.3 Borchers products

Set $\sigma(\Lambda) = 2 - b^-$. For $g \in O(\Lambda)$ and for a $\mathbb{C}[A_\Lambda]$ -valued function $F : \mathbb{H} \rightarrow \mathbb{C}[A_\Lambda]$, the natural $O(\Lambda)$ -action on $\mathbb{C}[A_\Lambda]$ induces a $\mathbb{C}[A_\Lambda]$ -valued function $g(F) : \mathbb{H} \rightarrow \mathbb{C}[A_\Lambda]$.

Theorem 3.1.4. *Let*

$$F(\tau) = \sum_{\gamma \in A_\Lambda} \mathbf{e}_\gamma \sum_{k \in \mathbb{Z} + \gamma^2/2} c_\gamma(k) q^k, \quad q = e^{2\pi i \tau}$$

be an elliptic modular form of weight $\sigma(\Lambda)/2$ and of type ρ_Λ . Then there exists a meromorphic automorphic form $\Psi_\Lambda(\cdot, F)$ on Ω_Λ^+ with the following properties:

- (1) $\Psi_\Lambda(\cdot, F)$ is an automorphic form for $\text{Aut}(\Lambda, F) \cap O^+(\Lambda)$ of weight $c_0(0)/2$, where $\text{Aut}(\Lambda, F)$ is a finite index subgroup of $O(\Lambda)$ defined by

$$\text{Aut}(\Lambda, F) = \{g \in O(\Lambda); g(F) = F\}.$$

- (2) *The divisor of $\Psi_\Lambda(\cdot, F)$ is given by*

$$\text{div}(\Psi_\Lambda(\cdot, F)) = \sum_{\lambda \in \Lambda^\vee / \pm 1, \lambda^2 < 0} c_{\bar{\lambda}}\left(\frac{\lambda^2}{2}\right) H_\lambda,$$

where $H_\lambda = \{[\eta] \in \Omega_\Lambda^+; (\eta, \lambda) = 0\}$ and $\bar{\lambda} = \lambda + \Lambda \in A_\Lambda$.

- (3) *Choose a connected component $\mathcal{W}$ of*

$$\mathcal{C}_L^+ \setminus \bigcup_{\lambda \in L^\vee, \lambda^2 < 0, \sum_{n=0}^{N-1} c_{(\frac{n}{N}, 0, \bar{\lambda})}(\lambda^2/2) \neq 0} \lambda^\perp.$$

Regard $\Psi_\Lambda(\cdot, F)$ as a holomorphic function on $L_\mathbb{R} + i\mathcal{C}_L^+$. For $y \in L_\mathbb{R} + i\mathcal{W}$ with $\text{Im } y \gg 0$, the following infinite product expansion holds:

$$\Psi_\Lambda(y, F) = e^{2\pi i(\varrho, y)} \prod_{\lambda \in L^\vee} \prod_{n \in \mathbb{Z}/N\mathbb{Z}} (1 - e^{2\pi i(\lambda, y) + \frac{n}{N}})^{c_{(\frac{n}{N}, 0, \bar{\lambda})}(\lambda^2/2)},$$

where $\varrho \in L_\mathbb{Q}$ is called the Weyl vector corresponding to L , Λ , $\mathcal{W}$ and F .

Proof. See [21, Theorem 13.3]. The explicit formula for the Weyl vector is given in [21, Theorem 10.4] and [22, p.321 correction]. $\square$

3.2 Siegel modular forms

3.2.1 Definition

Fix a positive integer $g > 0$. Let $Sp(2g, \mathbb{Z})$ be the Siegel modular group defined by

$$Sp(2g, \mathbb{Z}) = \left\{ M \in GL(2g, \mathbb{Z}); M \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} {}^t M = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \right\},$$

where I_g is the $g \times g$ -identity matrix. We define the Siegel upper half space $\mathfrak{S}_g$ of degree g by

$$\mathfrak{S}_g = \{ \Omega \in M_g(\mathbb{C}); \quad {}^t \Omega = \Omega, \quad \text{Im } \Omega > 0 \},$$

where $\text{Im } \Omega$ is the imaginary part of Ω . The Siegel modular variety $\mathcal{A}_g$ of degree g is defined by

$$\mathcal{A}_g = \mathfrak{S}_g / Sp(2g, \mathbb{Z}),$$

where the $Sp(2g, \mathbb{Z})$ -action on $\mathfrak{S}_g$ is given by

$$\gamma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \quad \Omega \in \mathfrak{S}_g, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}).$$

Definition 3.2.1. A holomorphic function F on $\mathfrak{S}_g$ is called a Siegel modular form of weight k if the following identity holds:

$$F((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^k F(\Omega) \quad \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}), \quad \Omega \in \mathfrak{S}_g \right).$$

The Petersson norm of F is defined by

$$\|F(\Omega)\|^2 = (\det \text{Im } \Omega)^k |F(\Omega)|^2.$$

Then $\|F\|^2$ is a $Sp(2g, \mathbb{Z})$ -invariant C^∞ -function on $\mathfrak{S}_g$ and induces a C^∞ -function on $\mathcal{A}_g$.

3.2.2 The Riemann theta constant

Let $a, b \in \{0, \frac{1}{2}\}^g$ be two column vectors. We define

$$\theta_{a,b}(\Omega) = \sum_{n \in \mathbb{Z}^g} \exp\{\pi i {}^t(n+a)\Omega(n+a) + 2\pi i {}^t(n+a)b\}.$$

Since $\text{Im } \Omega > 0$, $\theta_{a,b}(\Omega)$ converges uniformly on compact subsets and defines a holomorphic function on $\mathfrak{S}_g$. The function $\theta_{a,b}(\Omega)$ is called the Riemann theta constant. By [60, Proposition 3.14], we have $\theta_{a,b}(\Omega) = (-1)^{4^t ab} \theta_{a,b}(\Omega)$. Therefore, if $(a, b) := 4^t ab \in \mathbb{Z}$ is an odd integer, we identically have $\theta_{a,b}(\Omega) = 0$.

We define holomorphic functions χ_g and Υ_g on $\mathfrak{S}_g$ by

$$\chi_g(\Omega) = \prod_{(a,b):\text{even}} \theta_{a,b}(\Omega), \quad \Upsilon_g(\Omega) = \chi_g(\Omega)^8 \sum_{(a,b):\text{even}} \theta_{a,b}(\Omega)^{-8}.$$

Note that χ_g^8 and Υ_g are the elementary symmetric polynomials of the theta constants $\theta_{a,b}(\Omega)^8$ of degree $2^{g-1}(2^g+1)$ and $2^{g-1}(2^g+1)-1$, respectively. By [41, p. 176 Corollary, p. 182 Theorem 3], for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z})$, there exists a permutation σ_γ on $\{0, \frac{1}{2}\}^{2g}$ such that

$$\theta_{\sigma_\gamma(a,b)}(\gamma \cdot \Omega)^8 = \det(C\Omega + D)^4 \theta_{a,b}(\Omega)^8.$$

Therefore, χ_g^8 and Υ_g are Siegel modular forms of weight $2^{g+1}(2^g+1)$ and $2(2^g-1)(2^g+2)$, respectively.

Chapter 4

Holomorphic torsion invariants

In this chapter, we recall two types of holomorphic torsion invariants. First, we recall the invariant of 2-elementary K3 surfaces constructed by Yoshikawa [82] and his explicit formula for the invariant in terms of automorphic forms. Second, we recall the BCOV invariant of Calabi-Yau manifolds constructed by Eriksson-Freixas i Montplet-Mourougane [25] and their explicit formula for the BCOV invariant of mirror Calabi-Yau hypersurfaces.

4.1 A holomorphic torsion invariant of 2-elementary K3 surfaces

4.1.1 2-elementary K3 surfaces

Let M_0 be a primitive hyperbolic 2-elementary sublattice of L_{K3} . Since L_{K3} is unimodular and since M_0 is 2-elementary, the involution

$$M_0 \oplus M_0^\perp \rightarrow M_0 \oplus M_0^\perp, \quad (m, n) \mapsto (m, -n)$$

extends uniquely to an involution $\iota_{M_0} \in O(L_{K3})$ by [62, Corollary 1.5.2].

Let Y be a K3 surface and $\sigma : Y \rightarrow Y$ be an antisymplectic involution on Y . Set

$$H^2(Y, \mathbb{Z})^\sigma = \{x \in H^2(Y, \mathbb{Z}); \sigma^*x = x\}.$$

Definition 4.1.1. *We call the pair (Y, σ) a 2-elementary K3 surface of type M_0 if there exists an isometry of lattices $\alpha : H^2(Y, \mathbb{Z}) \rightarrow L_{K3}$ such that $\alpha(H^2(Y, \mathbb{Z})^\sigma) = M_0$.*

Let $\alpha : H^2(Y, \mathbb{Z}) \rightarrow L_{K3}$ be an isometry of lattices such that $\alpha(H^2(Y, \mathbb{Z})^\sigma) = M_0$. By the Hodge-Riemann bilinear relation, we have $\alpha(H^0(Y, \Omega_Y^2)) \in \Omega_{M_0^\perp}$, where $\Omega_{M_0^\perp}$ is the period domain defined by

$$\Omega_{M_0^\perp} = \{[\eta] \in \mathbb{P}(M_{0, \mathbb{C}}^\perp); (\eta, \eta) = 0, (\eta, \bar{\eta}) > 0\}.$$

Definition 4.1.2. *The point $\pi_{M_0}(Y, \sigma, \alpha) := \alpha(H^0(Y, \Omega_Y^2))$ of $\Omega_{M_0^\perp}$ is called the period of (Y, σ, α) .*

Since $\text{sign}(M_0^\perp) = (2, \text{rk } M_0^\perp - 2)$, $\Omega_{M_0^\perp}$ consists of two connected components. We fix a connected component $\Omega_{M_0^\perp}^+$ of $\Omega_{M_0^\perp}$. We obtain the orthogonal modular variety

$$\mathcal{M}_{M_0} = \Omega_{M_0^\perp}^+ / O^+(M_0^\perp)$$

of dimension $20 - \text{rk}(M_0)$.

Let $\tilde{\mathcal{M}}_{M_0}$ be the isomorphism classes of 2-elementary K3 surfaces of M_0 . We define the period map $\bar{\pi}_{M_0} : \tilde{\mathcal{M}}_{M_0} \rightarrow \mathcal{M}_{M_0}$ by

$$\bar{\pi}_{M_0}(Y, \sigma) = \pi_{M_0}(Y, \sigma, \alpha),$$

where $\alpha : H^2(Y, \mathbb{Z}) \rightarrow L_{K3}$ is an isometry of lattices such that $\alpha(H^2(Y, \mathbb{Z})^\sigma) = M_0$.

We set $\Delta(M_0^\perp) = \{d \in M_0^\perp; d^2 = -2\}$, and

$$\mathcal{D}_{M_0^\perp} = \bigcup_{d \in \Delta(M_0^\perp)} H_d \subset \Omega_{M_0^\perp}^+.$$

Here H_d is the divisor on $\Omega_{M_0^\perp}^+$ defined in the same way as in (2.2.1). By [82, Proposition 1.9.], $\mathcal{D}_{M_0^\perp}$ is locally finite and is viewed as a reduced divisor on $\Omega_{M_0^\perp}^+$. Set

$$\bar{\mathcal{D}}_{M_0^\perp} = \mathcal{D}_{M_0^\perp} / O^+(M_0^\perp)$$

and

$$\mathcal{M}_{M_0}^\circ = \mathcal{M}_{M_0} \setminus \bar{\mathcal{D}}_{M_0^\perp}.$$

Theorem 4.1.3. *Via the period map $\bar{\pi}_{M_0}$, $\mathcal{M}_{M_0}^\circ$ is a coarse moduli space of 2-elementary K3 surfaces of type M_0 .*

Proof. See [82, Theorem 1.8.]. □

4.1.2 The invariant of 2-elementary K3 surfaces of Yoshikawa

Let us recall the invariant τ_{M_0} of 2-elementary K3 surface of type M_0 introduced in [82, Definition 5.1]. Let (Y, σ) be a 2-elementary K3 surface of type M_0 . Choose a σ -invariant Kähler metric h_Y on Y and denote its Kähler form by ω_Y . We define the volume of (Y, ω_Y) by

$$\text{Vol}(Y, \omega_Y) = \int_Y \frac{\omega_Y^2}{2!}.$$

The fixed locus of $\sigma : Y \rightarrow Y$ is denoted by Y^σ . Set $r = \text{rk } M_0$ and $l = \dim_{\mathbb{Z}_2} A_{M_0}$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. If $M_0 \neq E_8(2) \oplus U(2), E_8(2) \oplus U$, we set

$$g = g(M_0) = 11 - \frac{r+l}{2}, \quad k = k(M_0) = \frac{r-l}{2}. \quad (4.1.1)$$

Proposition 4.1.4. *The fixed locus Y^σ is the empty set or a disjoint union of smooth projective curves. More precisely,*

$$Y^\sigma = \begin{cases} \emptyset & (M_0 = E_8(2) \oplus U(2)), \\ E \sqcup F & (M_0 = E_8(2) \oplus U), \\ C \sqcup E_1 \sqcup \cdots \sqcup E_k & (\text{otherwise}), \end{cases}$$

where E, F are elliptic curves, C is a smooth curves of genus g , and E_i are (-2) -curves.

Proof. See [63, Theorem 4.2.2]. $\square$

Assume that $Y^\sigma \neq \emptyset$. Let $Y^\sigma = \sqcup_i C_i$ be the decomposition into the connected components. We define the volume of $(Y^\sigma, \omega_{Y^\sigma})$ by

$$\text{Vol}(Y^\sigma, \omega_{Y^\sigma}) = \prod_i \text{Vol}(C_i, \omega_Y|_{C_i}) = \prod_i \int_{C_i} \omega_Y|_{C_i}.$$

We define a positive number $A(Y, \sigma, h_Y) \in \mathbb{R}_{>0}$ by

$$A(Y, \sigma, h_Y) = \exp \left[\frac{1}{8} \sum_i \int_{C_i} \log \left(\frac{\eta \wedge \bar{\eta}}{\omega_Y^2/2!} \frac{\text{Vol}(Y, \omega_Y)}{\|\eta\|_{L^2}^2} \right) c_1(TC_i, h_Y|_{C_i}) \right],$$

where η is a holomorphic 2-form on Y and $c_1(TC_i, h_Y|_{C_i})$ is the first Chern form of the holomorphic hermitian line bundle $(TC_i, h_Y|_{C_i})$. If $Y^\sigma = \emptyset$, we set $\text{Vol}(Y^\sigma, \omega_{Y^\sigma}) = A(Y, \sigma, h_Y) = 1$.

Let $\tau_\sigma(\bar{\mathcal{O}}_Y)$ be the equivariant analytic torsion of the trivial line bundle $\mathcal{O}_Y$ with respect to the Kähler metric h_Y and the trivial metric on $\mathcal{O}_Y$. Let $\tau(\bar{\mathcal{O}}_{C_i})$ be the analytic torsion of the trivial line bundle $\mathcal{O}_{C_i}$ with respect to the Kähler metric $h_Y|_{C_i}$ and the trivial metric on $\mathcal{O}_{C_i}$. We set

$$\tau(\bar{\mathcal{O}}_{Y^\sigma}) = \prod_i \tau(\bar{\mathcal{O}}_{C_i}).$$

If $Y^\sigma = \emptyset$, we set $\tau(\bar{\mathcal{O}}_{Y^\sigma}) = 1$.

Definition 4.1.5. *Let (Y, σ) be 2-elementary K3 surface of type M_0 and let h_Y be a σ -invariant Kähler metric on Y . We define a real number $\tau_{M_0}(Y, \sigma)$ by*

$$\tau_{M_0}(Y, \sigma) = \tau_\sigma(\bar{\mathcal{O}}_Y) \text{Vol}(Y, \omega_Y)^{\frac{14 - \text{rk}(M_0)}{4}} \tau(\bar{\mathcal{O}}_{Y^\sigma}) \text{Vol}(Y^\sigma, \omega_{Y^\sigma}) A(Y, \sigma, h_Y).$$

Theorem 4.1.6. *The real number $\tau_{M_0}(Y, \sigma)$ is independent of the choice of h_Y and is an invariant of (Y, σ) .*

Proof. See [82, Theorem 5.7]. $\square$

4.1.3 An explicit formula for τ_{M_0}

In the rest of this subsection, we set

$$\Lambda = M_0^\perp,$$

where the orthogonal complement is considered in L_{K3} . Then Λ is an even primitive 2-elementary sublattice of L_{K3} of signature $(2, \text{rk}(\Lambda) - 2)$. By [63, p.1434 table 1] (cf. [28] or [52]), there exist an integer $N \in \{1, 2\}$ and a 2-elementary hyperbolic sublattice L such that $\Lambda = U(N) \oplus L$.

For $\tau \in \mathbb{H}$, set $q = e^{2\pi i \tau}$. Let $\eta(\tau)$ be the Dedekind η -function. Namely, $\eta(\tau)$ is the holomorphic function on $\mathbb{H}$ defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Recall that the theta series of the A_1 -lattice (or equivalently the Jacobi theta series) are the holomorphic functions on $\mathbb{H}$ defined by

$$\theta_{A_1^+}(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad \theta_{A_1^+ + 1/2}(\tau) = \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2} \quad (\tau \in \mathbb{H}).$$

We define the holomorphic functions $f_k^{(0)}(\tau)$, $f_k^{(1)}(\tau)$ and the series $\{c_k^{(0)}(l)\}_{l \in \mathbb{Z}}$, $\{c_k^{(1)}(l)\}_{l \in \frac{k}{4} + \mathbb{Z}}$ as follows

$$\begin{aligned} f_k^{(0)}(\tau) &= \frac{\eta(2\tau)^8 \theta_{A_1^+}(\tau)^k}{\eta(\tau)^8 \eta(4\tau)^8} = \sum_{l \in \mathbb{Z}} c_k^{(0)}(l) q^l, \\ f_k^{(1)}(\tau) &= -16 \frac{\eta(4\tau)^8 \theta_{A_1^+ + \frac{1}{2}}(\tau)^k}{\eta(2\tau)^{16}} = 2 \sum_{l \in \frac{k}{4} + \mathbb{Z}} c_k^{(1)}(l) q^l. \end{aligned}$$

For $i \in \mathbb{Z}/4\mathbb{Z}$, we define the holomorphic function $g_k^{(i)}(\tau)$ on $\mathbb{H}$ by

$$g_k^{(i)}(\tau) = \sum_{l \equiv i \pmod{4}} c_k^{(0)}(l) q^{\frac{l}{4}}.$$

Recall that $\sigma(\Lambda) = 2 - b^- = 4 - \text{rk}(\Lambda)$. Set $l(\Lambda) = \dim_{\mathbb{Z}_2} A_\Lambda$.

Let $q_\Lambda : A_\Lambda \rightarrow \mathbb{Q}/2\mathbb{Z}$ be the discriminant form defined by

$$q_\Lambda(\lambda + \Lambda) = \lambda^2 \pmod{2\mathbb{Z}} \quad (\lambda \in \Lambda^\vee).$$

Since Λ is 2-elementary, the image of q_Λ is contained in $\mathbb{Z}/2\mathbb{Z}$ and q_Λ is $\mathbb{Z}/2\mathbb{Z}$ -linear.

Let $b_\Lambda : A_\Lambda \times A_\Lambda \rightarrow \mathbb{Q}/\mathbb{Z}$ be a bilinear form defined by

$$b_\Lambda(\lambda + \Lambda, \lambda' + \Lambda) = (\lambda, \lambda') \pmod{\mathbb{Z}} \quad (\lambda, \lambda' \in \Lambda^\vee).$$

Then b_Λ is non-degenerate. Therefore, there exists a unique element $\mathbf{1}_\Lambda \in A_\Lambda$ such that

$$b_\Lambda(\gamma, \mathbf{1}_\Lambda) \equiv \gamma^2 \pmod{\mathbb{Z}} \quad (\gamma \in A_\Lambda).$$

Let $\mathbf{e}_{\mathbf{1}_\Lambda} \in \{\mathbf{e}_\gamma\}_{\gamma \in A_\Lambda}$ be the vector corresponding to $\mathbf{1}_\Lambda \in A_\Lambda$ (cf. Section 3.1.1).

Definition 4.1.7. We define the $\mathbb{C}[A_\Lambda]$ -valued holomorphic function $F_\Lambda(\tau)$ on $\mathbb{H}$ by

$$F_\Lambda(\tau) = f_{8+\sigma(\Lambda)}^{(0)}(\tau)\mathbf{e}_0 + 2^{\{4-\sigma(\Lambda)-l(\Lambda)\}/2} \sum_{\gamma \in A_\Lambda} g_{8+\sigma(\Lambda)}^{(2\gamma^2)}(\tau)\mathbf{e}_\gamma + f_{8+\sigma(\Lambda)}^{(1)}(\tau)\mathbf{e}_{1_\Lambda}.$$

Theorem 4.1.8. The function $F_\Lambda(\tau)$ is an elliptic modular form of type ρ_Λ of weight $\sigma(\Lambda)/2$. Moreover, $\text{Aut}(F_\Lambda, \Lambda) = O(\Lambda)$.

Proof. See [87, Theorem 7.7]. □

Recall that $r = \text{rk}(M_0)$.

Definition 4.1.9. We define an elliptic modular form f_Λ of type ρ_Λ of weight $\sigma(\Lambda)/2$ as follows:

- (1) If $r \neq 2$, then f_Λ is defined by $\delta_{r,10}F_\Lambda(\tau)$.
- (2) If $M_0 = U$, then f_Λ is defined by

$$f_\Lambda(\tau) = E_4(\tau)/\eta(\tau)^{24},$$

where $E_4(\tau)$ be the Eisenstein series of weight 4.

- (3) If $M_0 = U(2)$, then f_Λ is defined by

$$f_\Lambda(\tau) = 8 \sum_{\gamma \in A_\Lambda} \left\{ \eta\left(\frac{\tau}{2}\right)^{-8} \eta(\tau)^{-8} + (-1)^{q_\Lambda(\gamma)} \eta\left(\frac{\tau+1}{2}\right)^{-8} \eta(\tau+1)^{-8} \right\} \mathbf{e}_\gamma + \eta(\tau)^{-8} \eta(2\tau)^{-8} \mathbf{e}_{00},$$

where $\mathbf{e}_{00} \in \{\mathbf{e}_\gamma\}_{\gamma \in A_\Lambda}$ corresponds to the zero in A_Λ .

By Theorem 3.1.4, for a $O^+(\Lambda)$ -invariant elliptic modular form F of type ρ_Λ of weight $1 - (20 - r)/2$, its Borcherds product $\Psi_\Lambda(\cdot, F)$ is a meromorphic modular form for a finite index subgroup of $O^+(\Lambda)$. If $r \leq 17$, it follows from Remark 3.1.3 that the Petersson norm $\|\Psi_\Lambda(\cdot, F)\|^2$ can be regarded as a C^∞ -function on the orthogonal modular variety $\mathcal{M}_{M_0} = \Omega_{M_0^\perp}^+ / O^+(M_0^\perp)$.

Let g be the total genus of Y^σ for $(Y, \sigma) \in \mathcal{M}_{M_0}^\circ$. Namely, if $Y^\sigma = \sqcup_i C_i$, then

$$g = \sum_i g(C_i) = 11 - \frac{r(M_0) + l(M_0)}{2},$$

where $g(C_i)$ is the genus of the curve C_i . Define a holomorphic map $J_{M_0} : \mathcal{M}_{M_0}^\circ \rightarrow \mathcal{A}_g$ by

$$J_{M_0}(Y, \sigma) = [\Omega(Y^\sigma)],$$

where $\Omega(Y^\sigma) \in \mathfrak{S}_g$ is the period matrix of Y^σ . Recall that the Petersson norms $\|\chi_g\|^2$ and $\|\Upsilon_g\|^2$ are the C^∞ -functions on the Siegel modular variety $\mathcal{A}_g$ in the sense of

orbifolds. Therefore, the pullbacks $J_{M_0}^* \|\chi_g\|^2$ and $J_{M_0}^* \|\Upsilon_g\|^2$ are the C^∞ -functions on $\mathcal{M}_{M_0}^\circ$.

Let $\delta = \delta_\Lambda \in \{0, 1\}$ be the parity of the discriminant form q_Λ . Namely,

$$\delta = \begin{cases} 0 & (\lambda^2 \in \mathbb{Z} \text{ for all } \lambda \in \Lambda^\vee), \\ 1 & (\text{otherwise}). \end{cases}$$

Theorem 4.1.10. *Let M_0 be a primitive hyperbolic 2-elementary sublattice of L_{K3} of rank r and let Λ be its orthogonal complement. Then there exists a constant $C_{M_0} > 0$ depending only on M_0 such that the following identities hold:*

(1) *if $(r, \delta) \neq (2, 0), (10, 0)$, then*

$$\tau_{M_0}^{-2^g(2^g+1)} = C_{M_0} \|\Psi_\Lambda(\cdot, 2^{g-1}F_\Lambda + f_\Lambda)\| \cdot J_{M_0}^* \|\chi_g\|;$$

(2) *if $(r, \delta) = (2, 0)$ or $(10, 0)$, then*

$$\tau_{M_0}^{-(2^g-1)(2^g+2)} = C_{M_0} \|\Psi_\Lambda(\cdot, 2^{g-1}F_\Lambda + f_\Lambda)\| \cdot J_{M_0}^* \|\Upsilon_g\|.$$

Proof. See [52, Theorem 0.1]. □

4.2 The BCOV invariants

Following Eriksson-Freixas i Montplet-Mourougane [25], [26], let us briefly recall the notion of the BCOV invariant of Calabi-Yau manifolds and some of its basic properties.

4.2.1 The construction of the BCOV invariant

Definition 4.2.1. *A compact Kähler manifold Z of dimension m is a Calabi-Yau manifold if $K_Z \cong \mathcal{O}_Z$ and*

$$H^q(Z, \mathcal{O}_Z) = 0 \quad (q = 1, \dots, m-1).$$

Let h_Z be a Kähler metric on Z . Its associated Kähler form is denoted by ω_Z . The hermitian metric on Ω_Z^p induced from h_Z is also denoted by h_Z . Set $\overline{\Omega}_Z^p = (\Omega_Z^p, h_Z)$. Let $\tau(\overline{\Omega}_Z^p)$ be the analytic torsion of the holomorphic hermitian vector bundle $\overline{\Omega}_Z^p$.

Define the BCOV torsion $\mathcal{T}_{BCOV}(Z, \omega_Z)$ by

$$\mathcal{T}_{BCOV}(Z, \omega_Z) = \prod_{p \geq 0} \tau(\overline{\Omega}_Z^p)^{(-1)^p p}.$$

Let η be a generator of $H^0(Z, \Omega_Z^m) \cong \mathbb{C}$. We set

$$A(Z, \omega_Z) = \exp \left\{ -\frac{1}{12} \int_Z (\log \varphi) c_m(TZ, h_Z) \right\},$$

where φ is a smooth function on Z defined by

$$\varphi = \frac{i^{m^2} \eta \wedge \bar{\eta}}{\omega_Z^m / m!} \frac{1}{\|\eta\|_{L^2}^2}.$$

Note that if h_Z is a Ricci-flat Kähler metric, then we have

$$A(Z, \omega_Z) = \text{Vol}(Z, \omega_Z)^{\frac{\chi(Z)}{12}},$$

where $\chi(Z)$ is a topological Euler characteristic of Z .

The covolume of the lattice $\text{Im}(H^k(Z, \mathbb{Z}) \rightarrow H^k(Z, \mathbb{R}))$ of $H^k(Z, \mathbb{R})$ with respect to the L^2 -metric induced from h_Z is denoted by $\text{Vol}_{L^2}(H^k(Z, \mathbb{Z}), \omega_Z)$. Namely, we define

$$\text{Vol}_{L^2}(H^k(Z, \mathbb{Z}), \omega_Z) = \det(\langle e_i, e_j \rangle_{L^2}),$$

where $e_1, \dots, e_{b_k(Z)}$ is an integral basis of $\text{Im}(H^k(Z, \mathbb{Z}) \rightarrow H^k(Z, \mathbb{R}))$. We set

$$B(Z, \omega_Z) = \prod_{k=1}^{2m} \text{Vol}_{L^2}(H^k(Z, \mathbb{Z}), \omega_Z)^{(-1)^{k+1} k/2}.$$

Definition 4.2.2. Let Z be a Calabi-Yau m -fold and Let ω_Z be a Kähler form on Z . The BCOV invariant of Z is defined by

$$\tau_{BCOV}(Z) = \frac{A(Z, \omega_Z)}{B(Z, \omega_Z)} \mathcal{T}_{BCOV}(Z, \omega_Z).$$

Theorem 4.2.3. The BCOV invariant $\tau_{BCOV}(Z)$ is independent of the choice of ω_Z .

Proof. See [25, Proposition 5.8]. □

4.2.2 The curvature formula for the BCOV invariant

Let $h : \mathcal{Z} \rightarrow S$ be a proper holomorphic submersion of connected complex manifolds whose fibers are Calabi-Yau manifolds. Set $Z_s = h^{-1}(s)$ for $s \in S$. Let $h_{\mathcal{Z}/S}$ be a hermitian metric on $T\mathcal{Z}/S$ which is fiberwise Kähler. Since $\dim H^q(Z_s, \Omega_{Z_s}^p)$ is independent of $s \in S$, $R^q h_* \Omega_{\mathcal{Z}/S}^p$ is a locally free sheaf and we regard it as a holomorphic vector bundle on S . By Hodge theory, $R^q h_* \Omega_{\mathcal{Z}/S}^p$ is equipped with the hermitian metric. This is called the L^2 -metric and is denoted by h_{L^2} .

Let ω_{H^k} and ω_{WP} be smooth $(1, 1)$ -forms on S defined by

$$\omega_{H^k} = \sum_{p+q=k} p c_1(R^q h_* \Omega_{\mathcal{Z}/S}^p, h_{L^2}) \quad \text{and} \quad \omega_{WP} = c_1(h_* K_{\mathcal{Z}/S}, h_{L^2}).$$

Let $\chi = \sum_k (-1)^k \dim H^k(Z_s, \mathbb{C})$ be the topological Euler characteristic of the fibers Z_s .

Define a function $\tau_{BCOV, \mathcal{Z}/S}$ on S by

$$\tau_{BCOV, \mathcal{Z}/S}(s) = \tau_{BCOV}(Z_s) \quad (s \in S).$$

Theorem 4.2.4. *The following identity of $(1,1)$ -forms on S holds:*

$$dd^c \log \tau_{BCOV, \mathcal{X}/S} = \sum_{k=0}^8 (-1)^k \omega_{H^k} - \frac{\chi}{12} \omega_{WP}.$$

Proof. See [25, Proposition 5.10]. □

4.2.3 Degeneration of Calabi-Yau manifolds and BCOV invariants

Let us recall the algebraicity of the singularity of the BCOV invariant of Calabi-Yau manifolds.

Theorem 4.2.5. *Let $f : \mathcal{X} \rightarrow \Delta$ be a projective holomorphic surjection that can be extendable to a holomorphic map from a projective manifold to a projective curve. Assume that the restriction $f|_{f^{-1}(\Delta^*)} : f^{-1}(\Delta^*) \rightarrow \Delta^*$ is a family of Calabi-Yau manifolds. Define a function $\tau_{BCOV, \mathcal{X}/\Delta}$ on Δ^* by*

$$\tau_{BCOV, \mathcal{X}/\Delta}(s) = \tau_{BCOV}(Z_s) \quad (s \in \Delta^*).$$

Then there exists a constant $\kappa_f \in \mathbb{Q}$ such that

$$\log \tau_{BCOV, \mathcal{X}/\Delta}(s) = \kappa_f \log |s|^2 + O(\log \log |s|^{-1}) \quad (s \rightarrow 0).$$

Proof. See [25, Theorem 6.5]. □

We refer the reader to [25] for some explicit formulas for κ_f for some classes of degenerations.

4.3 The BCOV invariant of the mirror Calabi-Yau hypersurfaces

Let us consider the BCOV invariant of the mirror Calabi-Yau hypersurfaces. This class of manifolds is the only example of higher-dimensional strict Calabi-Yau manifolds whose BCOV invariant is determined.

4.3.1 The mirror Calabi-Yau hypersurfaces

The element of $\mathbb{P}^n \times \mathbb{P}^1$ is denoted by $([z_0, \dots, z_n], \psi)$. For $\psi \in \mathbb{C}$, define the polynomial $F_\psi(z_0, \dots, z_n) \in \mathbb{C}[z_0, \dots, z_n]$ by

$$F_\psi(z_0, \dots, z_n) = \sum_{j=0}^n z_j^{n+1} - (n+1)\psi z_0 \dots z_n.$$

Let $\mathcal{X}$ be the hypersurface of $\mathbb{P}^n \times \mathbb{P}^1$ defined by

$$\mathcal{X} = \{([z], \psi) \in \mathbb{P}^n \times \mathbb{P}^1; F_\psi(z_0, \dots, z_n) = 0\}$$

and let $h : \mathcal{X} \rightarrow \mathbb{P}^1$ be the projection to the second factor.

Let μ_{n+1} be the group of the $(n+1)$ -th roots of unity. Define the subgroup $G(\cong (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1})$ of $PGL(n+1)$ by

$$G = \{\text{diag}(g_0, \dots, g_n); g_i \in \mu_{n+1}, g_0 \dots g_n = 1\} / \mu_{n+1} I.$$

Then G acts on $\mathcal{X}$ by $g \cdot ([z], \psi) = ([g \cdot z], \psi)$. Set $\mathcal{Y} = \mathcal{X}/G$ and let $g : \mathcal{Y} \rightarrow \mathbb{P}^1$ be the morphism induced from $h : \mathcal{X} \rightarrow \mathbb{P}^1$.

Proposition 4.3.1. *There exists a projective birational morphism $\mathcal{Z} \rightarrow \mathcal{Y}$ such that*

- (1) $\mathcal{Z}$ is smooth.
- (2) Let $f : \mathcal{Z} \rightarrow \mathbb{P}^1$ be the composition of $g : \mathcal{Y} \rightarrow \mathbb{P}^1$ and the birational morphism $\mathcal{Z} \rightarrow \mathcal{Y}$. If $\psi^{n+1} = 1$, then the fiber $Z_\psi = f^{-1}(\psi)$ has a single ordinary double point singularity.
- (3) If $\psi = \infty$, then Z_∞ is a simple normal crossing divisor on $\mathcal{Z}$.
- (4) If $\psi^{n+1} \neq 1$ and $\psi \neq \infty$, then $Z_\psi \rightarrow Y_\psi$ is a crepant resolution of the singularities of Y_ψ , and Z_ψ is a smooth Calabi-Yau manifold.

Proof. See [26, Lemma 3.3]. □

The family of Calabi-Yau $(n-1)$ -folds

$$f : \mathcal{Z} \rightarrow \mathbb{P}^1$$

is called a mirror family of a Calabi-Yau hypersurface of $\mathbb{P}^n$.

4.3.2 The sections η_k of the Hodge bundle

Define a holomorphic n -form Ω on $\mathbb{C}^{n+1}$ by

$$\Omega = \sum_{i=0}^n (-1)^{i-1} z_i dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_n.$$

Let $w_1, \dots, w_n$ be a system of local coordinates on $\mathbb{P}^n$ such that $\text{div } w_1 = F_\psi$. Let $\Omega_{\mathbb{P}^n}(\log X_\psi)$ be the sheaf of logarithmic differential forms which are locally generated by $dw_1/w_1, dw_2, \dots, dw_n$. We define $\Omega_{\mathbb{P}^n}^p(\log X_\psi) = \wedge^p \Omega_{\mathbb{P}^n}(\log X_\psi)$. Then we have $\Omega/F_\psi \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^n(\log X_\psi))$. Let

$$\theta_0(\psi) = \text{Res}_{X_\psi} \frac{\Omega}{F_\psi} \in H^0(X_\psi, \Omega_{X_\psi}^{n-1})$$

be its Poincaré residue.

Define the polynomial $H(z_0, \dots, z_n)$ by

$$H(z_0, \dots, z_n) = z_0 \dots z_n.$$

Then $H^k \Omega / F_\psi^{k+1}$ is a holomorphic n -form on $V := \mathbb{P}^n \setminus X_\psi$. We denote by

$$\theta_k(\psi) = \text{Res}_{X_\psi} \frac{k! H^k \Omega}{F_\psi^{k+1}} \in H^{n-1}(X_\psi, \mathbb{C})$$

the image of the homomorphism $H^n(V, \mathbb{C}) \rightarrow H^{n-1}(X_\psi, \mathbb{C})$ induced from the Gysin exact sequence. Let $F^\bullet$ be the Hodge filtration of the cohomology $H^{n-1}(X_\psi, \mathbb{C})$ (cf. [80, p.158]). Let $H^{n-1}(X_\psi, \mathbb{C})_{\text{prim}}^G$ be the G -invariant subspace of the primitive cohomology $H^{n-1}(X_\psi, \mathbb{C})_{\text{prim}}$ with respect to the polarization $\mathcal{O}_{\mathbb{P}^n}(1)$ on X_ψ .

By [26, Lemma 3.9], we have

$$\theta_k(\psi) = \text{Res}_{X_\psi} \frac{k! H^k \Omega}{F_\psi^{k+1}} \in F^{n-1-k} H^{n-1}(X_\psi, \mathbb{C})_{\text{prim}}^G \setminus F^{n-k} H^{n-1}(X_\psi, \mathbb{C})_{\text{prim}}^G$$

for each $k = 0, \dots, n-1$. By [26, Lemma 3.9], $H^{n-1-k,k}(X_\psi)_{\text{prim}}^G$ is one-dimensional and is generated by the image of $\theta_k(\psi)$ under the canonical isomorphism

$$H^{n-1-k,k}(X_\psi)_{\text{prim}}^G \cong \frac{F^{n-1-k} H^{n-1}(X_\psi, \mathbb{C})_{\text{prim}}^G}{F^{n-k} H^{n-1}(X_\psi, \mathbb{C})_{\text{prim}}^G}.$$

Let $\eta_k^\circ \in H^{n-1-k,k}(X_\psi)_{\text{prim}}^G$ be the image of $\theta_k(\psi)$.

Definition 4.3.2. Define $\eta_k(\psi) \in H^{n-1-k,k}(X_\psi)_{\text{prim}}^G$ by

$$\eta_k(\psi) = \psi^{k+1} \eta_k^\circ(\psi).$$

4.3.3 The explicit formula of Eriksson-Freixas i Montplet-Mourougane

Let χ be the topological Euler characteristic of Z_ψ . By [26, Lemma 3.4], we have

$$\chi = (-1)^{n-1} \left(\frac{(-n)^{n+1} - 1}{n+1} + n+1 \right).$$

Set

$$a = (-1)^{n-1} \frac{n(n-1)}{6} - \frac{\chi}{12(n+1)}, \quad b = (-1)^{n-1} \frac{n(3n-5)}{24}.$$

Theorem 4.3.3. *There exists a constant C_n depending only on n such that the following identity of functions on $\mathbb{C} \setminus \mu_{n+1}$ holds:*

$$\tau_{BCOV}(Z_\psi) = C_n \left| \frac{(\psi^{n+1})^a}{(1 - \psi^{n+1})^b} \right|^2 \frac{\|\eta_0(\psi)\|_{L^2}^{\frac{\chi}{6}}}{\left(\prod_{k=0}^{n-1} \|\eta_k(\psi)\|_{L^2}^{2(n-1-k)} \right)^{(-1)^{n-1}}},$$

where $\|\cdot\|_{L^2}$ is the L^2 -metric on $H^{p,q}(X_\psi)$ with respect to the polarization $\mathcal{O}_{\mathbb{P}^n}(1)|_{X_\psi}$.

Proof. See [26, Theorem 5.1]. □

Remark 4.3.4. In [26, Definition 3.15], Eriksson-Freixas i Montplet-Mourougane define $\eta_k(\psi)$ as a cohomology class of $H^{n-1-k,k}(Z_\psi)_{prim}$. If we regard $H^{n-1}(X_\psi, \mathbb{Q})_{prim}^G$ as a subspace of $H^{n-1}(Z_\psi, \mathbb{Q})_{prim}$ via the injection $H^{n-1}(X_\psi, \mathbb{Q})_{prim}^G \hookrightarrow H^{n-1}(Z_\psi, \mathbb{Q})_{prim}$ in [26, Lemma 3.11], our cohomology class $\eta_k(\psi)$ coincides with the one of Eriksson-Freixas i Montplet-Mourougane, up to a constant which is independent of ψ . Since the L^2 -metrics on $H^{n-1-k,k}(X_\psi)_{prim}^G$ and $H^{n-1-k,k}(Z_\psi)_{prim}$ are the intersection pairings, the L^2 -norm $\|\eta_k(\psi)\|_{L^2}$ in Theorem 4.3.3 coincides with the one in [26, Theorem 5.1], up to a universal constant depending only on n, k by [26, Proposition 3.7]. Therefore, the universal constant C_n in Theorem 4.3.3 is different from the one in [26, Theorem 5.1].

By using Theorem 4.3.3, Eriksson-Freixas i Montplet-Mourougane proved the mirror symmetry conjecture at genus one for the Calabi-Yau hypersurface of $\mathbb{P}^n$ (cf. [26, Theorem 5.9]).

Chapter 5

An invariant of $K3^{[2]}$ -type manifolds with involution

In this chapter, we construct an invariant of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$ and prove the curvature formula for the invariant.

Recall that L_2 is the Beauville-Bogomolov-Fujiki lattice for $K3^{[2]}$ -type manifolds (cf. Section 2.1.1). Throughout this chapter, we fix an admissible sublattice M of L_2 and a Kähler-type chamber $\mathcal{K} \in \text{KT}(M)$.

5.1 Some fundamental properties of hyperkähler manifolds

Let (X, ι) be a manifold of $K3^{[2]}$ -type with involution of type $(M, \mathcal{K})$. Choose $\eta \in H^0(X, \Omega_X^2)$ and $\theta \in H^2(X, \mathcal{O}_X)$.

Let $h_{X,0}$ be an ι -invariant Ricci-flat Kähler metric on X with Kähler form $\omega_{X,0}$. It is locally written as

$$\omega_{X,0} = \frac{i}{2} \sum_{j,k} h_{X,0} \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k} \right) dz^j \wedge d\bar{z}^k.$$

The Riemannian metric associated with $h_{X,0}$ is denoted by g . The hermitian metric on $\wedge^{p,q} T^*X$ attached to the Ricci-flat Kähler metric is also denoted by $h_{X,0}$.

Let I be the complex structure of X . Since (X, g) is hyperkähler, there are complex structures J and K of X such that (X, I, g) , (X, J, g) and (X, K, g) are manifolds of $K3^{[2]}$ -type and $IJ = -JI = K$. The Kähler forms with respect to J and K are given by

$$\omega_J = g(-, J(-)), \quad \text{and} \quad \omega_K = g(-, K(-)),$$

respectively. Set $\sigma_I = \omega_J + i\omega_K$. This is a holomorphic 2-form on X (cf. [36, §23]). Since $H^0(X, \Omega_X^2) = \mathbb{C}\eta$, there exists a complex number $\lambda \in \mathbb{C}$ such that $\eta = \frac{\lambda}{2}\sigma_I$. Note

that the L^2 -norm of η^2 is given by

$$\|\eta^2\|_{L^2}^2 = \int_X h_{X,0}(\eta^2, \eta^2) \frac{\omega_{X,0}^4}{4!} = \int_X \eta^2 \wedge \bar{\eta}^2,$$

and the volume of $(X, \omega_{X,0})$ is defined by

$$\text{Vol}(X, \omega_{X,0}) = \int_X \frac{\omega_{X,0}^4}{4!}.$$

Since $\omega_{X,0}$ is Ricci-flat, it follows from [36, Corollary 23.9] that

$$|\lambda|^2 = \frac{1}{2} \left(\frac{\eta^2 \wedge \bar{\eta}^2}{\omega_{X,0}^4/4!} \right)^{\frac{1}{2}} = \frac{1}{2} \left(\frac{\|\eta^2\|_{L^2}^2}{\text{Vol}(X, \omega_{X,0})} \right)^{\frac{1}{2}}. \quad (5.1.1)$$

Similarly, there exists a complex number $\mu \in \mathbb{C}$ such that $\theta = \frac{\mu}{2} \bar{\sigma}_I$ in $H^2(X, \mathcal{O}_X)$. We identify the cohomology class θ with its harmonic representative. The L^2 -norm of θ^2 is given by

$$\|\theta^2\|_{L^2}^2 = \int_X h_{X,0}(\theta^2, \theta^2) \frac{\omega_{X,0}^4}{4!} = \int_X \theta^2 \wedge \bar{\theta}^2,$$

and we have

$$|\mu|^2 = \frac{1}{2} \left(\frac{\|\theta^2\|_{L^2}^2}{\text{Vol}(X, \omega_{X,0})} \right)^{\frac{1}{2}}. \quad (5.1.2)$$

Lemma 5.1.1. *For any $\alpha \in A^{1,1}(X)$, the following identity holds:*

$$h_{X,0}(\theta \wedge \alpha, \theta \wedge \alpha) = |\mu|^2 h_{X,0}(\alpha, \alpha).$$

Proof. Fix $p \in X$. Since (X, g) is hyperkähler, the real tangent space $T_{p,\mathbb{R}}X$ at p is equipped with the structure of a quaternionic hermitian vector space (see [36, §23.2]). Therefore, there exist two tangent vector $e, f \in T_{p,\mathbb{R}}X$ such that

$$e, Ie, Je, Ke, f, If, Jf, Kf$$

form an orthonormal basis of $(T_{p,\mathbb{R}}X, g)$. Set

$$v_1 = \frac{e - iIe}{2}, \quad v_2 = \frac{-Ke - iJe}{2}, \quad v_3 = \frac{f - iIf}{2}, \quad v_4 = \frac{-Kf - iJf}{2}.$$

Then v_1, v_2, v_3, v_4 are of type $(1, 0)$ with respect to the complex structure I and form a $\mathbb{C}$ -basis of the holomorphic tangent space T_pX at p . Moreover, we have

$$h_{X,0}(v_i, v_j) = \frac{1}{2} \delta_{ij}.$$

Let v^1, v^2, v^3, v^4 be the dual basis. Then we have

$$h_{X,0}(v^i, v^j) = 2\delta_{ij}, \quad \text{and} \quad \sigma_I = iv^1 \wedge v^2 + iv^3 \wedge v^4.$$

Let $\alpha = \frac{1}{2} \sum_{i,j} \alpha_{ij} v^i \wedge \bar{v}^j \in \wedge^{1,1} T_p^* X$. Then we have

$$h_{X,0}(\bar{\sigma}_I \wedge \alpha, \bar{\sigma}_I \wedge \alpha) = 4 \sum_{i,j} |\alpha_{ij}|^2 = 4h_{X,0}(\alpha, \alpha),$$

which completes the proof. $\square$

5.2 Construction of an invariant

We now turn to the case where the ι -invariant Kähler metric h_X is not necessarily Ricci-flat. Its associated Kähler form is denoted by ω_X .

Let $\tau_\iota(\bar{\Omega}_X^1)$ be the equivariant analytic torsion of the cotangent bundle $\bar{\Omega}_X^1 = (\Omega_X^1, h_X)$ endowed with the hermitian metric induced from h_X .

Recall that X^ι is a possibly disconnected compact complex surface. Let $X^\iota = \sqcup_i Z_i$ be the decomposition into the connected components. Let $\tau(\bar{\mathcal{O}}_{X^\iota})$ be the analytic torsion of the trivial bundle $\bar{\mathcal{O}}_{X^\iota}$ with respect to the canonical metric. Here $\tau(\bar{\mathcal{O}}_{X^\iota})$ is given by

$$\tau(\bar{\mathcal{O}}_{X^\iota}) = \prod_i \tau(\bar{\mathcal{O}}_{Z_i}),$$

where $\tau(\bar{\mathcal{O}}_{Z_i})$ is the analytic torsion of the trivial bundle $\bar{\mathcal{O}}_{Z_i}$ with respect to the canonical metric.

The volume of (X, ω_X) is defined by

$$\text{Vol}(X, \omega_X) = \int_X \frac{\omega_X^4}{4!}.$$

Set $\omega_{X^\iota} = \omega_X|_{X^\iota}$. This is a Kähler form on X^ι attached to $h_{X^\iota} = h_X|_{X^\iota}$. We define the volume of $(X^\iota, \omega_{X^\iota})$ by

$$\text{Vol}(X^\iota, \omega_{X^\iota}) = \prod_i \text{Vol}(Z_i, \omega_X|_{Z_i}) = \prod_i \int_{Z_i} \frac{(\omega_X|_{Z_i})^2}{2!}.$$

The covolume of the lattice $\text{Im}(H^1(X^\iota, \mathbb{Z}) \rightarrow H^1(X^\iota, \mathbb{R}))$ with respect to the L^2 -metric induced from h_X is denoted by $\text{Vol}_{L^2}(H^1(X^\iota, \mathbb{Z}), \omega_{X^\iota})$. Namely,

$$\text{Vol}_{L^2}(H^1(X^\iota, \mathbb{Z}), \omega_{X^\iota}) = \det(\langle e_i, e_j \rangle_{L^2}),$$

where $e_1, \dots, e_{b_1(X^\iota)}$ is an integral basis of $\text{Im}(H^1(X^\iota, \mathbb{Z}) \rightarrow H^1(X^\iota, \mathbb{R}))$.

We define a real-valued function φ on X^ι by

$$\varphi = \frac{\omega_X^4/4!}{\eta^2 \wedge \bar{\eta}^2 \text{Vol}(X, \omega_X)} \Big|_{X^\iota}.$$

Obviously, φ is independent of the choice of η .

Definition 5.2.1. We define a positive number $A(X, \iota, h_X) \in \mathbb{R}_{>0}$ by

$$A(X, \iota, h_X) = \exp \left[\frac{1}{48} \int_{X^\iota} (\log \varphi) \Omega \right], \quad (5.2.1)$$

where Ω is a characteristic form on X^ι defined by

$$\Omega = c_1(TX^\iota, h_{X^\iota})^2 - 8c_2(TX^\iota, h_{X^\iota}) - c_1(TX, h_X)|_{X^\iota}^2 + 3c_2(TX, h_X)|_{X^\iota}.$$

Here we denote by $c_i(TX, h_X)$, $c_i(TX^\iota, h_{X^\iota})$ the i -th Chern form of the hermitian holomorphic vector bundles (TX, h_X) , (TX^ι, h_{X^ι}) , respectively. Note that if h_X is Ricci-flat, then we have $\varphi = 1$ and $A(X, \iota, h_X) = 1$.

Recall that $t = \text{Tr}(\iota^*|_{H^{1,1}(X)})$. By the definition of the admissible sublattice (M, ι_M) , we have $t = \text{Tr}(\iota_M) + 2$. Therefore t depends only on (M, ι_M) and is independent of (X, ι) itself.

Definition 5.2.2. We define a real number $\tau_{M, \mathcal{K}}(X, \iota)$ by

$$\begin{aligned} \tau_{M, \mathcal{K}}(X, \iota) &= \tau_\iota(\bar{\Omega}_X^1) \text{Vol}(X, \omega_X)^{\frac{(t-1)(t-7)}{16}} A(X, \iota, h_X) \\ &\quad \cdot \tau(\bar{\mathcal{O}}_{X^\iota})^{-2} \text{Vol}(X^\iota, \omega_{X^\iota})^{-2} \text{Vol}_{L^2}(H^1(X^\iota, \mathbb{Z}), \omega_{X^\iota}). \end{aligned}$$

In the next section, we prove that $\tau_{M, \mathcal{K}}(X, \iota)$ is an invariant of (X, ι) , as well as its variational formula. We also prove that $\tau_{M, \mathcal{K}}$ can be viewed as a function on the orthogonal modular variety $\mathcal{M}_{M, \mathcal{K}}^\circ$ introduced in Section 2.2.3.

5.3 Properties of $\tau_{M, \mathcal{K}}$

Let $\mathcal{X}$ be a complex manifold with holomorphic involution $\iota : \mathcal{X} \rightarrow \mathcal{X}$, let S be a complex manifold, and let $f : (\mathcal{X}, \iota) \rightarrow S$ be a family of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$.

The fixed locus of $\iota : \mathcal{X} \rightarrow \mathcal{X}$ is denoted by $\mathcal{X}^\iota = \{x \in \mathcal{X}; \iota(x) = x\}$. The restriction of $f : \mathcal{X} \rightarrow S$ to $\mathcal{X}^\iota$ is also denoted by $f : \mathcal{X}^\iota \rightarrow S$ and it is a family of smooth complex surfaces.

Let $h_{\mathcal{X}/S}$ be an ι -invariant fiberwise Kähler metric on $T\mathcal{X}/S$. For $s \in S$, the Kähler form associated with the Kähler metric $h_s = h_{\mathcal{X}/S}|_{X_s}$ is denoted by ω_s . We set $\omega_{\mathcal{X}/S} = \{\omega_s\}_{s \in S}$. Let $h_{\mathcal{X}^\iota/S}$ and h_N be the induced and quotient metric on $T\mathcal{X}^\iota/S$ and $N_{\mathcal{X}^\iota/\mathcal{X}}$, respectively. The ι -invariant hermitian metric on $\Omega_{\mathcal{X}/S}^1$ induced from $h_{\mathcal{X}/S}$ is also denoted by $h_{\mathcal{X}/S}$.

Let $E(\pm 1)$ be the (± 1) -eigenbundle of the μ_2 -action of $T\mathcal{X}/S|_{\mathcal{X}^\iota}$. They are holomorphic vector bundles on $\mathcal{X}^\iota$, and the decomposition $T\mathcal{X}/S|_{\mathcal{X}^\iota} = E(+1) \oplus E(-1)$ is orthogonal with respect to the metric $h_{\mathcal{X}/S}$. The restriction of $h_{\mathcal{X}/S}$ to $E(\pm 1)$ is denoted by $h_\pm$. Then we have two isometries

$$(T\mathcal{X}^\iota/S, h_{\mathcal{X}^\iota/S}) \cong (E(+1), h_+) \quad \text{and} \quad (N_{\mathcal{X}^\iota/\mathcal{X}}, h_N) \cong (E(-1), h_-). \quad (5.3.1)$$

Hence the following short exact sequence of holomorphic hermitian vector bundles splits

$$0 \rightarrow (T\mathcal{X}^\iota/S, h_{\mathcal{X}^\iota/S}) \rightarrow (T\mathcal{X}/S, h_{\mathcal{X}/S})|_{\mathcal{X}^\iota} \rightarrow (N_{\mathcal{X}^\iota/\mathcal{X}}, h_N) \rightarrow 0. \quad (5.3.2)$$

We set

$$\bar{T}\mathcal{X}^\iota/S = (T\mathcal{X}^\iota/S, h_{\mathcal{X}^\iota/S}), \bar{T}\mathcal{X}/S = (T\mathcal{X}/S, h_{\mathcal{X}/S}), \text{ and } \bar{N}_{\mathcal{X}^\iota/\mathcal{X}} = (N_{\mathcal{X}^\iota/\mathcal{X}}, h_N).$$

Let

$$c_i(\bar{T}\mathcal{X}^\iota/S), \quad c_i(\bar{T}\mathcal{X}/S), \quad \text{and} \quad c_i(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}})$$

be their Chern forms, respectively.

Lemma 5.3.1. *The following identities hold:*

$$c_1(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}}) = -c_1(\bar{T}\mathcal{X}^\iota/S) + c_1(\bar{T}\mathcal{X}/S)|_{\mathcal{X}^\iota},$$

$$c_2(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}}) = c_1(\bar{T}\mathcal{X}^\iota/S)^2 - c_2(\bar{T}\mathcal{X}^\iota/S) - c_1(\bar{T}\mathcal{X}^\iota/S)c_1(\bar{T}\mathcal{X}/S)|_{\mathcal{X}^\iota} + c_2(\bar{T}\mathcal{X}/S)|_{\mathcal{X}^\iota}.$$

Proof. Since the short exact sequence (5.3.2) splits, we have

$$c(\bar{T}\mathcal{X}/S)|_{\mathcal{X}^\iota} = c(\bar{T}\mathcal{X}^\iota/S)c(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}}),$$

and this implies the statement. $\square$

We define a characteristic form $\Omega \in A^{2,2}(\mathcal{X}^\iota)$ by

$$\Omega = c_1(\bar{T}\mathcal{X}^\iota/S)^2 - 8c_2(\bar{T}\mathcal{X}^\iota/S) - c_1(\bar{T}\mathcal{X}/S)|_{\mathcal{X}^\iota}^2 + 3c_2(\bar{T}\mathcal{X}/S)|_{\mathcal{X}^\iota}.$$

Lemma 5.3.2. *The following identities holds in $A^{3,3}(\mathcal{X}^\iota)$:*

$$\left[Td_\iota(\bar{T}\mathcal{X}/S)ch_\iota(\bar{\Omega}_{\mathcal{X}/S}^1) \right]^{(3,3)} = 2 \left[Td(\bar{T}\mathcal{X}^\iota/S) \right]^{(3,3)} + \frac{1}{48}c_1(\bar{T}\mathcal{X}/S)|_{\mathcal{X}^\iota} \wedge \Omega.$$

Proof. By the construction of the equivariant Todd form (1.2.2) and by the isometries (5.3.1), we have

$$Td_\iota(\bar{T}\mathcal{X}/S) = Td(\bar{T}\mathcal{X}^\iota/S) \det \left(\frac{I}{I + \exp(+\frac{R_-}{2\pi i})} \right), \quad (5.3.3)$$

where R_- is the curvature form of $(N_{\mathcal{X}^\iota/\mathcal{X}}, h_N)$. By the isometries (5.3.1), $\Omega_{\mathcal{X}^\iota/S}^1$ is the $(+1)$ -eigenbundle of $\Omega_{\mathcal{X}/S}^1|_{\mathcal{X}^\iota}$ and $N_{\mathcal{X}^\iota/\mathcal{X}}^\vee$ is the (-1) -eigenbundle of $\Omega_{\mathcal{X}/S}^1|_{\mathcal{X}^\iota}$. By the construction of the equivariant Chern character form (1.2.3), we have

$$ch_\iota(\bar{\Omega}_{\mathcal{X}/S}^1) = ch(\bar{\Omega}_{\mathcal{X}^\iota/S}^1) - ch(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}}^\vee). \quad (5.3.4)$$

By the definition of Todd form and Chern form, we have

$$\begin{aligned} Td(\bar{T}\mathcal{X}'/S) &= 1 + \frac{1}{2}c_1(\bar{T}\mathcal{X}'/S) + \frac{1}{12}\{c_1(\bar{T}\mathcal{X}'/S)^2 + c_2(\bar{T}\mathcal{X}'/S)\} \\ &\quad + \frac{1}{24}c_1(\bar{T}\mathcal{X}'/S)c_2(\bar{T}\mathcal{X}'/S) + \text{higher degree terms.} \end{aligned} \quad (5.3.5)$$

Since $\frac{1}{1+e^{-x}} = \frac{1}{2} + \frac{x}{4} + 0x^2 + \dots$, the following identity of functions of 2×2 matrices holds:

$$\det\left(\frac{I}{I + \exp(-A)}\right) = \frac{1}{4} + \frac{1}{8}c_1(A) + \frac{1}{16}c_2(A) + \text{higher degree terms.}$$

Therefore, we have

$$\det\left(\frac{I}{I + \exp(+\frac{R_-}{2\pi i})}\right) = \frac{1}{4} + \frac{1}{8}c_1(\bar{N}_{\mathcal{X}'/\mathcal{X}}) + \frac{1}{16}c_2(\bar{N}_{\mathcal{X}'/\mathcal{X}}) + \text{higher degree terms.} \quad (5.3.6)$$

By (5.3.3), (5.3.5), (5.3.6), we have

$$\begin{aligned} [Td_\iota(\bar{T}\mathcal{X}/S)]^{(0,0)} &= \frac{1}{4}, \\ [Td_\iota(\bar{T}\mathcal{X}/S)]^{(1,1)} &= \frac{1}{8}\{c_1(\bar{T}\mathcal{X}'/S) + c_1(\bar{N}_{\mathcal{X}'/\mathcal{X}})\}, \\ [Td_\iota(\bar{T}\mathcal{X}/S)]^{(2,2)} &= \frac{1}{48}\{c_1(\bar{T}\mathcal{X}'/S)^2 + c_2(\bar{T}\mathcal{X}'/S) + 3c_1(\bar{T}\mathcal{X}'/S)c_1(\bar{N}_{\mathcal{X}'/\mathcal{X}}) + 3c_2(\bar{N}_{\mathcal{X}'/\mathcal{X}})\}. \end{aligned} \quad (5.3.7)$$

On the other hand, we have

$$\begin{aligned} ch(\bar{\Omega}_{\mathcal{X}'/S}^1) &= 2 - c_1(\bar{T}\mathcal{X}'/S) + \frac{1}{2}\{c_1(\bar{T}\mathcal{X}'/S)^2 - 2c_2(\bar{T}\mathcal{X}'/S)\} \\ &\quad - \frac{1}{6}\{c_1(\bar{T}\mathcal{X}'/S)^3 - 3c_1(\bar{T}\mathcal{X}'/S)c_2(\bar{T}\mathcal{X}'/S)\} + \text{higher degree terms,} \end{aligned} \quad (5.3.8)$$

and

$$\begin{aligned} ch(\bar{N}_{\mathcal{X}'/\mathcal{X}}^\vee) &= 2 - c_1(\bar{N}_{\mathcal{X}'/\mathcal{X}}) + \frac{1}{2}\{c_1(\bar{N}_{\mathcal{X}'/\mathcal{X}})^2 - 2c_2(\bar{N}_{\mathcal{X}'/\mathcal{X}})\} \\ &\quad - \frac{1}{6}\{c_1(\bar{N}_{\mathcal{X}'/\mathcal{X}})^3 - 3c_1(\bar{N}_{\mathcal{X}'/\mathcal{X}})c_2(\bar{N}_{\mathcal{X}'/\mathcal{X}})\} + \text{higher degree terms} \end{aligned} \quad (5.3.9)$$

By (5.3.4), (5.3.8), (5.3.9), we have

$$\begin{aligned}
[ch_\iota(\bar{\Omega}_{\mathcal{X}/S}^1)]^{(0,0)} &= 0, \\
[ch_\iota(\bar{\Omega}_{\mathcal{X}/S}^1)]^{(1,1)} &= -c_1(\bar{T}\mathcal{X}^\iota/S) + c_1(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}}), \\
[ch_\iota(\bar{\Omega}_{\mathcal{X}/S}^1)]^{(2,2)} &= \frac{1}{2}\{c_1(\bar{T}\mathcal{X}^\iota/S)^2 - 2c_2(\bar{T}\mathcal{X}^\iota/S) - c_1(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}})^2 + 2c_2(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}})\}, \\
[ch_\iota(\bar{\Omega}_{\mathcal{X}/S}^1)]^{(3,3)} &= -\frac{1}{6}\{c_1(\bar{T}\mathcal{X}^\iota/S)^3 - 3c_1(\bar{T}\mathcal{X}^\iota/S)c_2(\bar{T}\mathcal{X}^\iota/S) \\
&\quad - c_1(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}})^3 + 3c_1(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}})c_2(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}})\}.
\end{aligned} \tag{5.3.10}$$

Combining the formulas (5.3.7) and (5.3.10), we obtain

$$\begin{aligned}
&\left[Td_\iota(\bar{T}\mathcal{X}/S)ch_\iota(\bar{\Omega}_{\mathcal{X}/S}^1)\right]^{(3,3)} \\
&= \frac{1}{48}\{c_1(\bar{T}\mathcal{X}^\iota/S)^2c_1(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}}) - c_1(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}})^3 - c_1(\bar{T}\mathcal{X}^\iota/S)c_2(\bar{T}\mathcal{X}^\iota/S) \\
&\quad + 3c_1(\bar{T}\mathcal{X}^\iota/S)c_2(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}}) + 3c_1(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}})c_2(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}}) - 5c_1(\bar{N}_{\mathcal{X}^\iota/\mathcal{X}})c_2(\bar{T}\mathcal{X}^\iota/S)\}.
\end{aligned}$$

By Lemma 5.3.1 and by (5.3.5), we obtain the desired formula. $\square$

Consider the direct image sheaf $f_*K_{\mathcal{X}/S}$ of the relative canonical bundle $K_{\mathcal{X}/S}$. This is a holomorphic line bundle on S and is equipped with the L^2 -metric h_{L^2} . On the other hand, $K_{\mathcal{X}/S}$ is equipped with the hermitian metric h induced from the fiberwise Kähler metric $h_{\mathcal{X}/S}$. We define a smooth function φ on S by

$$\varphi = \frac{\omega_{\mathcal{X}/S}^4/4!}{\eta^2 \wedge \bar{\eta}^2 \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S})} \|\eta\|_{L^2}^2,$$

where η is a locally defined nowhere-vanishing holomorphic section of $f_*\Omega_{\mathcal{X}/S}^2$. Then φ is independent of the choice of η . The evaluation map $f^*f_*K_{\mathcal{X}/S} \rightarrow K_{\mathcal{X}/S}$ is an isomorphism and we have

$$c_1(K_{\mathcal{X}/S}, h) = f^*c_1(f_*K_{\mathcal{X}/S}, h_{L^2}) + dd^c \log \varphi + dd^c \log \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S}) \tag{5.3.11}$$

in $A^{1,1}(\mathcal{X}^\iota)$. We define a function $A(\mathcal{X}/S)$ on S by

$$A(\mathcal{X}/S)(s) = A(X_s, \iota_s, h_s) \quad (s \in S).$$

Lemma 5.3.3. *The following identity holds:*

$$\int_{\mathcal{X}^\iota/S} \Omega = -3(t^2 + 7),$$

where $\int_{\mathcal{X}^\iota/S}$ denotes the integration along the fibers of $f : \mathcal{X}^\iota \rightarrow S$.

Proof. By Theorem 2.2.5, we have

$$\int_{\mathcal{X}^\iota/S} c_1(T\mathcal{X}^\iota/S)^2 = t^2 - 1, \quad \int_{\mathcal{X}^\iota/S} c_2(T\mathcal{X}^\iota/S) = \frac{t^2 + 23}{2}. \quad (5.3.12)$$

Since the canonical bundle of an irreducible holomorphic symplectic manifold is trivial, we have

$$\int_{\mathcal{X}^\iota/S} c_1(T\mathcal{X}/S)|_{\mathcal{X}^\iota}^2 = 0. \quad (5.3.13)$$

Since a holomorphic symplectic form induces an isomorphism $N_{\mathcal{X}^\iota/\mathcal{X}} \cong \Omega_{\mathcal{X}^\iota/S}^1$, we have $c_1(N_{\mathcal{X}^\iota/\mathcal{X}}) = -c_1(T\mathcal{X}^\iota/S)$. By Lemma 5.3.1, we have $c_2(T\mathcal{X}/S)|_{\mathcal{X}^\iota} = 2c_2(T\mathcal{X}^\iota/S) - c_1(T\mathcal{X}^\iota/S)^2$. Hence

$$\int_{\mathcal{X}^\iota/S} c_2(T\mathcal{X}/S)|_{\mathcal{X}^\iota} = 24. \quad (5.3.14)$$

By (5.3.12), (5.3.13), (5.3.14), we have the desired result. $\square$

In the next lemma, we write f_* for the integration along the fibers of $f : \mathcal{X}^\iota \rightarrow S$ instead of $\int_{\mathcal{X}^\iota/S}$.

Proposition 5.3.4. *The following identity holds:*

$$\begin{aligned} & \left[f_* \left(Td_\iota(\overline{T}\mathcal{X}/S) ch_\iota(\overline{\Omega}_{\mathcal{X}/S}^1) \right) \right]^{(1,1)} \\ &= 2 \left[f_* Td(\overline{T}\mathcal{X}^\iota/S) \right]^{(1,1)} - dd^c \log A(\mathcal{X}/S) \\ & \quad + \frac{t^2 + 7}{16} c_1(f_* K_{\mathcal{X}/S}, h_{L^2}) + \frac{t^2 + 7}{16} dd^c \log \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S}). \end{aligned}$$

Proof. By Lemma 5.3.2 and by the formula (5.3.11), we have

$$\begin{aligned} \left[Td_\iota(\overline{T}\mathcal{X}/S) ch_\iota(\overline{\Omega}_{\mathcal{X}/S}^1) \right]^{(3,3)} &= 2 \left[Td(\overline{T}\mathcal{X}^\iota/S) \right]^{(3,3)} - \frac{1}{48} f_* c_1(f_* K_{\mathcal{X}/S}, h_{L^2})|_{\mathcal{X}^\iota} \wedge \Omega \\ & \quad - \frac{1}{48} dd^c(\log \varphi) \Omega - \frac{1}{48} dd^c(\log \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S})) \Omega. \end{aligned}$$

By the projection formula, we have

$$\begin{aligned} \left[f_* \left(Td_\iota(\overline{T}\mathcal{X}/S) ch_\iota(\overline{\Omega}_{\mathcal{X}/S}^1) \right) \right]^{(1,1)} &= 2 \left[f_* Td(\overline{T}\mathcal{X}^\iota/S) \right]^{(1,1)} - \frac{1}{48} \int_{\mathcal{X}^\iota/S} dd^c(\log \varphi) \Omega \\ & \quad - \frac{1}{48} \left(\int_{\mathcal{X}^\iota/S} \Omega \right) c_1(f_* K_{\mathcal{X}/S}, h_{L^2}) - \frac{1}{48} \left(\int_{\mathcal{X}^\iota/S} \Omega \right) dd^c \log \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S}). \end{aligned}$$

By Lemma 5.3.3, the proof is completed. $\square$

Recall that each fiber X of $f : \mathcal{X} \rightarrow S$ is a manifold of $K3^{[2]}$ -type. By [37, Main Theorem], we have

$$h^{1,q}(X) = \begin{cases} 0 & (q = 0, 2, 4) \\ 21 & (q = 1, 3) \end{cases} \quad (5.3.15)$$

By [36, Proposition 24.1], the homomorphism

$$H^1(X, \Omega_X^1) \otimes H^2(X, \mathcal{O}_X) \rightarrow H^3(X, \Omega_X^1), \quad \alpha \otimes \theta \mapsto \alpha \wedge \theta$$

is an isomorphism. Therefore we have an isomorphism of holomorphic vector bundles

$$R^1 f_* \Omega_{\mathcal{X}/S}^1 \otimes R^2 f_* \mathcal{O}_{\mathcal{X}} \cong R^3 f_* \Omega_{\mathcal{X}/S}^1. \quad (5.3.16)$$

For $q = 1, 3$, let $(R^q f_* \Omega_{\mathcal{X}/S}^1)_{\pm}$ be the (± 1) -eigenbundle of the μ_2 -action of $R^q f_* \Omega_{\mathcal{X}/S}^1$. By the definition of t , we have

$$\text{rank}(R^1 f_* \Omega_{\mathcal{X}/S}^1)_{\pm} = \text{rank}(R^3 f_* \Omega_{\mathcal{X}/S}^1)_{\mp} = \frac{21 \pm t}{2}.$$

Since ι is fiberwise antisymplectic, we deduce from (5.3.16) the following isomorphism

$$(R^1 f_* \Omega_{\mathcal{X}/S}^1)_{\pm} \otimes R^2 f_* \mathcal{O}_{\mathcal{X}} \rightarrow (R^3 f_* \Omega_{\mathcal{X}/S}^1)_{\mp}.$$

Lemma 5.3.5. *The following identity holds:*

$$\begin{aligned} & c_1((R^1 f_* \Omega_{\mathcal{X}/S}^1)_{\pm}, h_{L^2}) - c_1((R^3 f_* \Omega_{\mathcal{X}/S}^1)_{\mp}, h_{L^2}) \\ &= -\frac{21 \pm t}{4} c_1(R^4 f_* \mathcal{O}_{\mathcal{X}}, h_{L^2}) - \frac{21 \pm t}{4} dd^c \log \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S}). \end{aligned}$$

Proof. We only prove the identity

$$\begin{aligned} & c_1((R^1 f_* \Omega_{\mathcal{X}/S}^1)_{+}, h_{L^2}) - c_1((R^3 f_* \Omega_{\mathcal{X}/S}^1)_{-}, h_{L^2}) \\ &= -\frac{21 + t}{4} c_1(R^4 f_* \mathcal{O}_{\mathcal{X}}, h_{L^2}) - \frac{21 + t}{4} dd^c \log \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S}). \end{aligned}$$

The other identity can be shown in the same manner. Set $N = \frac{21+t}{2}$. Let $s_1, \dots, s_N$ be a local holomorphic frame of $(R^1 f_* \Omega_{\mathcal{X}/S}^1)_{+}$. Then $s_1 \wedge \dots \wedge s_N$ is a nowhere vanishing holomorphic section of $\det(R^1 f_* \Omega_{\mathcal{X}/S}^1)_{+}$. Let θ be a nowhere vanishing holomorphic section of $R^2 f_* \mathcal{O}_{\mathcal{X}}$. Then $s_1 \wedge \theta, \dots, s_N \wedge \theta$ are local holomorphic frame of $(R^3 f_* \Omega_{\mathcal{X}/S}^1)_{-}$ and $(s_1 \wedge \theta) \wedge \dots \wedge (s_N \wedge \theta)$ is a nowhere vanishing holomorphic section of $\det(R^3 f_* \Omega_{\mathcal{X}/S}^1)_{-}$.

It suffices to show that

$$\begin{aligned} & -\log \|(s_1 \wedge \theta) \wedge \dots \wedge (s_N \wedge \theta)\|_{L^2}^2 \\ &= -\frac{N}{2} \log \|\theta\|_{L^2}^2 - \log \|s_1 \wedge \dots \wedge s_N\|_{L^2}^2 + \log \left(2^N \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S})^{\frac{N}{2}} \right). \end{aligned} \quad (5.3.17)$$

We may assume that S consists of one point, and we set $X = \mathcal{X}$.

Since the L^2 -metrics on $H^q(X, \Omega_X^1)$ ($q = 1, 3$) and $H^2(X, \mathcal{O}_X)$ depend only on the choice of the Kähler class $[\omega_X]$ and are independent of the Kähler form ω_X itself (see Lemma 5.4.2 below), we may assume that $\omega_X = \omega_{X,0}$ is Ricci-flat.

By Lemma 5.1.1, we have

$$h_{X,0}(\theta \wedge \alpha, \theta \wedge \alpha) = |\mu|^2 h_{X,0}(\alpha, \alpha)$$

for each $\alpha \in A^{1,1}(X)$. By integrating both sides, we have

$$\|\theta \wedge \alpha\|_{L^2}^2 = |\mu|^2 \|\alpha\|_{L^2}^2.$$

Therefore we have an isometry

$$(\det H^1(X, \Omega_X^1)_+, |\mu|^{2N} \det h_{L^2}) \cong (\det H^3(X, \Omega_X^1)_-, \det h_{L^2}).$$

By the formula (5.1.2), we obtain the formula (5.3.17). $\square$

Proposition 5.3.6. *The following identity holds:*

$$\sum_{q \geq 0} (-1)^q [ch_\iota(R^q f_* \Omega_{\mathcal{X}/S}^1, h_{L^2})]^{(1,1)} = -\frac{t}{2} c_1(f_* K_{\mathcal{X}/S}, h_{L^2}) + \frac{t}{2} dd^c \log \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S}).$$

Proof. By Lemma 5.3.5 and by (5.3.15), we have

$$\begin{aligned} & \sum_{q \geq 0} (-1)^q [ch_\iota(R^q f_* \Omega_{\mathcal{X}/S}^1, h_{L^2})]^{(1,1)} \\ &= -[ch_\iota(R^1 f_* \Omega_{\mathcal{X}/S}^1, h_{L^2})]^{(1,1)} - [ch_\iota(R^3 f_* \Omega_{\mathcal{X}/S}^1, h_{L^2})]^{(1,1)} \\ &= -c_1((R^1 f_* \Omega_{\mathcal{X}/S}^1)_+, h_{L^2}) + c_1((R^1 f_* \Omega_{\mathcal{X}/S}^1)_-, h_{L^2}) \\ & \quad - c_1((R^3 f_* \Omega_{\mathcal{X}/S}^1)_+, h_{L^2}) + c_1((R^3 f_* \Omega_{\mathcal{X}/S}^1)_-, h_{L^2}) \\ &= \frac{t}{2} c_1(R^4 f_* \mathcal{O}_{\mathcal{X}}, h_{L^2}) + \frac{t}{2} dd^c \log \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S}). \end{aligned}$$

By the Serre duality (see Lemma 5.4.1 below), we have

$$c_1(R^4 f_* \mathcal{O}_{\mathcal{X}}, h_{L^2}) = -c_1(f_* K_{\mathcal{X}/S}, h_{L^2}),$$

and we obtain the desired formula. $\square$

Definition 5.3.7. *We define the characteristic form $\omega_{H^\bullet(\mathcal{X}^\iota/S)} \in A^{1,1}(S)$ by*

$$\omega_{H^\bullet(\mathcal{X}^\iota/S)} = c_1(f_* \Omega_{\mathcal{X}^\iota/S}^1, h_{L^2}) - c_1(R^1 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) - 2c_1(f_* K_{\mathcal{X}^\iota/S}, h_{L^2}). \quad (5.3.18)$$

We define a function $\text{Vol}_{L^2}(R^1 f_* \mathbb{Z}, \omega_{\mathcal{X}^\iota/S})$ on S by

$$\text{Vol}_{L^2}(R^1 f_* \mathbb{Z}, \omega_{\mathcal{X}^\iota/S})(s) = \text{Vol}_{L^2}(H^1(X_s^\iota, \mathbb{Z}), \omega_s|_{X_s^\iota}).$$

Proposition 5.3.8. *The following identity holds:*

$$\sum_{q \geq 0} (-1)^q [ch(R^q f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2})]^{(1,1)} = -\frac{1}{2} dd^c \log \{ \text{Vol}(\mathcal{X}^\iota/S, \omega_{\mathcal{X}^\iota/S})^2 \text{Vol}_{L^2}(R^1 f_* \mathbb{Z}, \omega_{\mathcal{X}^\iota/S})^{-1} \} \\ + \frac{1}{2} \omega_{H^*(\mathcal{X}^\iota/S)}.$$

Proof. By the Serre duality (see Lemma 5.4.1 below),

$$\begin{aligned} c_1(f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) &= -dd^c \log \text{Vol}(\mathcal{X}^\iota/S, \omega_{\mathcal{X}^\iota/S}), \\ c_1(R^2 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) &= -c_1(f_* K_{\mathcal{X}^\iota/S}, h_{L^2}). \end{aligned} \quad (5.3.19)$$

Moreover, since

$$\begin{aligned} c_1(f_* \Omega_{\mathcal{X}^\iota/S}^1, h_{L^2}) + c_1(R^1 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) &= c_1(R^1 f_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) \\ &= -dd^c \log \text{Vol}_{L^2}(R^1 f_* \mathbb{Z}, \omega_{\mathcal{X}^\iota/S}), \end{aligned}$$

we have

$$\begin{aligned} c_1(R^1 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) &= \frac{1}{2} c_1(R^1 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) - \frac{1}{2} c_1(f_* \Omega_{\mathcal{X}^\iota/S}^1, h_{L^2}) \\ &\quad - \frac{1}{2} dd^c \log \text{Vol}_{L^2}(R^1 f_* \mathbb{Z}, \omega_{\mathcal{X}^\iota/S}). \end{aligned} \quad (5.3.20)$$

By the formulas (5.3.19) and (5.3.20), the proof is completed. $\square$

We show that a family of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$ is locally projective. For the proof, we follow [82, Lemma 2.7 and Theorem 5.6].

Lemma 5.3.9. *Let $f : \mathcal{X} \rightarrow S$ be a family of $K3^{[2]}$ -type manifolds and let $\sigma \in H^0(S, R^2 f_* \mathbb{Z})$. If $\sigma(s) = \sigma|_{X_s} \in H^{1,1}(X_s, \mathbb{R})$ for each $s \in S$, then we have $\sigma \in \text{Im}(H^0(S, R^1 f_* \mathcal{O}_{\mathcal{X}}^*) \rightarrow H^0(S, R^2 f_* \mathbb{Z}))$.*

Proof. We may assume that S is a polydisk. Since $h^2(X_s, \mathcal{O}_{X_s}) = 1$ for each $s \in S$, $R^2 f_* \mathcal{O}_{\mathcal{X}}$ is an invertible sheaf and we may regard $R^2 f_* \mathcal{O}_{\mathcal{X}}$ as a holomorphic line bundle on S . Since any holomorphic line bundle on the polydisk S is trivial, there exists a nowhere-vanishing holomorphic section $\xi \in H^0(S, R^2 f_* \mathcal{O}_{\mathcal{X}})$ such that

$$R^2 f_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_S \cdot \xi \quad \text{and} \quad H^2(X_s, \mathcal{O}_{X_s}) = \mathbb{C} \xi|_{X_s} \quad (s \in S).$$

The exponential sequence on $\mathcal{X}$

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{X}}^* \rightarrow 0$$

induces the following exact sequence:

$$H^0(S, R^1 f_* \mathcal{O}_{\mathcal{X}}^*) \xrightarrow{\phi} H^0(S, R^2 f_* \mathbb{Z}) \xrightarrow{\psi} H^0(S, R^2 f_* \mathcal{O}_{\mathcal{X}}). \quad (5.3.21)$$

Since $\psi(\sigma) \in H^0(S, R^2 f_* \mathcal{O}_{\mathcal{X}})$, there exists a holomorphic function F on S such that $\psi(\sigma) = F \cdot \xi$. By our assumption of $\sigma|_{X_s} \in H^{1,1}(X_s, \mathbb{R})$,

$$0 = \psi(\sigma)|_{X_s} = F(s) \cdot \xi|_{X_s} \quad (s \in S).$$

Therefore $F = 0$ and $\sigma \in \text{Ker } \psi$. By the exact sequence (5.3.21), we have $\sigma \in \text{Im } \phi$. $\square$

Lemma 5.3.10. *Let $f : (\mathcal{X}, \iota) \rightarrow S$ be a family of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$. Then $f : \mathcal{X} \rightarrow S$ is locally projective.*

Proof. Fix a point $s \in S$. By Proposition 2.2.3 (4), there exists a μ_2 -equivariant ample line bundle L_s on X_s . We may assume that S is a polydisk. Choose an isomorphism $\alpha : R^2 f_* \mathbb{Z} \rightarrow L_{2,S}$ such that for $s \in S$, $\alpha_s : H^2(X_s, \mathbb{Z}) \rightarrow L_2$ is an admissible marking for $(M, \mathcal{K})$. Then we have $\alpha_s(c_1(L_s)) \in M$. If S is sufficiently small, it follows from Lemma 5.3.9 that there exists a holomorphic line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}|_{X_s} = L_s$. Since ampleness is an open condition, there exists an open neighborhood $U \subset S$ of s such that $\mathcal{L}|_{f^{-1}(U)}$ is relatively ample. $\square$

In particular, a family $f : (\mathcal{X}, \iota) \rightarrow S$ of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$ is locally Kähler.

Theorem 5.3.11. *We define a real-valued function $\tau_{M, \mathcal{K}, \mathcal{X}/S}$ on S by*

$$\tau_{M, \mathcal{K}, \mathcal{X}/S}(s) = \tau_{M, \mathcal{K}}(X_s, \iota_s) \quad (s \in S).$$

Then $\tau_{M, \mathcal{K}, \mathcal{X}/S}$ is smooth and satisfies

$$-dd^c \log \tau_{M, \mathcal{K}, \mathcal{X}/S} = \frac{(t+1)(t+7)}{16} c_1(f_* K_{\mathcal{X}/S}, h_{L^2}) + \omega_{H^1(\mathcal{X}^\iota/S)}.$$

Proof. By Proposition 5.3.4, we have

$$\begin{aligned} & [f_*(Td_\iota(T\mathcal{X}/S, h_{\mathcal{X}/S})ch_\iota(\Omega_{\mathcal{X}/S}^1, h_{\mathcal{X}/S}^*))]^{(1,1)} \\ &= 2[f_*Td(T\mathcal{X}^\iota/S, h_{\mathcal{X}^\iota/S})]^{(1,1)} + dd^c \log A(\mathcal{X}/S, h_{\mathcal{X}/S}) \\ &+ \frac{t^2+7}{16} c_1(f_* K_{\mathcal{X}/S}, h_{L^2}) + \frac{t^2+7}{16} dd^c \log \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S}). \end{aligned}$$

Moreover, by Proposition 5.3.6, we have

$$\sum_{q \geq 0} (-1)^q [ch_\iota(R^q f_* \Omega_{\mathcal{X}/S}^1)]^{(1,1)} = -\frac{t}{2} c_1(f_* K_{\mathcal{X}/S}, h_{L^2}) + \frac{t}{2} dd^c \log \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S}).$$

Therefore, by the curvature formula for the equivariant Quillen metric (Theorem 1.2.3), we have

$$\begin{aligned} dd^c \log \tau_\iota(\bar{\Omega}_{\mathcal{X}/S}^1) &= 2[f_*Td(\bar{T}_{\mathcal{X}^\iota/S})]^{(1,1)} + dd^c \log A(\mathcal{X}/S, \omega_{\mathcal{X}/S}) \\ &+ \frac{(t+7)(t+1)}{16} c_1(f_* K_{\mathcal{X}/S}, h_{L^2}) + \frac{(t-7)(t-1)}{16} dd^c \log \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S}). \end{aligned} \tag{5.3.22}$$

On the other hand, by Proposition 5.3.8, we have

$$\begin{aligned} \sum_{q \geq 0} (-1)^q [ch(R^q f_* \bar{\mathcal{O}}_{\mathcal{X}^\iota})]^{(1,1)} &= \frac{1}{2} dd^c \log \{ \text{Vol}(\mathcal{X}^\iota/S, \omega_{\mathcal{X}^\iota/S})^{-2} \text{Vol}_{L^2}(R^1 f_* \mathbb{Z}, \omega_{\mathcal{X}^\iota/S}) \} \\ &\quad + \frac{1}{2} \omega_{H^\cdot(\mathcal{X}^\iota/S)}. \end{aligned}$$

Therefore, by the curvature formula of the Quillen metric (Theorem 1.2.3), we have

$$\begin{aligned} [f_* Td(\bar{T}_{\mathcal{X}^\iota/S})]^{(1,1)} &= \frac{1}{2} dd^c \log \{ \tau(\bar{\mathcal{O}}_{\mathcal{X}^\iota})^{-2} \text{Vol}(\mathcal{X}^\iota/S, \omega_{\mathcal{X}^\iota/S})^{-2} \text{Vol}_{L^2}(R^1 f_* \mathbb{Z}, \omega_{\mathcal{X}^\iota/S}) \} \\ &\quad + \frac{1}{2} \omega_{H^\cdot(\mathcal{X}^\iota/S)}. \end{aligned} \tag{5.3.23}$$

By (5.3.22) and (5.3.23) and by the definition of $\tau_{M,\mathcal{K}}$, we obtain the desired formula. $\square$

Lemma 5.3.12. *The form $\omega_{H^\cdot(\mathcal{X}^\iota/S)}$ is independent of the choice of an ι -invariant fiberwise Kähler metric $\omega_{\mathcal{X}/S}$.*

Proof. Set

$$\lambda(\mathcal{X}^\iota/S) = \bigotimes_{p,q} \left(\det R^q f_* \Omega_{\mathcal{X}^\iota/S}^p \right)^{(-1)^{p+q}p},$$

and

$$\lambda_{dR}(\mathcal{X}^\iota/S) = \bigotimes_{k=0}^{2n} \left(\det R^k f_* \mathbb{C} \otimes \mathcal{O}_S \right)^{(-1)^k k}.$$

Then $\lambda(\mathcal{X}^\iota/S)$ is a holomorphic line bundle on S , and $\lambda_{dR}(\mathcal{X}^\iota/S)$ is a flat holomorphic line bundle on S . Moreover there exists a canonical isomorphism of smooth line bundles

$$\lambda_{dR}(\mathcal{X}^\iota/S) \cong \lambda(\mathcal{X}^\iota/S) \otimes \overline{\lambda(\mathcal{X}^\iota/S)} \tag{5.3.24}$$

induced by Hodge decomposition.

For a sufficiently small open subset $U \subset S$, let $\tau \in H^0(U, \lambda(\mathcal{X}^\iota/S))$ be a nowhere-vanishing holomorphic section of $\lambda(\mathcal{X}^\iota/S)$ on U , and $\tau_0 \in \Gamma(U, \lambda_{dR}(\mathcal{X}^\iota/S))$ be a non-zero flat section of $\lambda_{dR}(\mathcal{X}^\iota/S)$ on U . By the isomorphism (5.3.24), there is a smooth function $f : U \rightarrow \mathbb{C}$ such that

$$\tau_0 = e^f \tau \otimes \bar{\tau}.$$

Note that the 2-form $dd^c \text{Re } f$ is independent of the choice of the sections τ and τ_0 . The Hodge form ω_H is a real smooth differential form on S of type $(1,1)$ defined by

$$\omega_H|_U = dd^c \text{Re } f$$

on U . By construction, this is independent of the choice of $\omega_{\mathcal{X}/S}$.

By [88, Section 1.2 in particular Proposition 1.1], the following formula holds:

$$\omega_H = \frac{1}{2} \sum_{0 \leq p, q \leq 2} (-1)^{p+q} (p-q) c_1(R^q f_* \Omega_{\mathcal{X}^\iota/S}^p, h_{L^2}) = -\omega_{H^\cdot(\mathcal{X}^\iota/S)},$$

where we used the Serre duality (see Lemma 5.4.1 below) to get the second equality. This completes the proof. $\square$

Theorem 5.3.13. *Let (X, ι) be a manifold of $K3^{[2]}$ -type with antisymplectic involution of type $(M, \mathcal{K})$. Then $\tau_{M, \mathcal{K}}(X, \iota)$ is independent of the choice of an ι -invariant Kähler metric. In particular, it is an invariant of (X, ι) .*

Proof. We follow [82, Theorem 5.7]. We regard $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Let $h_{X,0}, h_{X,\infty}$ be two ι -invariant Kähler metrics on X . Set $\mathcal{X} = X \times \mathbb{P}^1$, and let $f : \mathcal{X} \rightarrow \mathbb{P}^1$ be the projection. For each $z \in \mathbb{P}^1$, we define an ι_z -invariant Kähler metric h_z on the fiber X_z by

$$h_z = \frac{1}{|z|^2 + 1} h_{X,0} + \frac{|z|^2}{|z|^2 + 1} h_{X,\infty}.$$

We may regard $(h_z)_{z \in \mathbb{P}^1}$ as an ι -invariant fiberwise Kähler metric on $T\mathcal{X}/\mathbb{P}^1$.

Since the family $f : \mathcal{X} \rightarrow \mathbb{P}^1$ does not change the complex structures, we have $c_1(f_* K_{\mathcal{X}/\mathbb{P}^1}, h_{L^2}) = \omega_{H^\cdot(\mathcal{X}^\iota/\mathbb{P}^1)} = 0$. By Theorem 5.3.11, we have $dd^c \log \tau_{M, \mathcal{K}, \mathcal{X}/\mathbb{P}^1} = 0$, and $\log \tau_{M, \mathcal{K}, \mathcal{X}/\mathbb{P}^1}$ is a pluriharmonic function on $\mathbb{P}^1$. Since $\mathbb{P}^1$ is compact, it is constant and $\tau_{M, \mathcal{K}, \mathcal{X}/\mathbb{P}^1}(0) = \tau_{M, \mathcal{K}, \mathcal{X}/\mathbb{P}^1}(\infty)$. Thus $\tau_{M, \mathcal{K}}(X, \iota)$ is independent of the choice of an ι -invariant Kähler metric. $\square$

By Proposition 2.2.14, the image of the period map $P_{M, \mathcal{K}} : \tilde{\mathcal{M}}_{M, \mathcal{K}} \rightarrow \mathcal{M}_{M, \mathcal{K}}$ is $\mathcal{M}_{M, \mathcal{K}}^\circ$.

Lemma 5.3.14. *Let $p \in \mathcal{M}_{M, \mathcal{K}}^\circ$, and we define a real number $\tau_{M, \mathcal{K}}(p)$ by*

$$\tau_{M, \mathcal{K}}(p) = \tau_{M, \mathcal{K}}(X, \iota) \quad ((X, \iota) \in P_{M, \mathcal{K}}^{-1}(p)).$$

Then $\tau_{M, \mathcal{K}}(p)$ is well-defined. Namely, it is independent of the choice of $(X, \iota) \in P_{M, \mathcal{K}}^{-1}(p)$.

Proof. Let $(X, \iota), (X', \iota') \in P_{M, \mathcal{K}}^{-1}(p)$, and let $\pi : (\mathcal{X}, \iota) \rightarrow \text{Def}(X, \iota)$ and $\pi' : (\mathcal{X}', \iota') \rightarrow \text{Def}(X', \iota')$ be the Kuranishi families, respectively. By Lemma 2.2.18, there exist sequences $\{p_m\}_{m=1}^\infty$ in $\text{Def}(X, \iota)$ and $\{p'_m\}_{m=1}^\infty$ in $\text{Def}(X', \iota')$ such that $\pi^{-1}(p_m) \cong \pi'^{-1}(p'_m)$ for all $m \geq 1$ and $p_m \rightarrow p, p'_m \rightarrow p$ ($m \rightarrow \infty$). Therefore we have

$$\tau_{M, \mathcal{K}}(X, \iota) = \lim_{m \rightarrow \infty} \tau_{M, \mathcal{K}, \mathcal{X}/\text{Def}(X, \iota)}(p_m) = \lim_{m \rightarrow \infty} \tau_{M, \mathcal{K}, \mathcal{X}'/\text{Def}(X', \iota')}(p'_m) = \tau_{M, \mathcal{K}}(X', \iota'),$$

which completes the proof. $\square$

By Lemma 5.3.14, we obtain a real-valued smooth function $\tau_{M,\mathcal{K}}$ on $\mathcal{M}_{M,\mathcal{K}}^\circ$.

Fix a vector $l \in M_{\mathbb{R}}$ with $l^2 \geq 0$. Recall that $\Omega_{M^\perp}^+$ is a bounded symmetric domain of type IV. The Bergman metric $\omega_{\Omega_{M^\perp}^+}$ on $\Omega_{M^\perp}^+$ is defined by

$$\omega_{\Omega_{M^\perp}^+}([\eta]) = -dd^c \log B_{M^\perp}([\eta]) \quad ([\eta] \in \Omega_{M^\perp}^+), \quad (5.3.25)$$

where

$$B_{M^\perp}([\eta]) = \frac{(\eta, \bar{\eta})}{|(\eta, l)^2|} \quad ([\eta] \in \Omega_{M^\perp}^+).$$

Since $\omega_{\Omega_{M^\perp}^+}$ is $\Gamma_{M^\perp, \mathcal{K}}$ -invariant, it induces an orbifold Kähler form $\omega_{\mathcal{M}_{M,\mathcal{K}}}$ on $\mathcal{M}_{M,\mathcal{K}}$.

Lemma 5.3.15. *There exists a smooth $(1,1)$ -form $\sigma_{M,\mathcal{K}}$ on $\mathcal{M}_{M,\mathcal{K}}^\circ$ such that for any $(X, \iota) \in \tilde{\mathcal{M}}_{M,\mathcal{K}}$ we have*

$$P_{M,\mathcal{K}}^* \sigma_{M,\mathcal{K}} = \omega_{H^\cdot(\mathcal{X}^\iota / \text{Def}(X, \iota))},$$

where $P_{M,\mathcal{K}} : \text{Def}(X, \iota) \rightarrow \mathcal{M}_{M,\mathcal{K}}$ is the period map of the Kuranishi family $\pi : (\mathcal{X}, \iota) \rightarrow \text{Def}(X, \iota)$ of (X, ι) .

Proof. Let $(X, \iota) \in \tilde{\mathcal{M}}_{M,\mathcal{K}}$ and let $\pi : (\mathcal{X}, \iota) \rightarrow \text{Def}(X, \iota)$ be the Kuranishi family. Fix an isomorphism $\alpha : R^2\pi_*\mathbb{Z} \rightarrow L_{2, \text{Def}(X, \iota)}$ such that for each $s \in \text{Def}(X, \iota)$, $\alpha_s : H^2(X_s, \mathbb{Z}) \rightarrow L_2$ is an admissible marking for $(M, \mathcal{K})$. The period map $P_{M,\mathcal{K}} : \text{Def}(X, \iota) \rightarrow \Omega_{M^\perp}^+$ is defined by

$$P_{M,\mathcal{K}}(s) = \alpha_s(H^{2,0}(X_s)) \quad (s \in \text{Def}(X, \iota)).$$

By the local Torelli theorem, we may assume that $P_{M,\mathcal{K}}$ is an isomorphism onto its image. We set $\text{Def}(X, \iota, \alpha) = P_{M,\mathcal{K}}(\text{Def}(X, \iota))$ and we identify $\text{Def}(X, \iota, \alpha)$ with $\text{Def}(X, \iota)$. It suffices to show that $\{\omega_{H^\cdot(\mathcal{X}^\iota / \text{Def}(X, \iota, \alpha))}\}_{(X, \iota, \alpha)}$ patch together to a smooth $(1,1)$ -form τ on $\Omega_{M^\perp}^+ \setminus \mathcal{D}_{M^\perp}$ and that τ is $\Gamma_{M^\perp, \mathcal{K}}$ -invariant.

Let (X', ι', α') be another triple and its Kuranishi family is denoted by $\pi' : (\mathcal{X}', \iota') \rightarrow \text{Def}(X', \iota', \alpha')$. We assume that $\text{Def}(X, \iota, \alpha) \cap \text{Def}(X', \iota', \alpha') \neq \emptyset$. Set $U = \text{Def}(X, \iota, \alpha) \cap \text{Def}(X', \iota', \alpha')$ and $U^\circ = U \setminus \mathcal{D}_{\mathcal{K}}$. Here $\mathcal{D}_{\mathcal{K}}$ is a $\Gamma_{M^\perp, \mathcal{K}}$ -invariant effective reduced divisor on $\Omega_{M^\perp}^+$ defined in Theorem 2.2.15. By Theorem 2.2.15, $\pi : (\pi^{-1}(U^\circ), \iota|_{\pi^{-1}(U^\circ)}) \rightarrow U^\circ$ and $\pi' : ((\pi')^{-1}(U^\circ), \iota'|_{{(\pi')^{-1}(U^\circ)}}) \rightarrow U^\circ$ are locally isomorphic. By Lemma 5.3.12,

$$\omega_{H^\cdot(\mathcal{X}^\iota / \text{Def}(X, \iota, \alpha))}|_{U^\circ} = \omega_{H^\cdot((\mathcal{X}')^\iota / \text{Def}(X', \iota', \alpha'))}|_{U^\circ}.$$

Since U° is dense in U and since both $\omega_{H^\cdot(\mathcal{X}^\iota / \text{Def}(X, \iota, \alpha))}$ and $\omega_{H^\cdot((\mathcal{X}')^\iota / \text{Def}(X', \iota', \alpha'))}$ are smooth, we have

$$\omega_{H^\cdot(\mathcal{X}^\iota / \text{Def}(X, \iota, \alpha))}|_U = \omega_{H^\cdot((\mathcal{X}')^\iota / \text{Def}(X', \iota', \alpha'))}|_U.$$

Since $\{\text{Def}(X, \iota, \alpha)\}$ is an open covering of $\Omega_{M^\perp}^+ \setminus \mathcal{D}_{M^\perp}$, there exists a smooth $(1,1)$ -form τ on $\Omega_{M^\perp}^+ \setminus \mathcal{D}_{M^\perp}$ such that

$$\tau|_{\text{Def}(X, \iota, \alpha)} = \omega_{H^\cdot(\mathcal{X}^\iota / \text{Def}(X, \iota, \alpha))}$$

for each (X, ι, α) .

For $\gamma \in \Gamma_{M^\perp, \mathcal{K}}$, there exists $g \in \Gamma(\mathcal{K})$ such that $\gamma = g|_{M^\perp}$. Since

$$\gamma^* \omega_{H^\cdot}(\mathcal{X}^\iota / \text{Def}(X, \iota, \alpha)) = \omega_{H^\cdot}(\mathcal{X}^\iota / \text{Def}(X, \iota, g^{-1} \circ \alpha)),$$

the form τ is $\Gamma_{M^\perp, \mathcal{K}}$ -invariant. Therefore τ induces a smooth $(1, 1)$ -form $\sigma_{M, \mathcal{K}}$ on $\mathcal{M}_{M, \mathcal{K}}^\circ$. $\square$

Theorem 5.3.16. *The following equation of differential forms on $\mathcal{M}_{M, \mathcal{K}}^\circ$ holds:*

$$-dd^c \log \tau_{M, \mathcal{K}, \mathcal{X}/S} = \frac{(t+1)(t+7)}{8} \omega_{\mathcal{M}_{M, \mathcal{K}}} + \sigma_{M, \mathcal{K}}.$$

Proof. Let (X, ι, α) be as in Lemma 5.3.15. Applying Theorem 5.3.11 to the Kuranishi family $\pi : (\mathcal{X}, \iota) \rightarrow \text{Def}(X, \iota, \alpha)$, the following equation on $\text{Def}(X, \iota, \alpha)$ holds:

$$-dd^c \log \tau_{M, \mathcal{K}, \mathcal{X}/\text{Def}(X, \iota, \alpha)} = \frac{(t+1)(t+7)}{16} c_1(\pi_* K_{\mathcal{X}/\text{Def}(X, \iota, \alpha)}, h_{L^2}) + \omega_{H^\cdot}(\mathcal{X}^\iota / \text{Def}(X, \iota, \alpha)).$$

By (5.3.25), we have $c_1(\pi_* K_{\mathcal{X}/\text{Def}(X, \iota, \alpha)}, h_{L^2}) = 2\omega_{\Omega_{M^\perp}^+}|_{\text{Def}(X, \iota, \alpha)}$. By Lemma 5.3.15, we obtain the desired equation. $\square$

Remark 5.3.17. Let (X, ι) be a manifold of $K3^{[2]}$ -type with involution of type $(M, \mathcal{K})$ and let $f : (\mathcal{X}, \iota) \rightarrow S$ be a deformation of (X, ι) . Consider the equivariant analytic torsion $\tau_\iota(\overline{\mathcal{O}_{\mathcal{X}}})$ of the trivial line bundle on $\mathcal{X}$ with trivial metric. In the same manner as in this subsection §5.3, we have

$$dd^c \log \{ \tau_\iota(\overline{\mathcal{O}_{\mathcal{X}}}) A_0(\mathcal{X}/S, h_{\mathcal{X}/S}) \} = 0,$$

where $A_0(\mathcal{X}/S, h_{\mathcal{X}/S})$ is a function on S defined by

$$A_0(\mathcal{X}/S, h_{\mathcal{X}/S})(s) = A_0(X_s, \iota_s, h_s) := \exp \left[\frac{1}{96} \int_{X_s^\iota} \log \left\{ \frac{\omega_s^4/4!}{\eta_s^2 \wedge \bar{\eta}_s^2} \frac{\|\eta_s^2\|_{L^2}^2}{\text{Vol}(X_s, \omega_s)} \right\} \Omega_0|_{X_s^\iota} \right],$$

$$\Omega_0 = c_1(\overline{T\mathcal{X}^\iota}/S)^2 - 2c_2(\overline{T\mathcal{X}^\iota}/S) - c_1(\overline{T\mathcal{X}}/S)|_{\mathcal{X}^\iota}^2 + 3c_2(\overline{T\mathcal{X}}/S)|_{\mathcal{X}^\iota}.$$

It is very likely that $\tau_\iota(\overline{\mathcal{O}_{\mathcal{X}}})$ does not induce an interesting invariant which reflects the complex structure of a $K3^{[2]}$ -type manifold with antisymplectic involution.

Similarly, consider the equivariant analytic torsion $\tau_\iota(\overline{\Omega_{\mathcal{X}/S}^2})$ of $\Omega_{\mathcal{X}/S}^2$ on $\mathcal{X}$ with the μ_2 -invariant hermitian metric induced from the fiberwise Kähler metric $h_{\mathcal{X}/S}$. In the same manner as in this subsection §5.3, we have

$$dd^c \log \tau_\iota(\overline{\Omega_{\mathcal{X}/S}^2}) = 0.$$

Again, it is very likely that $\tau_\iota(\overline{\Omega_{\mathcal{X}/S}^2})$ does not induce an interesting invariant which reflects the complex structure of a $K3^{[2]}$ -type manifold with antisymplectic involution.

We conjecture that $\tau_\iota(\overline{\mathcal{O}_X}) A_0(X, \iota, h_X)$ and $\tau_\iota(\overline{\Omega_X^2})$ are constant functions on the space $\mathcal{M}_{M, \mathcal{K}}^\circ$. These are the reason why we study the equivariant analytic torsion $\tau_\iota(\overline{\Omega_X^1})$ of holomorphic cotangent bundle with μ_2 -invariant hermitian metric induced from the fiberwise Kähler metric h_X .

As an application of this invariant, we show the isotriviality of families of $K3^{[2]}$ -type manifolds with involution under a certain assumption. We assume the following properties of $(M, \mathcal{K})$:

- (1) $t \neq -1, -7$.
- (2) Each manifold of $K3^{[2]}$ -type (X, ι) with antisymplectic involution of type $(M, \mathcal{K})$ satisfies $q(X^\iota) = p_g(X^\iota) = 0$.

Here $q(X^\iota) = \dim H^1(X^\iota, \mathcal{O}_{X^\iota})$ is the irregularity of X^ι , and $p_g(X^\iota) = \dim H^2(X^\iota, \mathcal{O}_{X^\iota})$ is the geometric genus of X^ι .

Example 5.3.18. Let $M_0 = U(2) \oplus E_8(2)$ or $\Lambda_k(2)^\perp$ ($k = 0, \dots, 9$), where

$$\Lambda_k = I_2 \oplus -I_{10-k} \quad (k \neq 8), \quad \Lambda_8 = I_2 \oplus -I_2 \text{ or } U \oplus U \quad (k = 8).$$

By [63, Theorem 4.2.2.], the fixed locus Y^σ of a 2-elementary K3 surface (Y, σ) of type M_0 is the empty set or consists of smooth rational curves. Let $M = M_0 \oplus \mathbb{Z}e$ and let $\mathcal{K} \in \text{KT}(M)$ be a natural chamber (See Definition 6.3.2 below). Then $(M, \mathcal{K})$ satisfies the above assumption by (6.3.3).

Theorem 5.3.19. *Suppose that $(M, \mathcal{K})$ satisfies the above assumptions (1), (2). Then there exists no irreducible projective curve on $\mathcal{M}_{M,\mathcal{K}}^\circ$. Moreover if $f : (\mathcal{X}, \iota) \rightarrow S$ is a family of $K3^{[2]}$ -type manifolds with antisymplectic involution of type $(M, \mathcal{K})$ and S is compact, then f is isotrivial. Namely, any two fibers of f are isomorphic.*

Proof. Let $(X, \iota) \in \tilde{\mathcal{M}}_{M,\mathcal{K}}$ and let $\pi : (\mathcal{X}, \iota) \rightarrow \text{Def}(X, \iota)$ be the Kuranishi family. Since $q(X_s^\iota) = p_g(X_s^\iota) = 0$ for each fiber (X_s, ι_s) , we have $\omega_{H^*(\mathcal{X}^\iota/\text{Def}(X, \iota))} = 0$. By the construction of $\sigma_{M,\mathcal{K}}$, we have $\sigma_{M,\mathcal{K}} = 0$ on $\mathcal{M}_{M,\mathcal{K}}^\circ$. By Theorem 5.3.16, we have the formula

$$-dd^c \log \tau_{M,\mathcal{K}} = \frac{(t+1)(t+7)}{8} \omega_{\mathcal{M}_{M,\mathcal{K}}} \quad (5.3.26)$$

on $\mathcal{M}_{M,\mathcal{K}}^\circ$. Since $\omega_{\mathcal{M}_{M,\mathcal{K}}}$ is a positive form, $-\frac{8}{(t+1)(t+7)} \log \tau_{M,\mathcal{K}}$ is a plurisubharmonic function on $\mathcal{M}_{M,\mathcal{K}}^\circ$.

Suppose that there is an irreducible projective curve C on $\mathcal{M}_{M,\mathcal{K}}^\circ$. Then the function $\left(-\frac{8}{(t+1)(t+7)} \log \tau_{M,\mathcal{K}}\right)|_C$ is a subharmonic function on C . By the maximal principle, it is a constant function, which contradicts the formula (5.3.26) since $\omega_{\mathcal{M}_{M,\mathcal{K}}}$ is a Kähler form.

Let $f : (\mathcal{X}, \iota) \rightarrow S$ be a family of $K3^{[2]}$ -type manifolds with antisymplectic involution of type $(M, \mathcal{K})$. Assume that S is compact. The period map $P_{M,\mathcal{K}} : S \rightarrow \mathcal{M}_{M,\mathcal{K}}^\circ$ is a proper holomorphic map. By the Remmert proper mapping theorem, its image $P_{M,\mathcal{K}}(S)$ is an analytic subset of the quasi-projective variety $\mathcal{M}_{M,\mathcal{K}}^\circ$. Suppose that f is not isotrivial. Then $P_{M,\mathcal{K}}$ is not a constant map and $P_{M,\mathcal{K}}(S)$ contains an irreducible projective curve. This contradicts to the first statements. $\square$

5.4 The L^2 -metric on direct image sheaves

Let us show the fundamental properties of the L^2 -metric on direct image sheaves which we used in the previous section.

Let $\mathcal{M}$ and B be complex manifolds of dimension $m+n$ and n , respectively. Let $\pi : \mathcal{M} \rightarrow B$ be a proper holomorphic submersion. Set $M_b = \pi^{-1}(b)$ for $b \in B$. Let $h_{\mathcal{M}/B}$ be a fiberwise Kähler metric on the relative tangent bundle $T\mathcal{M}/B$. The hermitian metric on $\Omega_{\mathcal{M}/B}^p$ induced from $h_{\mathcal{M}/B}$ is also denoted by $h_{\mathcal{M}/B}$.

Since $\dim H^q(M_b, \Omega_{M_b}^p)$ is independent of $b \in B$, the direct image sheaf $R^q\pi_*\Omega_{\mathcal{M}/B}^p$ is locally free. We regard $R^q\pi_*\Omega_{\mathcal{M}/B}^p$ as a holomorphic vector bundle on B . Recall that $R^q\pi_*\Omega_{\mathcal{M}/B}^p$ is equipped with the hermitian metric called the L^2 -metric. This metric is denoted by h_{L^2} . The dual metric on the dual $\left(R^q\pi_*\Omega_{\mathcal{M}/B}^p\right)^*$ is denoted by $h_{L^2}^*$. By the Serre duality, there exists an isomorphism of holomorphic vector bundles

$$\Phi : R^q\pi_*\Omega_{\mathcal{M}/B}^p \rightarrow \left(R^{m-q}\pi_*\Omega_{\mathcal{M}/B}^{m-p}\right)^*.$$

Lemma 5.4.1. *The following identity holds:*

$$c_1(R^q\pi_*\Omega_{\mathcal{M}/B}^p, h_{L^2}) = -c_1(R^{m-q}\pi_*\Omega_{\mathcal{M}/B}^{m-p}, h_{L^2}).$$

Proof. Let M be a fiber of $\pi : \mathcal{M} \rightarrow B$. The restriction $h_M := h_{\mathcal{M}/B}|_M$ is a Kähler metric on M . We identify a cohomology class $\alpha \in H^q(M, \Omega_M^p)$ with its harmonic representative. It suffices to show that the Serre duality

$$\Phi : H^q(M, \Omega_M^p) \cong H^{m-q}(M, \Omega_M^{m-p})^*, \quad \alpha \mapsto \left(\beta \mapsto \int_M \alpha \wedge \beta \right)$$

is an isometry with respect to the hermitian inner products h_{L^2} and $h_{L^2}^*$.

It is well-known that the Hodge $*$ -operator

$$* : (\wedge^{p,q} T^*M, h_M) \rightarrow (\wedge^{m-q, m-p} T^*M, h_M)$$

is an isometry (cf. [40, Proposition 1.2.20]). Then $*$ induces an isometry

$$* : (H^q(M, \Omega_M^p), h_{L^2}) \rightarrow (H^{m-q}(M, \Omega_M^{m-p}), h_{L^2}).$$

Let $e_1, \dots, e_m$ be an orthonormal basis of $(H^q(M, \Omega_M^p), h_{L^2})$. If we set $f_i = *e_i$ ($i = 1, \dots, m$), then $f_1, \dots, f_m$ is an orthonormal basis of $(H^{m-q}(M, \Omega_M^{m-p}), h_{L^2})$. By the construction of the dual metric $h_{L^2}^*$ on $(H^{m-q}(M, \Omega_M^{m-p})^*, h_{L^2}^*)$, its dual basis $f_1^*, \dots, f_m^*$ is also an orthonormal basis.

For each $i = 1, \dots, m$, since $\Phi(e_i) \in H^{m-q}(M, \Omega_M^{m-p})^*$, there exist complex numbers $a_{ij} \in \mathbb{C}$ ($j = 1, \dots, m$) such that

$$\Phi(e_i) = \sum_j a_{ij} f_j^*.$$

For each i, j , we have

$$a_{ij} = \left(\sum_k a_{ik} f_k^* \right) (f_j) = \Phi(e_i)(f_j) = \int_M e_i \wedge f_j. \quad (5.4.1)$$

By the definition of f_j , we have

$$\int_M e_i \wedge f_j = \int_M e_i \wedge * \bar{e}_j = h_{L^2}(e_i, e_j) = \delta_{ij}. \quad (5.4.2)$$

By (5.4.1) and (5.4.2), we have $\Phi(e_i) = f_i^*$. Since Φ maps the orthonormal basis $e_1, \dots, e_m$ to the orthonormal basis $f_1^*, \dots, f_m^*$, Φ is an isometry. This completes the proof. $\square$

Lemma 5.4.2. *Let M be a compact Kähler manifold of dimension 4. Then the L^2 -inner product h_{L^2} on the cohomology $H^1(X, \Omega_X^p)$ ($p = 1, 3$) depends only on the choice of the Kähler class and is independent of the choice of a Kähler metric itself.*

Proof. Let h_M be a Kähler metric on M . Its associated Kähler form is denoted by ω_M . Let L be the Lefschetz operator defined by the multiplication of ω_M . Let $\alpha \in A^{p,1}(M)$ ($p = 1, 3$) and consider the Lefschetz decomposition $\alpha = L\beta + \gamma$, where β is a primitive $(p-1, 0)$ -form, and γ is a primitive $(p, 1)$ -form. Then we have

$$\|\alpha\|_{L^2}^2 = \|L\beta\|_{L^2}^2 + \|\gamma\|_{L^2}^2. \quad (5.4.3)$$

Here, by [40, Proposition 1.2.31], we have

$$\|L\beta\|_{L^2}^2 = \int_M L\beta \wedge \left\{ \frac{(-1)^{\frac{p(p-1)}{2}} i^{-p+1}}{(4-p)!} L^{4-p} \bar{\beta} \right\} = \frac{(-1)^{\frac{p(p-1)}{2}} i^{-p+1}}{(4-p)!} \int_M \omega_M^{5-p} \wedge \beta \wedge \bar{\beta}. \quad (5.4.4)$$

On the other hand, by [40, Proposition 1.2.31], we have

$$\|\gamma\|_{L^2}^2 = \int_M \gamma \wedge \left\{ \frac{(-1)^{\frac{(p+1)(p+2)}{2}} i^{-p+1}}{(3-p)!} L^{3-p} \bar{\gamma} \right\} = \frac{(-1)^{\frac{(p+1)(p+2)}{2}} i^{-p+1}}{(3-p)!} \int_M \omega_M^{3-p} \wedge \gamma \wedge \bar{\gamma}. \quad (5.4.5)$$

By (5.4.3), (5.4.4) and (5.4.5), we have

$$\|\alpha\|_{L^2}^2 = \frac{(-1)^{\frac{p(p-1)}{2}} i^{-p+1}}{(4-p)!} \int_M \omega_M^{5-p} \wedge \beta \wedge \bar{\beta} + \frac{(-1)^{\frac{(p+1)(p+2)}{2}} i^{-p+1}}{(3-p)!} \int_M \omega_M^{3-p} \wedge \gamma \wedge \bar{\gamma}. \quad (5.4.6)$$

Let $\alpha \in H^1(M, \Omega_M^p)$ and consider the Lefschetz decomposition $\alpha = L\beta + \gamma$, where $\beta \in H^0(M, \Omega_M^{p-1})$ and $\gamma \in H^1(M, \Omega_M^p)$ are primitive. If we identify cohomology classes

α, β, γ with their harmonic representatives, the L^2 -norm of α satisfies the above formula (5.4.6). Since the Lefschetz operator $L : H^q(M, \Omega_M^p) \rightarrow H^{q+1}(M, \Omega_M^{p+1})$ depends only on the choice of the Kähler class, the Lefschetz decomposition depends only on the choice of the Kähler class. Let ω'_M be another Kähler form on M with $[\omega_M] = [\omega'_M]$ in $H^2(M, \mathbb{R})$. By the $\partial\bar{\partial}$ -Poincaré lemma (see [40, Corollary 3.2.10]), there exist differential forms δ, ε such that

$$\beta' = \beta + dd^c\delta, \quad \gamma' = \gamma + dd^c\varepsilon \quad (5.4.7)$$

are harmonic representatives of the cohomology classes $[\beta], [\gamma]$ with respect to the Kähler form ω'_M , respectively. Hence $\omega'_M \wedge \beta' + \gamma'$ is the harmonic representative of α with respect to ω'_M . By (5.4.6) applied to α and ω'_M , we get

$$\|\alpha\|_{L^2, \omega'_M}^2 = \frac{(-1)^{\frac{p(p-1)}{2}} i^{-p+1}}{(4-p)!} \int_M \omega'_M{}^{5-p} \wedge \beta' \wedge \bar{\beta}' + \frac{(-1)^{\frac{(p+1)(p+2)}{2}} i^{-p+1}}{(3-p)!} \int_M \omega'_M{}^{3-p} \wedge \gamma' \wedge \bar{\gamma}'.$$

Thus, by Stokes' formula and (5.4.7), we get $\|\alpha\|_{L^2, \omega_M}^2 = \|\alpha\|_{L^2, \omega'_M}^2$. Hence, these L^2 -metrics on $H^1(M, \Omega_M^p)$ ($p = 1, 3$) depends only on the Kähler class. $\square$

Chapter 6

The singularity of the invariant $\tau_{M,\mathcal{K}}$

In this chapter, we show the algebraicity of the singularity of the invariant $\tau_{M,\mathcal{K}}$. As an application, we prove that in some cases, our invariant coincides with the invariant of 2-elementary K3 surfaces in Section 4.1 up to a universal constant.

6.1 The singularity of equivariant analytic torsion

Let $\mathcal{X}$ be a smooth projective manifold of dimension $n + 1$ and let $\iota : \mathcal{X} \rightarrow \mathcal{X}$ be a holomorphic involution. In what follows, we consider the μ_2 -action on $\mathcal{X}$ induced by ι . Let C be a smooth projective curve with trivial μ_2 -action, and let $f : \mathcal{X} \rightarrow C$ be a μ_2 -equivariant holomorphic surjection. Then the μ_2 -action preserves the fibers of f .

Let $\Sigma_f = \{x \in \mathcal{X} ; df(x) = 0\}$ be the critical locus of f and let $\Delta_f = f(\Sigma_f)$ be the discriminant locus of f . The fiber of f at $s \in C$ is denoted by X_s . We fix a general fiber X_∞ . We set $C^\circ = C \setminus \Delta_f$, $\mathcal{X}^\circ = f^{-1}(C^\circ)$, and $f^\circ = f|_{\mathcal{X}^\circ}$. Then $f^\circ : \mathcal{X}^\circ \rightarrow C^\circ$ is a family of projective manifolds with μ_2 -action.

Let $h_{\mathcal{X}}$ be a μ_2 -invariant Kähler metric on $\mathcal{X}$. We assume that the Kähler form of $h_{\mathcal{X}}$ is integral. The relative tangent bundle $T\mathcal{X}/C$ is the μ_2 -equivariant subbundle of $T\mathcal{X}|_{\mathcal{X} \setminus \Sigma_f}$ defined by

$$T\mathcal{X}/C = \text{Ker}(f_*|_{\mathcal{X} \setminus \Sigma_f} : T\mathcal{X}|_{\mathcal{X} \setminus \Sigma_f} \rightarrow f^*TC|_{\mathcal{X} \setminus \Sigma_f}).$$

We set $h_{\mathcal{X}/C} = h_{\mathcal{X}}|_{T\mathcal{X}/C}$.

Fix $0 \in \Delta_f$ and let (Δ, s) be a local coordinate on C centered at 0 such that $\Delta \cap \Delta_f = \{0\}$ and that $s(\Delta)$ is the unit disk in $\mathbb{C}$. We set $X = f^{-1}(\Delta)$ and $\Delta^* = \Delta \setminus \{0\}$.

Assumption 6.1.1. We assume that the singular fiber X_0 is simple normal crossing. Namely, for each point of X_0 there exists a coordinate neighborhood $(U, z_1, \dots, z_{n+1})$ on $\mathcal{X}$ such that $X_0 \cap U$ is given by $z_1 \cdots z_k = 0$ and each irreducible component of X_0 is smooth.

In this section, we prove that $\log \tau_i(\Omega_{X_s}^1, h_{\mathcal{X}}|_{X_s})$ has a logarithmic singularity as $s \rightarrow 0$. For this, we study the singularity of the equivariant Quillen metrics and the L^2 -metrics on the equivariant determinant of the cohomology.

6.1.1 Kähler extension and the singularity of equivariant Quillen metrics

The μ_2 -invariant hermitian metric on $\Omega_{\mathcal{X}}^1$ induced from the metric $h_{\mathcal{X}}$ on $T\mathcal{X}$ is also denoted by $h_{\mathcal{X}}$. The μ_2 -invariant hermitian metric on $\Omega_{\mathcal{X}/C}^1 = \Omega_{\mathcal{X}}^1|_{\mathcal{X} \setminus \Sigma_f} / f^* \Omega_C^1|_{\mathcal{X} \setminus \Sigma_f}$ induced from the metric $h_{\mathcal{X}/C}$ on $T\mathcal{X}/C$ is also denoted by $h_{\mathcal{X}/C}$.

We define a hermitian metric $h_{f^* \Omega_C^1}$ on $f^* \Omega_C^1|_{\mathcal{X} \setminus \Sigma_f}$ by $h_{f^* \Omega_C^1} = h_{\mathcal{X}}|_{f^* \Omega_C^1|_{\mathcal{X} \setminus \Sigma_f}}$. Let h_C be a hermitian metric on Ω_C^1 such that $h_C(ds, ds) = 1$ on Δ . Then $f^* h_C$ is a hermitian metric on $f^* \Omega_C^1$. We identify f with $s \circ f$. Then $df (= d(s \circ f))$ is a holomorphic 1-form, and

$$f^* h_C(df, df) = f^*(h_C(ds, ds)) = 1, \quad h_{f^* \Omega_C^1}(df, df) = h_{\mathcal{X}}(df, df) = \|df\|^2.$$

Therefore we have

$$h_{f^* \Omega_C^1} = \|df\|^2 f^* h_C. \quad (6.1.1)$$

Let $\tilde{\Omega}_{\mathcal{X}/C}^1$ be the complex of locally free sheaves on $\mathcal{X}$ defined by

$$\tilde{\Omega}_{\mathcal{X}/C}^1 : f^* \Omega_C^1 \rightarrow \Omega_{\mathcal{X}}^1.$$

Since

$$0 \rightarrow f^* \Omega_C^1|_{\mathcal{X}^\circ} \rightarrow \Omega_{\mathcal{X}^\circ}^1 \rightarrow \Omega_{\mathcal{X}^\circ/C^\circ}^1 \rightarrow 0$$

is a short exact sequence of μ_2 -equivariant vector bundles on $\mathcal{X}^\circ$, we have the following canonical isomorphism of holomorphic vector bundles of rank 2 on C°

$$\lambda_{\mu_2}(\Omega_{\mathcal{X}^\circ/C^\circ}^1) \cong \lambda_{\mu_2}(\Omega_{\mathcal{X}^\circ}^1) \otimes \lambda_{\mu_2}(f^* \Omega_C^1)^{-1}|_{C^\circ}.$$

Namely,

$$\lambda_{\pm}(\Omega_{\mathcal{X}^\circ/C^\circ}^1) \cong \lambda_{\pm}(\Omega_{\mathcal{X}^\circ}^1) \otimes \lambda_{\pm}(f^* \Omega_C^1)^{-1}|_{C^\circ}.$$

Definition 6.1.2. We define the Kähler extension of $\lambda_{\mu_2}(\Omega_{\mathcal{X}^\circ/C^\circ}^1)$ by

$$\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1) = \lambda_+(\tilde{\Omega}_{\mathcal{X}/C}^1) \oplus \lambda_-(\tilde{\Omega}_{\mathcal{X}/C}^1), \quad \lambda_{\pm}(\tilde{\Omega}_{\mathcal{X}/C}^1) = \lambda_{\pm}(\Omega_{\mathcal{X}}^1) \otimes \lambda_{\pm}(f^* \Omega_C^1)^{-1}. \quad (6.1.2)$$

Clearly, we have the canonical isomorphism of holomorphic vector bundles on C°

$$\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1)|_{C^\circ} \cong \lambda_{\mu_2}(\Omega_{\mathcal{X}^\circ/C^\circ}^1). \quad (6.1.3)$$

Let $\Pi : \mathbb{P}(\Omega_{\mathcal{X}}^1) \rightarrow \mathcal{X}$ be the projective bundle associated with $\Omega_{\mathcal{X}}^1$ and $\Pi^\vee : \mathbb{P}(T\mathcal{X})^\vee \rightarrow \mathcal{X}$ be the projective bundle associated with $T\mathcal{X}$, where the fiber $\mathbb{P}(T_x \mathcal{X})^\vee$ is the set of all hyperplanes of $T_x \mathcal{X}$.

The Gauss maps $\nu : \mathcal{X} \setminus \Sigma_f \rightarrow \mathbb{P}(\Omega_{\mathcal{X}}^1)$ and $\mu : \mathcal{X} \setminus \Sigma_f \rightarrow \mathbb{P}(T\mathcal{X})^\vee$ are defined by

$$\nu(x) = [df] = \left[\sum_{i=0}^n \frac{\partial f}{\partial z_i}(x) dz_i \right], \quad \mu(x) = [T_x X_{f(x)}].$$

Under the canonical isomorphism $\mathbb{P}(\Omega_{\mathcal{X}}^1) \cong \mathbb{P}(T\mathcal{X})^\vee$, we have $\nu = \mu$.

Let $L = \mathcal{O}_{\mathbb{P}(\Omega_{\mathcal{X}}^1)}(-1) \subset \Pi^*\Omega_{\mathcal{X}}^1$ be the tautological line bundle on $\mathbb{P}(\Omega_{\mathcal{X}}^1)$ and set $Q = \Pi^*\Omega_{\mathcal{X}}^1/L$. We have the following short exact sequence of holomorphic vector bundles on $\Pi^*\Omega_{\mathcal{X}}^1$:

$$\mathcal{S} : 0 \rightarrow L \rightarrow \Pi^*\Omega_{\mathcal{X}}^1 \rightarrow Q \rightarrow 0.$$

Let $H = \mathcal{O}_{\mathbb{P}(T\mathcal{X})^\vee}(1)$ and let U be the universal hyperplane bundle of $(\Pi^\vee)^*T\mathcal{X}$. We have the dual of $\mathcal{S}$ given by

$$\mathcal{S}^\vee : 0 \rightarrow U \rightarrow (\Pi^\vee)^*T\mathcal{X} \rightarrow H \rightarrow 0.$$

The vector bundle $\Pi^*\Omega_{\mathcal{X}}^1$ is endowed with the hermitian metric $\Pi^*h_{\mathcal{X}}$. Let h_L and h_Q be its induced and quotient metrics on L and Q , respectively. The vector bundle $(\Pi^\vee)^*T\mathcal{X}$ is endowed with the hermitian metric $(\Pi^\vee)^*h_{\mathcal{X}}$. Let h_U and h_H be its induced and quotient metrics on U and H , respectively. We denote by $\overline{\mathcal{S}}$ the short exact sequence of holomorphic hermitian vector bundles

$$\overline{\mathcal{S}} = (\mathcal{S}, (h_L, \Pi^*h_{\mathcal{X}}, h_Q)).$$

We have the following isometries of holomorphic hermitian vector bundles over $\mathcal{X} \setminus \Sigma_f$:

$$\overline{\mathcal{F}} = \nu^*\overline{\mathcal{S}}, \quad (T\mathcal{X}/C, h_{\mathcal{X}/C}) = \mu^*(U, h_U), \quad (6.1.4)$$

where $\overline{\mathcal{F}}$ is the short exact sequence of holomorphic hermitian vector bundles over $\mathcal{X} \setminus \Sigma_f$ given by

$$\mathcal{F} : 0 \rightarrow f^*\Omega_C^1 \rightarrow \Omega_{\mathcal{X}}^1 \rightarrow \Omega_{\mathcal{X}/C}^1 \rightarrow 0 \quad \text{and} \quad \overline{\mathcal{F}} = (\mathcal{F}, (h_{f^*\Omega_C^1}, h_{\Omega_{\mathcal{X}}^1}, h_{\mathcal{X}/C})).$$

Since df is a nowhere vanishing holomorphic section of $\nu^*L|_{\mathcal{X} \setminus \Sigma_f}$, we have the following equation of $(1, 1)$ -forms on $\mathcal{X} \setminus \Sigma_f$:

$$\nu^*c_1(L, h_L) = -dd^c \log \|df\|^2. \quad (6.1.5)$$

Note that Σ_f is a proper analytic subset of $\mathcal{X}$. By [64, Theorem 4.5.3], the Gauss maps ν and μ extend to meromorphic maps $\nu : \mathcal{X} \dashrightarrow \mathbb{P}(\Omega_{\mathcal{X}}^1)$ and $\mu : \mathcal{X} \dashrightarrow \mathbb{P}(T\mathcal{X})^\vee$. Then there exist a smooth projective manifold $\widetilde{\mathcal{X}}$, a normal crossing divisor E on $\widetilde{\mathcal{X}}$, a bimeromorphic morphism $q : \widetilde{\mathcal{X}} \rightarrow \mathcal{X}$, and two holomorphic maps $\widetilde{\nu} : \widetilde{\mathcal{X}} \rightarrow \mathbb{P}(\Omega_{\mathcal{X}}^1)$, $\widetilde{\mu} : \widetilde{\mathcal{X}} \rightarrow \mathbb{P}(T\mathcal{X})^\vee$ such that

- $q^{-1}(\Sigma_f) = E$,
- the restriction $q|_{\widetilde{\mathcal{X}} \setminus E} : \widetilde{\mathcal{X}} \setminus E \rightarrow \mathcal{X} \setminus \Sigma_f$ is an isomorphism,
- on $\widetilde{\mathcal{X}} \setminus E$, $\widetilde{\nu} = \nu \circ q$ and $\widetilde{\mu} = \mu \circ q$.

We set $E_0 = E \cap q^{-1}(X_0)$. The fixed locus $\mathcal{X}^\iota$ admits the decomposition $\mathcal{X}^\iota = \mathcal{X}_H^\iota \sqcup \mathcal{X}_V^\iota$, where any connected component of $\mathcal{X}_H^\iota$ is flat over C and $\mathcal{X}_V^\iota \subset f^{-1}(\Delta_f)$. Let $\widetilde{\mathcal{X}}_H^\iota \subset \widetilde{\mathcal{X}}$ be the proper transform of $\mathcal{X}_H^\iota$. We define topological invariants of the germs $(f : (\mathcal{X}, X_0) \rightarrow (C, 0), \Omega_{\mathcal{X}}^1)$ and $(f : (\mathcal{X}, X_0) \rightarrow (C, 0), f^*\Omega_C^1)$ by

$$\begin{aligned} \alpha_\iota(X_0, \Omega_{\mathcal{X}}^1) &= \int_{E_0 \cap \widetilde{\mathcal{X}}_H^\iota} \widetilde{\nu}^* \left\{ \frac{1 - Td(H)^{-1}}{c_1(H)} \right\} q^* \{Td_\iota(T\mathcal{X})ch_\iota(\Omega_{\mathcal{X}}^1)\} \\ &\quad - \int_{\mathcal{X}_V^\iota \cap X_0} Td_\iota(T\mathcal{X})ch_\iota(\Omega_{\mathcal{X}}^1), \\ \alpha_\iota(X_0, f^*\Omega_C^1) &= \int_{E_0 \cap \widetilde{\mathcal{X}}_H^\iota} \widetilde{\nu}^* \left\{ \frac{1 - Td(H)^{-1}}{c_1(H)} \right\} q^* \{Td_\iota(T\mathcal{X})ch_\iota(f^*\Omega_C^1)\} \\ &\quad - \int_{\mathcal{X}_V^\iota \cap X_0} Td_\iota(T\mathcal{X})ch_\iota(f^*\Omega_C^1), \end{aligned}$$

The equivariant Quillen metric on $\lambda_{\mu_2}(\Omega_{\mathcal{X}^\circ/C^\circ}^1)$ with respect to $h_{\mathcal{X}/C}$ is denoted by $\|\cdot\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}^\circ/C^\circ}^1), Q, h_{\mathcal{X}/C}}^2(\iota)$, the equivariant Quillen metric on $\lambda_{\mu_2}(\Omega_{\mathcal{X}}^1)$ with respect to $h_{\mathcal{X}}$ is denoted by $\|\cdot\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}}^1), Q, h_{\mathcal{X}}}^2(\iota)$, the equivariant Quillen metric on $\lambda_{\mu_2}(f^*\Omega_C^1)|_{C^\circ}$ with respect to $h_{f^*\Omega_C^1}$ is denoted by $\|\cdot\|_{\lambda_{\mu_2}(f^*\Omega_C^1), Q, h_{f^*\Omega_C^1}}^2(\iota)$, and the equivariant Quillen metric on $\lambda_{\mu_2}(f^*\Omega_C^1)$ with respect to f^*h_C is denoted by $\|\cdot\|_{\lambda_{\mu_2}(f^*\Omega_C^1), Q, f^*h_C}^2(\iota)$.

Definition 6.1.3. (1) We denote by $\|\cdot\|_{\lambda_{\mu_2}(\widetilde{\Omega}_{\mathcal{X}/C}^1), Q, h_{\mathcal{X}/C}}^2(\iota)$ the equivariant Quillen metric on $\lambda_{\mu_2}(\widetilde{\Omega}_{\mathcal{X}/C}^1)|_{C^\circ}$ induced from $\|\cdot\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}^\circ/C^\circ}^1), Q, h_{\mathcal{X}/C}}^2(\iota)$ by the isomorphism (6.1.3).

(2) Let $\|\cdot\|_{\lambda_{\mu_2}(\widetilde{\Omega}_{\mathcal{X}/C}^1), Q, h_{f^*\Omega_C^1}}^2(\iota)$ be the μ_2 -equivariant metric on $\lambda_{\mu_2}(\widetilde{\Omega}_{\mathcal{X}/C}^1)|_{C^\circ}$ defined by

$$\|\cdot\|_{\lambda_{\mu_2}(\widetilde{\Omega}_{\mathcal{X}/C}^1), Q, h_{f^*\Omega_C^1}}^2(\iota) = \|\cdot\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}}^1), Q, h_{\mathcal{X}}}^2(\iota) \otimes \|\cdot\|_{\lambda_{\mu_2}(f^*\Omega_C^1), Q, h_{f^*\Omega_C^1}}^{-2}(\iota)$$

via the canonical isomorphism (6.1.2).

(3) Let $\|\cdot\|_{\lambda_{\mu_2}(\widetilde{\Omega}_{\mathcal{X}/C}^1), Q, f^*h_C}^2(\iota)$ be the μ_2 -equivariant metric on $\lambda_{\mu_2}(\widetilde{\Omega}_{\mathcal{X}/C}^1)$ defined by

$$\|\cdot\|_{\lambda_{\mu_2}(\widetilde{\Omega}_{\mathcal{X}/C}^1), Q, f^*h_C}^2(\iota) = \|\cdot\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}}^1), Q, h_{\mathcal{X}}}^2(\iota) \otimes \|\cdot\|_{\lambda_{\mu_2}(f^*\Omega_C^1), Q, f^*h_C}^{-2}(\iota).$$

via the canonical isomorphism (6.1.2).

Let $\mathcal{E}$ be a μ_2 -equivariant holomorphic vector bundle on $X = f^{-1}(\Delta)$ with $\dim H^q(X_s, \mathcal{E}|_{X_s})$ constant. Recall that a holomorphic section $\sigma = (\sigma_+, \sigma_-)$ of the determinant of the cohomology $\lambda_{\mu_2}(\mathcal{E}) = \lambda_+(\mathcal{E}) \oplus \lambda_-(\mathcal{E})$ is called an admissible section if both σ_+ and σ_- are nowhere vanishing on Δ .

Let $\sigma_{\mathcal{X}} = (\sigma_{\mathcal{X}}^+, \sigma_{\mathcal{X}}^-)$ be an admissible section of $\lambda_{\mu_2}(\Omega_{\mathcal{X}}^1)|_{\Delta}$. Let $\sigma_C = (\sigma_C^+, \sigma_C^-)$ be an admissible section of $\lambda_{\mu_2}(f^*\Omega_C^1)|_{\Delta}$. We define an admissible section σ of $\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1)|_{\Delta}$ by $\sigma = \sigma_{\mathcal{X}} \otimes \sigma_C^{-1}$.

For $f, g \in C^\infty(\Delta^*)$, we write $f \equiv g$ if $f - g \in C^0(\Delta)$.

Proposition 6.1.4. *The following equalities hold:*

$$\begin{aligned} \log \|\sigma_{\mathcal{X}}(s)\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}}^1), Q, h_{\mathcal{X}}}^2(\iota) &\equiv \alpha_\iota(X_0, \Omega_{\mathcal{X}}^1) \log |s|^2, \\ \log \|\sigma_C(s)\|_{\lambda_{\mu_2}(f^*\Omega_C^1), Q, f^*h_C}^2(\iota) &\equiv \alpha_\iota(X_0, f^*\Omega_C^1) \log |s|^2. \end{aligned}$$

Proof. This proposition follows from Theorem 6.5.4 below. $\square$

Let f_* be the integration along the fibers of $f : \mathcal{X}^\iota \rightarrow C$.

Proposition 6.1.5. *The following equality of functions on Δ holds:*

$$\log \left(\frac{\|\cdot\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1), Q, h_{\mathcal{X}/C}}^2(\iota)}{\|\cdot\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1), Q, h_{f^*\Omega_C^1}}^2(\iota)} \right) \equiv 0.$$

Proof. Note that $\mathcal{F}$ is acyclic on $\mathcal{X}^\circ$. By the anomaly formula for equivariant Quillen metrics (Theorem 1.2.8), we have

$$\log \left(\frac{\|\cdot\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1), Q, h_{\mathcal{X}/C}}^2(\iota)}{\|\cdot\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1), Q, h_{f^*\Omega_C^1}}^2(\iota)} \right) = \left[f_* \left(Td_\iota(T\mathcal{X}/C, h_{\mathcal{X}/C}) \tilde{ch}_\iota(\overline{\mathcal{F}}) \right) \right]^{(0,0)},$$

where $\tilde{ch}_\iota(\overline{\mathcal{F}})$ is the Bott-Chern secondary class (Theorem 1.2.5) such that

$$-dd^c \tilde{ch}_\iota(\overline{\mathcal{F}}) = ch_\iota(f^*\Omega_C^1, h_{f^*\Omega_C^1}) + ch_\iota(\Omega_{\mathcal{X}/C}^1, h_{\mathcal{X}/C}) - ch_\iota(\Omega_{\mathcal{X}}^1, h_{\Omega_{\mathcal{X}}^1}).$$

We set $\tilde{f} = f \circ q : \tilde{\mathcal{X}} \rightarrow C$. The integration along the fibers of f and $\tilde{f}$ are denoted by f_* and $(\tilde{f})_*$, respectively. For any differential form α on $\tilde{\mathcal{X}}^\iota \setminus q^{-1}(\Sigma_f)$, we have $(\tilde{f})_*(\alpha) = f_*((q^{-1})^*\alpha)$. On Δ^* , we deduce from (6.1.4) that

$$\begin{aligned} f_* \left(Td_\iota(T\mathcal{X}/C, h_{\mathcal{X}/C}) \tilde{ch}_\iota(\overline{\mathcal{F}}) \right) &= f_* \left(\mu^* Td_\iota(U, h_U) \nu^* \tilde{ch}_\iota(\overline{\mathcal{S}}) \right) \\ &= f_*(q^{-1})^* \left(\tilde{\mu}^* Td_\iota(U, h_U) \tilde{\nu}^* \tilde{ch}_\iota(\overline{\mathcal{S}}) \right) \\ &= (\tilde{f})_* \left(\tilde{\mu}^* Td_\iota(U, h_U) \tilde{\nu}^* \tilde{ch}_\iota(\overline{\mathcal{S}}) \right). \end{aligned}$$

Since $\tilde{\mu}^* Td_\iota(U, h_U) \tilde{\nu}^* \tilde{ch}_\iota(\overline{\mathcal{S}})$ is a smooth differential form on $\tilde{\mathcal{X}}^\iota$ and since $\tilde{f} : \tilde{\mathcal{X}}^\iota \rightarrow C$ is a proper holomorphic map, $\left[(\tilde{f})_* \left(\tilde{\mu}^* Td_\iota(U, h_U) \tilde{\nu}^* \tilde{ch}_\iota(\overline{\mathcal{S}}) \right) \right]^{(0,0)}$ is a continuous function on Δ by [4, Théorème 4 bis.]. Therefore the function $\left[f_* \left(Td_\iota(T\mathcal{X}/C, h_{\mathcal{X}/C}) \tilde{ch}_\iota(\overline{\mathcal{F}}) \right) \right]^{(0,0)}$ on Δ^* extends to a continuous function on Δ . This completes the proof. $\square$

We set

$$\beta_\iota(X_0, \tilde{\Omega}_{\mathcal{X}/C}^1) = \int_{E_0 \cap \tilde{\mathcal{X}}_H^\iota} \tilde{\mu}^* Td_\iota(U, h_U) \frac{\exp(\tilde{\nu}^* c_1(L, h_L)) - 1}{\tilde{\nu}^* c_1(L, h_L)}.$$

Proposition 6.1.6. *The following equality of functions on Δ holds:*

$$\log \left(\frac{\|\cdot\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1, Q, h_{f^*\Omega_C^1})}^2(\iota)}{\|\cdot\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1, Q, f^*h_C)}^2(\iota)} \right) \equiv \beta_\iota(X_0, \tilde{\Omega}_{\mathcal{X}/C}^1) \log |s|^2.$$

Proof. Let $\tilde{ch}_\iota(f^*\Omega_C^1; f^*h_C, h_{f^*\Omega_C^1})$ and $\tilde{ch}(f^*\Omega_C^1; f^*h_C, h_{f^*\Omega_C^1})$ be the Bott-Chern secondary classes (Theorem 1.2.5) such that

$$\begin{aligned} dd^c \tilde{ch}_\iota(f^*\Omega_C^1; f^*h_C, h_{f^*\Omega_C^1}) &= ch_\iota(f^*\Omega_C^1, f^*h_C) - ch_\iota(f^*\Omega_C^1, h_{f^*\Omega_C^1}), \\ dd^c \tilde{ch}(f^*\Omega_C^1; f^*h_C, h_{f^*\Omega_C^1}) &= ch(f^*\Omega_C^1, f^*h_C) - ch(f^*\Omega_C^1, h_{f^*\Omega_C^1}) \end{aligned}$$

hold on $\mathcal{X}^\iota \setminus \Sigma_f$ and $\mathcal{X} \setminus \Sigma_f$, respectively. By the anomaly formula for equivariant Quillen metrics (Theorem 1.2.8), we have

$$\begin{aligned} \log \left(\frac{\|\cdot\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1, Q, h_{f^*\Omega_C^1})}^2(\iota)}{\|\cdot\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1, Q, f^*h_C)}^2(\iota)} \right) &= \log \left(\frac{\|\cdot\|_{\lambda_{\mu_2}(f^*\Omega_C^1, Q, f^*h_C)}^2(\iota)}{\|\cdot\|_{\lambda_{\mu_2}(f^*\Omega_C^1, Q, h_{f^*\Omega_C^1})}^2(\iota)} \right) \\ &= \left[f_* \left(Td_\iota(T\mathcal{X}/C, h_{\mathcal{X}/C}) \tilde{ch}_\iota(f^*\Omega_C^1; f^*h_C, h_{f^*\Omega_C^1}) \right) \right]^{(0,0)}. \end{aligned}$$

Since ι acts trivially on $f^*\Omega_C^1|_{\mathcal{X}^\iota \setminus \Sigma_f}$, we have

$$\tilde{ch}_\iota(f^*\Omega_C^1; f^*h_C, h_{f^*\Omega_C^1}) = \tilde{ch}(f^*\Omega_C^1; f^*h_C, h_{f^*\Omega_C^1})|_{\mathcal{X}^\iota \setminus \Sigma_f}.$$

By [27, (5.5)], we have

$$\tilde{ch}(f^*\Omega_C^1; f^*h_C, h_{f^*\Omega_C^1}) = \frac{\exp(\nu^* c_1(L, h_L)) - 1}{\nu^* c_1(L, h_L)} \log \|df\|^2. \quad (6.1.6)$$

By (6.1.4) and (6.1.6), we have

$$\begin{aligned} & \left[f_* \left(Td_\iota(T\mathcal{X}/C, h_{\mathcal{X}/C}) \tilde{ch}_\iota(f^*\Omega_C^1; f^*h_C, h_{f^*\Omega_C^1}) \right) \right]^{(0,0)} \\ &= \left[f_* \left(\mu^* Td_\iota(U, h_U) \frac{\exp(\nu^* c_1(L, h_L)) - 1}{\nu^* c_1(L, h_L)} \log \|df\|^2 \right) \right]^{(0,0)} \\ &= \left[\tilde{f}_* \left(\tilde{\mu}^* Td_\iota(U, h_U) \frac{\exp(\tilde{\nu}^* c_1(L, h_L)) - 1}{\tilde{\nu}^* c_1(L, h_L)} q^* \log \|df\|^2 \right) \right]^{(0,0)} \\ &\equiv \beta_\iota(X_0, \tilde{\Omega}_{\mathcal{X}/C}^1) \log |s|^2, \end{aligned}$$

where the last equivalence holds by [83, Lemma 4.4 and Corollary 4.6]. $\square$

We define a topological invariant $\gamma_\iota(X_0, \tilde{\Omega}_{\mathcal{X}/C}^1)$ by

$$\gamma_\iota(X_0, \tilde{\Omega}_{\mathcal{X}/C}^1) = \alpha_\iota(X_0, \Omega_{\mathcal{X}}^1) - \alpha_\iota(X_0, f^*\Omega_C^1) + \beta_\iota(X_0, \tilde{\Omega}_{\mathcal{X}/C}^1).$$

By Definition 6.1.3 (3) and by Propositions 6.1.4, 6.1.5, and 6.1.6, we obtain the following result.

Proposition 6.1.7. *The following equality of functions on Δ holds:*

$$\log \|\sigma(s)\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1, Q, h_{\mathcal{X}/C})}^2(\iota) \equiv \gamma_\iota(X_0, \tilde{\Omega}_{\mathcal{X}/C}^1) \log |s|^2.$$

6.1.2 Logarithmic extension and singularity of equivariant L^2 -metrics

Let $\mathcal{V}$ be a flat vector bundle on the punctured disk Δ^* . The Gauss-Manin connection ∇ is defined by

$$\nabla(v \otimes f) = v \otimes df,$$

where v is a flat section of $\mathcal{V}$ and f is a holomorphic function. The lower Deligne extension $(\mathcal{V}_{\log}, \tilde{\nabla})$ is a unique extension of $(\mathcal{V}, \nabla)$ such that

- (1) $\mathcal{V}_{\log}$ is a holomorphic vector bundle on Δ with $\mathcal{V}_{\log}|_{\Delta^*} = \mathcal{V}$,
- (2) $\tilde{\nabla} : \mathcal{V}_{\log} \rightarrow \mathcal{V}_{\log} \otimes \Omega_{\Delta}^1(\log[0])$ is a logarithmic connection with $\tilde{\nabla}|_{\Delta^*} = \nabla$,
- (3) If α is an eigenvalue of the residue map $\text{res}_0(\nabla)$, then $\alpha \in \mathbb{Q} \cap [0, 1)$.

Here the residue map $\text{res}_0(\nabla) : \mathcal{V}_{\log}(0) \rightarrow \mathcal{V}_{\log}(0)$ is induced by

$$(1 \otimes R_0) \circ \tilde{\nabla} : \mathcal{V}_{\log} \rightarrow \mathcal{V}_{\log} \otimes \Omega_{\Delta}^1(\log[0]) \rightarrow \mathcal{V}_{\log},$$

where $\mathcal{V}_{\log}(0) = \mathcal{V}_{\log}/I_0\mathcal{V}_{\log}$ with $I_0 \subset \mathcal{O}_{\Delta}$ being the ideal sheaf of $0 \in \Delta$ and $R_0 : \Omega_{\Delta}^1(\log[0]) \rightarrow \mathbb{C}$ is the Poincaré residue map given by $R_0(f \frac{ds}{s}) = f(0)$.

Following Eriksson-Freixas i Montplet-Mourougane [25, §2], let us briefly recall the explicit construction of $\mathcal{V}_{\log}$. Let $q : \mathbb{H} \rightarrow \Delta^*$ be the universal covering defined by $q(\tau) = \exp(2\pi i\tau)$, where $\mathbb{H}$ is the upper half plane. Let $\nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_{\Delta^*}^1$ be the Gauss-Manin connection defined by

$$\nabla(v \otimes f) = v \otimes df,$$

where v is a flat section of $\mathcal{V}$ and f is a holomorphic function. We define the space V_{∞} of flat sections of $\mathcal{V}$ by

$$V_{\infty} = \text{Ker}(q^*\nabla : \Gamma(\mathbb{H}, q^*\mathcal{V}) \rightarrow \Gamma(\mathbb{H}, q^*\mathcal{V} \otimes \Omega_{\mathbb{H}}^1)).$$

The monodromy transformation $T \in \text{End } V_{\infty}$ is defined by $T\sigma(\tau) = \sigma(\tau + 1)$. By the monodromy theorem [73, (6.1)], T is quasi-unipotent. By the Jordan decomposition, we

have $T = T_s T_u = T_u T_s$, where T_s is a semi-simple endomorphism and T_u is a unipotent endomorphism. The logarithm of T_u is well-defined. We choose the lower branch of logarithm that takes values in $\mathbb{R} + 2\pi i(-1, 0]$ and denote it by ${}^l\log$. Then we can define ${}^l\log T_s$ and we set

$$\frac{1}{2\pi i} \log T = \frac{1}{2\pi i} {}^l\log T_s + \frac{1}{2\pi i} \log T_u.$$

For $e \in V_\infty$, we define $\tilde{e} \in \Gamma(\mathbb{H}, q^*\mathcal{V})$ by

$$\tilde{e}(\tau) = \exp(-\tau \log T) e(\tau).$$

Since $\tilde{e}(\tau + 1) = \tilde{e}(\tau)$, we may regard $\tilde{e}$ as the section of $\mathcal{V}$ over Δ^* . Therefore we have an injective $\mathbb{C}$ -linear map of vector spaces

$$\phi : V_\infty \rightarrow \Gamma(\Delta^*, \mathcal{V}), \quad e \mapsto \tilde{e}(q),$$

which induces an isomorphism of vector bundles on Δ^*

$$V_\infty \otimes \mathcal{O}_{\Delta^*} \cong \mathcal{V}. \quad (6.1.7)$$

We define $\mathcal{V}_{\log}$ by

$$\mathcal{V}_{\log} = \phi(V_\infty) \otimes \mathcal{O}_\Delta.$$

By taking the fiber of $\mathcal{V}_{\log}$ at 0, we have an isomorphism of vector spaces

$$\psi : V_\infty \rightarrow \mathcal{V}_{\log}(0), \quad e \mapsto \tilde{e}(0). \quad (6.1.8)$$

Consider the normal crossing degeneration $f : X = f^{-1}(\Delta) \rightarrow \Delta$. We recall Steenbrink's construction of the lower extension of the Hodge bundle induced from $f : X^\circ = f^{-1}(\Delta^*) \rightarrow \Delta^*$. For more details, we refer to [77, §2], [78, §2] and [25, §2].

We define locally free sheaves on $X = f^{-1}(\Delta)$ by

$$\Omega_{X/\Delta}^1(\log) = \Omega_X^1(\log X_0)/f^*\Omega_\Delta^1(\log[0]), \quad \Omega_{X/\Delta}^p(\log) = \wedge^p \Omega_{X/\Delta}^1(\log).$$

These sheaves are equipped with the relative exterior differential d and $(\Omega_{X/\Delta}^\bullet(\log), d)$ is called the logarithmic de Rham complex of f .

Let $\tilde{\mathcal{V}}_{\log} = R^k f_* \Omega_{X/\Delta}^\bullet(\log)$ be the k -th direct image sheaf of the complex of vector bundles $(\Omega_{X/\Delta}^\bullet(\log), d)$. By [77, (2.18) Theorem], $\tilde{\mathcal{V}}_{\log} = R^k f_* \Omega_{X/\Delta}^\bullet(\log)$ is a locally free sheaf. By [77, (2.2) Proposition], we have $\tilde{\mathcal{V}}_{\log}|_{\Delta^*} = \mathcal{V}$. Hence $\tilde{\mathcal{V}}_{\log}$ is a holomorphic vector bundle on Δ and is an extension of the flat vector bundle $\mathcal{V} = R^k f_* \mathbb{C} \otimes \mathcal{O}_{\Delta^*}$. By [77, (2.19), (2.20) Propositions], there exists a logarithmic connection $\tilde{\nabla} : \tilde{\mathcal{V}}_{\log} \rightarrow \tilde{\mathcal{V}}_{\log} \otimes \Omega_\Delta^1(\log[0])$ such that $\tilde{\nabla}|_{\Delta^*} = \nabla$, and such that if α is an eigenvalue of the residue map $\text{res}_0(\nabla)$, then $\alpha \in \mathbb{Q} \cap [0, 1)$. By the uniqueness of the Deligne extension, $\tilde{\mathcal{V}}_{\log}$ is the lower Deligne extension of $\mathcal{V}$.

In what follows, we identify $\tilde{\mathcal{V}}_{\log} = R^k f_* \Omega_{X/\Delta}^\bullet(\log)$ with $\mathcal{V}_{\log}$. Let $j : \Delta^* \hookrightarrow \Delta$ be the inclusion and let $\mathcal{F}^p \mathcal{V}$ be the Hodge filtration bundle of $\mathcal{V}$. We define the Hodge filtration bundle $\mathcal{F}_{\log}^p$ of $\mathcal{V}_{\log}$ by

$$\mathcal{F}_{\log}^p = j_*(\mathcal{F}^p \mathcal{V}) \cap \mathcal{V}_{\log} \subset j_* \mathcal{V}.$$

Namely, $\mathcal{F}_{\log}^p$ consists of sections of $\mathcal{F}^p \mathcal{V}$ which extends to sections of $\mathcal{V}_{\log}$.

By [78, (2.11) Theorem], $R^q f_* \Omega_{X/\Delta}^p(\log)$ is a locally free sheaf. Since the spectral sequence

$$E_1^{p,q} = R^q f_* \Omega_{X/\Delta}^p(\log) \Rightarrow R^k f_* \Omega_{X/\Delta}^\bullet(\log)$$

attached to the bête filtration $\Omega_{X/\Delta}^{\bullet \geq p}(\log)$ degenerates at E_1 (cf. [91, p. 130]), we have

$$\mathcal{F}_{\log}^p = \text{Im}(R^k f_* \Omega_{X/\Delta}^{\bullet \geq p}(\log) \rightarrow R^k f_* \Omega_{X/\Delta}^\bullet(\log)) \quad (6.1.9)$$

and

$$\mathcal{F}_{\log}^p / \mathcal{F}_{\log}^{p+1} = R^{k-p} f_* \Omega_{X/\Delta}^p(\log).$$

We have the following short exact sequence of coherent sheaves on Δ

$$0 \rightarrow \mathcal{F}_{\log}^{p+1} \rightarrow \mathcal{F}_{\log}^p \rightarrow R^{k-p} f_* \Omega_{X/\Delta}^p(\log) \rightarrow 0.$$

By the local freeness of $R^q f_* \Omega_{X/\Delta}^p(\log)$, $\mathcal{F}_{\log}^p$ is a locally free subsheaf of $\mathcal{V}_{\log}$.

Recall that $X^\circ = f^{-1}(\Delta^*)$ is equipped with the μ_2 -action induced from the involution ι and that Δ^* is equipped with trivial μ_2 -action. Since $f : X^\circ \rightarrow \Delta^*$ is μ_2 -equivariant, the μ_2 -action preserves the fibers of f . Therefore the μ_2 -action induces the μ_2 -equivariant structure on the flat vector bundle $\mathcal{V} = R^k f_* \mathbb{C} \otimes \mathcal{O}_{\Delta^*}$.

By the definition of the Gauss-Manin connection ∇ , it is a μ_2 -equivariant map and the μ_2 -action on $\Gamma(\mathbb{H}, q^* \mathcal{V})$ preserves V_∞ . Since the identification $(q^* \mathcal{V})_\tau \cong \mathcal{V}_{q(\tau)} \cong \mathcal{V}_{q(\tau+1)} \cong (q^* \mathcal{V})_{\tau+1}$ is μ_2 -equivariant, the monodromy operator T is also μ_2 -equivariant. By its construction, ϕ is μ_2 -equivariant and $\mathcal{V}_{\log} = \phi(V_\infty) \otimes \mathcal{O}_\Delta$ is endowed with the μ_2 -action induced from the μ_2 -action on V_∞ .

Lemma 6.1.8. *Let E, F be μ_2 -equivariant holomorphic vector bundles on Δ and let $\varphi : E \rightarrow F$ be an isomorphism of holomorphic vector bundles. If $\varphi|_{E|_{\Delta^*}} : E|_{\Delta^*} \rightarrow F|_{\Delta^*}$ is μ_2 -equivariant, then $\varphi : E \rightarrow F$ is an isomorphism of μ_2 -equivariant holomorphic vector bundles on Δ .*

Proof. Let r be the rank of E . The involutions of E and F induced from the μ_2 -action are denoted by $i : E \rightarrow E$ and $j : F \rightarrow F$, respectively. Since E and F are holomorphic vector bundles on Δ , E and F are trivial vector bundles and there are global holomorphic frames $e_1, \dots, e_r$ and $f_1, \dots, f_r$ of E and F , respectively. We define the $r \times r$ -matrix $A = (a_{pq})$ by

$$\varphi(i(e_p)) - j(\varphi(e_p)) = \sum_q a_{pq} f_q.$$

Then a_{pq} is a holomorphic function on Δ for each p, q . Since φ is μ_2 -equivariant over Δ^* , we have $a_{pq}|_{\Delta^*} = 0$. Since a_{pq} is continuous, we have $a_{pq} = 0$ on Δ . Therefore for each p, q ,

$$\varphi(i(e_p)) = j(\varphi(e_p)).$$

Thus φ is a μ_2 -equivariant isomorphism. This completes the proof. $\square$

Lemma 6.1.9. *The μ_2 -action on $\mathcal{V}_{\log}$ is the extension of the μ_2 -action on $\mathcal{V}$ and preserves the Hodge filtration $\mathcal{F}_{\log}^p$.*

Proof. Since ϕ is a μ_2 -equivariant homomorphism, (6.1.7) is a μ_2 -equivariant isomorphism. Therefore, $\mathcal{V}_{\log}|_{\Delta^*} = \phi(V_{\infty}) \otimes \mathcal{O}_{\Delta^*} \cong \mathcal{V}$ is a μ_2 -equivariant isomorphism and the μ_2 -action on $\mathcal{V}_{\log}$ is the extension of the μ_2 -action on $\mathcal{V}$. Consider the μ_2 -action of $R^k f_* \Omega_{X/\Delta}^{\bullet}(\log)$ induced from the μ_2 -action of $\Omega_{X/\Delta}^{\bullet}(\log)$. Since the μ_2 -action on $R^k f_* \Omega_{X/\Delta}^{\bullet}(\log)$ is the extension of the μ_2 -action on $\mathcal{V} = R^k f_* \mathbb{C} \otimes \mathcal{O}_{\Delta^*}$ via the identification $R^k f_* \Omega_{X^{\circ}/\Delta^*}^{\bullet}(\log) \cong R^k f_* \mathbb{C} \otimes \mathcal{O}_{\Delta^*}$, it follows from Lemma 6.1.8 that $\mathcal{V}_{\log} \cong R^k f_* \Omega_{X/\Delta}^{\bullet}(\log)$ is a μ_2 -equivariant isomorphism. Since the μ_2 -action on $R^k f_* \Omega_{X/\Delta}^{\bullet}(\log)$ preserves subbundle $\text{Im}(R^k f_* \Omega_{X/\Delta}^{\bullet \geq p}(\log) \rightarrow R^k f_* \Omega_{X/\Delta}^{\bullet}(\log))$, the corresponding subbundle $\mathcal{F}_{\log}^p$ of $\mathcal{V}_{\log}$ is a μ_2 -equivariant subbundle via the isomorphism (6.1.9). This completes the proof. $\square$

By Lemma 6.1.9, $\mathcal{V}_{\log}$, $\mathcal{F}_{\log}^p$, and $R^{k-p} f_* \Omega_{X/\Delta}^p(\log)$ are μ_2 -equivariant vector bundles on Δ . Their (± 1) -eigenbundles are denoted by $\mathcal{V}_{\log, \pm}$, $\mathcal{F}_{\log, \pm}^p$, and $R^{k-p} f_* \Omega_{X/\Delta}^p(\log)_{\pm}$, respectively. These are holomorphic vector bundles on Δ .

Consider a semistable reduction

$$\begin{array}{ccc} Y & \xrightarrow{r} & X \\ g \downarrow & & \downarrow f \\ \Delta' & \xrightarrow{\rho} & \Delta, \end{array}$$

where Δ' is the unit disk, t is a coordinate on Δ' , $g : Y \rightarrow \Delta$ is a semistable degeneration and $\rho(t) = t^l$. Let s be the coordinate on Δ with $s = t^l$. Since $g^{-1}(\Delta'^*)$ is endowed with the μ_2 -action induced from the μ_2 -action on X , $R^k g_* \mathbb{C} \otimes \mathcal{O}_{\Delta'^*}$ is a μ_2 -equivariant flat vector bundle on Δ'^* . In the same manner as above, its lower Deligne extension $\mathcal{U} = R^k g_* \Omega_{Y/\Delta'}^{\bullet}(\log)$ is a μ_2 -equivariant vector bundle on Δ' and it has a μ_2 -equivariant Hodge filtration bundle ${}^{\rho}\mathcal{F}^p$ such that

$${}^{\rho}\mathcal{F}^p / {}^{\rho}\mathcal{F}^{p+1} \cong R^{k-p} g_* \Omega_{Y/\Delta'}^p(\log)$$

is a μ_2 -equivariant isomorphism.

Let h_X be the μ_2 -invariant Kähler metric on X such that $h_X = h_{\mathcal{X}}|_X$ and let ω_X be the associated Kähler form. Set $Y^{\circ} = g^{-1}(\Delta^*)$. Since the restriction $r|_{Y^{\circ}} : Y^{\circ} \rightarrow X^{\circ}$ is a μ_2 -equivariant finite étale cover,

$$\omega_{Y^{\circ}} := (r^* \omega_X)|_{Y^{\circ}} \tag{6.1.10}$$

is a μ_2 -invariant Kähler form on Y° . The associated Kähler metric on Y° is denoted by h_{Y° .

We denote by $H^q(\Omega_{X_\infty}^p)_\pm$ the (± 1) -eigenspace of the μ_2 -action on $H^q(\Omega_{X_\infty}^p)$. We set $h_\pm^{p,q} = \dim H^q(\Omega_{X_\infty}^p)_\pm$ and $h_\pm^p = \sum_{i \geq p} h_\pm^{i,k-i}$. The pullback of differential forms from X to Y induces μ_2 -equivariant injections of locally free sheaves of equal rank

$$\rho^* \mathcal{V}_{\log} \hookrightarrow \mathcal{U}, \quad \rho^* \mathcal{F}_{\log}^p \hookrightarrow {}^\rho \mathcal{F}^p, \quad \rho^* R^q f_* \Omega_{X/\Delta}^p(\log) \hookrightarrow R^q g_* \Omega_{Y/\Delta'}^p(\log).$$

The quotients are torsion sheaves and we can write

$$\frac{{}^\rho \mathcal{F}_\pm^p}{{}^\rho \mathcal{F}_{\log,\pm}^p} \cong \bigoplus_{j=1}^{h_\pm^p} \frac{\mathcal{O}_{\Delta',0}}{t^{a_{j,\pm}^p} \mathcal{O}_{\Delta',0}}, \quad \frac{R^q g_* \Omega_{Y/\Delta'}^p(\log)_\pm}{\rho^* R^q f_* \Omega_{X/\Delta}^p(\log)_\pm} \cong \bigoplus_{j=1}^{h_\pm^{p,q}} \frac{\mathcal{O}_{\Delta',0}}{t^{b_{j,\pm}^{p,q}} \mathcal{O}_{\Delta',0}}$$

for some integers $a_{j,\pm}^p$ and $b_{j,\pm}^{p,q}$. By [25, Lemma 2.4], $0 \leq a_{j,\pm}^p \leq l-1$ and $0 \leq b_{j,\pm}^{p,q} \leq l-1$. Moreover $a_{j,\pm}^p/l$, $b_{j,\pm}^{p,q}/l$ are independent of the choice of semistable reduction. We set

$$\alpha_{j,\pm}^{p,q} = b_{j,\pm}^{p,q}/l \quad \text{and} \quad \alpha_\pm^{p,q} = \sum_j \alpha_{j,\pm}^{p,q}. \quad (6.1.11)$$

The space of flat sections of $\mathcal{U}|_{\Delta'^*}$ is denoted by U_∞ . There is a μ_2 -equivariant isomorphism between V_∞ and U_∞ defined by

$$\nu : V_\infty \rightarrow U_\infty \quad \nu(e)(\tau) = e(l\tau).$$

In the same way as (6.1.8), we have the μ_2 -equivariant isomorphism

$${}^\rho \psi : U_\infty \xrightarrow{\sim} \mathcal{U}(0),$$

and the Hodge filtration ${}^\rho \mathcal{F}^p$ induces a filtration $F_{\infty,\pm}^p$ on $V_{\infty,\pm}$ through the isomorphism ${}^\rho \psi \circ \nu$, which is called Steenbrink's limiting Hodge filtration.

By [25, Corollary 2.8], $\exp(-2\pi i \alpha_{j,\pm}^{p,q})$ forms the eigenvalues of the T_s -action on $\text{Gr}_{F_{\infty,\pm}^p}^p H^k(X_\infty)_\pm$ and

$$\alpha_\pm^{p,q} = -\frac{1}{2\pi i} \text{Tr}(l \log T_s |_{\text{Gr}_{F_{\infty,\pm}^p}^p H^{p+q}(X_\infty)_\pm}). \quad (6.1.12)$$

Lemma 6.1.10. *For every $p, q \geq 0$, there exists a nowhere vanishing holomorphic section $\theta_\pm^{p,q}$ of $\det R^q f_* \Omega_{X/\Delta}^p(\log)_\pm$ such that $t^{-\sum_j b_{j,\pm}^{p,q}} \rho^* \theta_\pm^{p,q}$ is a nowhere vanishing holomorphic section of $\det R^q g_* \Omega_{Y/\Delta'}^p(\log)_\pm$.*

Proof. We set

$$\begin{aligned} k &= h_+^{p+1} = \text{rank } \mathcal{F}_{\log,+}^{p+1}, & \ell &= h_+^p = \text{rank } \mathcal{F}_{\log,+}^p, \\ m &= h_-^{p+1} = \text{rank } \mathcal{F}_{\log,-}^{p+1}, & n &= h_-^p = \text{rank } \mathcal{F}_{\log,-}^p. \end{aligned}$$

Choose a local frame of $\mathcal{F}_{\log,+}^p$ and $\mathcal{F}_{\log,-}^p$ around 0 and denote them by

$$\sigma_{1,+}, \dots, \sigma_{\ell,+} \quad \text{and} \quad \sigma_{1,-}, \dots, \sigma_{n,-}$$

such that $\sigma_{1,+}, \dots, \sigma_{\ell,+}$ forms a local frame of $\mathcal{F}_{\log,+}^{p+1}$, such that $\sigma_{1,-}, \dots, \sigma_{n,-}$ forms a local frame of $\mathcal{F}_{\log,-}^{p+1}$, such that $\sigma_{k+1,+}, \dots, \sigma_{\ell,+}$ projects onto a local frame of $\mathcal{F}_{\log,+}^p / \mathcal{F}_{\log,+}^{p+1} \cong R^q f_* \Omega_{X/\Delta}^p(\log)_+$, and such that $\sigma_{m+1,-}, \dots, \sigma_{n,-}$ projects onto a local frame of $\mathcal{F}_{\log,-}^p / \mathcal{F}_{\log,-}^{p+1} \cong R^q f_* \Omega_{X/\Delta}^p(\log)_-$. Applying [25, Theorem 2.6 and the proof of Corollary 2.8] to the local frame $\sigma_{1,+}, \dots, \sigma_{\ell,+}, \sigma_{1,-}, \dots, \sigma_{n,-}$ of $\mathcal{F}_{\log}^p$, there is another frame

$$\theta_1, \dots, \theta_\ell, \theta_{\ell+1}, \dots, \theta_{\ell+n}$$

with the following properties:

- (1) $\bigoplus_{i=1}^k \mathcal{O}_\Delta \theta_i = \bigoplus_{i=1}^k \mathcal{O}_\Delta \sigma_{i,+} = \mathcal{F}_{\log,+}^{p+1}$ and $\bigoplus_{i=1}^\ell \mathcal{O}_\Delta \theta_i = \bigoplus_{i=1}^\ell \mathcal{O}_\Delta \sigma_{i,+} = \mathcal{F}_{\log,+}^p$,
- (2) there are integers $0 \leq a_i \leq l-1$ such that ${}^\rho \theta_i = t^{-a_i} \rho^* \theta_i$ forms a local frame of ${}^\rho \mathcal{F}^p = {}^\rho \mathcal{F}_+^p \oplus {}^\rho \mathcal{F}_-^p$,
- (3) $\sum_{i=k+1}^\ell a_i = \sum_j b_{j,+}^{p,q}$.

By (1) and (2), $\theta_{k+1}, \dots, \theta_\ell$ projects onto a local frame of the bundle $R^q f_* \Omega_{X/\Delta}^p(\log)_+$ and $t^{-a_{k+1}} \rho^* \theta_{k+1}, \dots, t^{-a_\ell} \rho^* \theta_\ell$ projects onto a local frame of the bundle $R^q g_* \Omega_{Y/\Delta'}^p(\log)_+$. Therefore we obtain a nowhere vanishing holomorphic section $\theta_+^{p,q}$ of $\det R^q f_* \Omega_{X/\Delta}^p(\log)_+$ such that $t^{-\sum_j b_{j,+}^{p,q}} \rho^* \theta_+^{p,q}$ is a nowhere vanishing holomorphic section of $\det R^q g_* \Omega_{Y/\Delta'}^p(\log)_+$.

Similarly, applying [25, Theorem 2.6.] to the local frame $\sigma_{1,-}, \dots, \sigma_{n,-}, \sigma_{1,+}, \dots, \sigma_{\ell,+}$ of $\mathcal{F}_{\log}^p$, we obtain a nowhere vanishing holomorphic section $\theta_-^{p,q}$ of $\det R^q f_* \Omega_{X/\Delta}^p(\log)_-$ such that $t^{-\sum_j b_{j,-}^{p,q}} \rho^* \theta_-^{p,q}$ is a nowhere vanishing holomorphic section of $\det R^q g_* \Omega_{Y/\Delta'}^p(\log)_-$. $\square$

Denote the L^2 -norms on $\det R^q f_* \Omega_{X/\Delta}^p(\log)_\pm|_{\Delta^*}$ and $\det R^q g_* \Omega_{Y/\Delta'}^p(\log)_\pm|_{\Delta'^*}$ with respect to $h_X|_{X^\circ}$ and h_{Y° by $\|\cdot\|_{L^2}^2$.

Lemma 6.1.11. *The following identity holds:*

$$\rho^* \log \|\theta_\pm^{p,q}\|_{L^2}^2 - \log \|t^{-\sum_j b_{j,\pm}^{p,q}} \rho^* \theta_\pm^{p,q}\|_{L^2}^2 = \alpha_\pm^{p,q} l \log |t|^2.$$

Proof. By (6.1.10), $\rho^* R^q f_* \Omega_{X/\Delta}^p(\log)|_{\Delta'^*} \cong R^q g_* \Omega_{Y/\Delta'}^p(\log)|_{\Delta'^*}$ is an isometry with respect to the L^2 -metrics. Therefore, we have

$$\log \|t^{-\sum_j b_{j,\pm}^{p,q}} \rho^* \theta_\pm^{p,q}\|_{L^2}^2 = - \sum_j b_{j,\pm}^{p,q} \log |t|^2 + \rho^* \log \|\theta_\pm^{p,q}\|_{L^2}^2.$$

By (6.1.11), we obtain the desired result. $\square$

Proposition 6.1.12. *Let $\sigma = (\sigma_+, \sigma_-)$ be an admissible section of $\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^p(\log))$ on Δ and let $\|\cdot\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^p(\log)), L^2}^2(t)$ be the L^2 -metric on $\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^p(\log))$. Then the following identity holds:*

$$\log \|\sigma(s)\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^p(\log)), L^2}^2(t) = \sum_q (-1)^q (\alpha_+^{p,q} - \alpha_-^{p,q}) \log |s|^2 + O(\log \log |s|^{-1})$$

Proof. Let $\sigma' = (\sigma'_+, \sigma'_-)$ be an admissible section of $\lambda_{\mu_2}(\Omega_{Y/\Delta'}^p(\log))$ on Δ . Since $g : Y \rightarrow \Delta'$ is a semistable degeneration, we get by [73, (6.6)], [69, Proposition 2.2.1.]

$$\log \|\sigma'_\pm(t)\|_{L^2}^2 = O(\log \log |t|^{-1}) \quad (t \rightarrow 0).$$

Since we can take $\sigma'_\pm = t^{-\sum_j b_{j,\pm}^{p,q}} \rho^* \theta_\pm^{p,q}$, this implies that

$$\log \|t^{-\sum_j b_{j,\pm}^{p,q}} \rho^* \theta_\pm^{p,q}\|_{L^2}^2 = O(\log \log |t|^{-1}) \quad (t \rightarrow 0). \quad (6.1.13)$$

Since we can take $\sigma_\pm = \otimes_q (\theta_\pm^{p,q})^{(-1)^q}$, we get by Lemma 6.1.11 and (6.1.13)

$$\begin{aligned} (\rho^* \log \|\sigma_\pm\|_{L^2}^2)(t) &= \sum_q (-1)^q \log \|t^{-\sum_j b_{j,\pm}^{p,q}} \rho^* \theta_\pm^{p,q}\|_{L^2}^2 + \sum_q (-1)^q \alpha_\pm^{p,q} \log |t|^2 \\ &= \sum_q (-1)^q \alpha_\pm^{p,q} \log |t|^2 + O(\log \log |t|^{-1}). \end{aligned} \quad (6.1.14)$$

Since $s = t^l = \rho(t)$, we get the result by (6.1.14). □

6.1.3 Comparison between the Kähler extension and the log-arithmetic extension

In this subsection, following Eriksson-Freixas i Montplet-Mourougane [25], we compare the Kähler extension with the logarithmic extension.

We write $X_0 = \sum_i m_i D_i$, where D_i is a reduced irreducible component of X_0 . Since X_0 is simply normal crossing, D_i is smooth. We set $D(1) = \sqcup_i D_i$. Let $a : D(1) \rightarrow X$ be the natural morphism.

For a μ_2 -equivariant vector bundle E over a complex manifold M , we set

$$\chi_\pm(E) = \sum_{q \geq 0} (-1)^q \dim H^q(M, E)_\pm.$$

We define integers $\mu_\pm$ by

$$\mu_\pm = \chi_\pm(\mathcal{O}_{X_\infty}) - \sum_i \chi_\pm(\mathcal{O}_{D_i}).$$

Throughout this subsection, tensor products of line bundles are denoted additively for lighter notation.

Proposition 6.1.13. *The following canonical isomorphisms hold:*

$$\lambda_{\pm}(\tilde{\Omega}_{X/\Delta}^1) = \lambda_{\pm}(\Omega_{\mathcal{X}/C}^1(\log)) + \mu_{\pm}\mathcal{O}_{\Delta}([0]).$$

Proof. We follow [25, Proposition 3.7]. By the projection formula, we have

$$\det(R^q f_* f^* \Omega_{\Delta}^1)_{\pm} = \text{rank}(R^q f_* \mathcal{O}_X)_{\pm} \cdot \Omega_{\Delta}^1 + \det(R^q f_* \mathcal{O}_X)_{\pm}.$$

Therefore

$$\lambda_{\pm}(\tilde{\Omega}_{X/\Delta}^1) = -\chi_{\pm}(\mathcal{O}_{X_{\infty}})\Omega_{\Delta}^1 + \lambda_{\pm}(\Omega_X^1) - \lambda_{\pm}(\mathcal{O}_X).$$

In the same manner, we can see that

$$\lambda_{\pm}(\Omega_{X/\Delta}^1(\log)) = -\chi_{\pm}(\mathcal{O}_{X_{\infty}})\Omega_{\Delta}^1(\log[0]) + \lambda_{\pm}(\Omega_X^1(\log X_0)) - \lambda_{\pm}(\mathcal{O}_X).$$

Therefore we have

$$\lambda_{\pm}(\Omega_{X/\Delta}^1(\log)) - \lambda_{\pm}(\tilde{\Omega}_{X/\Delta}^1) = -\chi_{\pm}(\mathcal{O}_{X_{\infty}})\mathcal{O}_{\Delta}([0]) + \lambda_{\pm}(\Omega_X^1(\log X_0)) - \lambda_{\pm}(\Omega_X^1).$$

By the short exact sequence

$$0 \rightarrow \Omega_X^1 \rightarrow \Omega_X^1(\log X_0) \rightarrow a_*\mathcal{O}_{D(1)} \rightarrow 0,$$

we have

$$\lambda_{\pm}(\Omega_X^1(\log X_0)) - \lambda_{\pm}(\Omega_X^1) = \lambda_{\pm}(a_*\mathcal{O}_{D(1)}) = \chi_{\pm}(\mathcal{O}_{D(1)})\mathcal{O}_{\Delta}([0]).$$

Thus we obtain the desired results. $\square$

We set

$$c(X_0, \Omega_{\mathcal{X}/C}^1) = \gamma_{\iota}(X_0, \tilde{\Omega}_{\mathcal{X}/C}^1) + (\mu_+ - \mu_-) - \sum_q (-1)^q (\alpha_+^{1,q} - \alpha_-^{1,q}). \quad (6.1.15)$$

Theorem 6.1.14. *The following identity holds:*

$$\log \tau_{\iota}(\bar{\Omega}_{\mathcal{X}/S}^1) = c(X_0, \Omega_{\mathcal{X}/C}^1) \log |s|^2 + O(\log \log |s|^{-1}) \quad (s \rightarrow 0).$$

Proof. Let $\|\cdot\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1), Q, h_{\mathcal{X}/C}}^2(\iota)$ be the equivariant Quillen metric on the Kähler extension $\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1)$ defined in Definition 6.1.3 and let $\sigma = (\sigma_+, \sigma_-)$ is an admissible section of $\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1)$. By Proposition 6.1.7, we have

$$\log \|\sigma\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1), Q, h_{\mathcal{X}/C}}^2(\iota) \equiv \gamma_{\iota}(X_0, \tilde{\Omega}_{\mathcal{X}/C}^1) \log |s|^2. \quad (6.1.16)$$

Let

$$\|\cdot\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^1(\log)), Q, h_{\mathcal{X}/C}}^2(\iota) := \tau_{\iota}(\bar{\Omega}_{\mathcal{X}/C}^1) \|\cdot\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^1(\log)), L^2, h_{\mathcal{X}/C}}^2(\iota)$$

be the equivariant Quillen metric on the logarithmic extension $\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^1(\log))$ with respect to the Kähler metric $h_{\mathcal{X}/C}$ and let $\sigma' = (\sigma'_+, \sigma'_-)$ be an admissible section of $\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^1(\log))$. By Proposition 6.1.13, there exist holomorphic functions $f_{\pm}$ on Δ such that

$$f_{\pm}(0) \neq 0 \quad \text{and} \quad \sigma_{\pm} = s^{-\mu_{\pm}} f_{\pm} \sigma'_{\pm} \quad (6.1.17)$$

By (6.1.16) and (6.1.17), we have

$$\begin{aligned} & \log \|\sigma'\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^1(\log)), Q, h_{\mathcal{X}/C}}^2(\iota) \\ &= \log \|\sigma'\|_{\lambda_{\mu_2}(\tilde{\Omega}_{\mathcal{X}/C}^1), Q, h_{\mathcal{X}/C}}^2(\iota) + (\mu_+ - \mu_-) \log |s|^2 + O(1) \\ &= \left(\gamma_{\iota}(X_0, \tilde{\Omega}_{\mathcal{X}/C}^1) + (\mu_+ - \mu_-) \right) \log |s|^2 + O(1). \end{aligned} \quad (6.1.18)$$

On the other hand, by Proposition 6.1.12, we have

$$\log \|\sigma'\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^1(\log)), L^2}^2(\iota) = \sum_{q \geq 0} (-1)^q (\alpha_+^{1,q} - \alpha_-^{1,q}) \log |s|^2 + O(\log \log |s|^{-1}) \quad (6.1.19)$$

By (6.1.18) and (6.1.19), we have

$$\begin{aligned} & \log \tau_{\iota}(\bar{\Omega}_{\mathcal{X}/C}^1) \\ &= \log \|\sigma'\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^1(\log)), Q, h_{\mathcal{X}/C}}^2(\iota) - \log \|\sigma'\|_{\lambda_{\mu_2}(\Omega_{\mathcal{X}/C}^1(\log)), L^2, h_{\mathcal{X}/C}}^2(\iota) \\ &= \left(\gamma_{\iota}(X_0, \tilde{\Omega}_{\mathcal{X}/C}^1) + (\mu_+ - \mu_-) - \sum_{q \geq 0} (-1)^q (\alpha_+^{1,q} - \alpha_-^{1,q}) \right) \log |s|^2 + O(\log \log |s|^{-1}). \end{aligned} \quad (6.1.20)$$

By (6.1.15), the proof is completed. $\square$

6.2 The algebraicity of the singularity of $\tau_{M,\mathcal{K}}$

In this section, we prove that $\tau_{M,\mathcal{K}}$ has an algebraic singularity.

6.2.1 Asymptotic behavior of $\tau_{M,\mathcal{K}}$

In this subsection, we keep the notation of the previous section §6.1.1. In addition, we assume that the fiber (X_s, ι_s) of $f : (\mathcal{X}, \iota) \rightarrow C$ at any point $s \in C^{\circ} = C \setminus \Delta_f$ is a manifold of $K3^{[2]}$ -type with involution of type $(M, \mathcal{K})$.

Fix $0 \in \Delta_f$. Let (Δ, s) be a coordinate on C centered at 0 such that $s(\Delta)$ is the unit disk and that $\Delta \cap \Delta_f = \{0\}$. The restriction of f to $X = f^{-1}(\Delta)$ is denoted by $f : X \rightarrow \Delta$. By Assumption 6.1.1, the singular fiber $X_0 = f^{-1}(0)$ is simple normal crossing.

Let $h_X := h_{\mathcal{X}}|_X$ be the μ_2 -invariant Kähler metric on X induced from the μ_2 -invariant Kähler metric $h_{\mathcal{X}}$ on $\mathcal{X}$ and its associated Kähler form is denoted by ω_X . The fiberwise Kähler metric on $TX/\Delta|_{\Delta^*}$ induced from h_X is denoted by $h_{X/\Delta}$ and we set $h_s = h_{X/\Delta}|_{X_s}$ and $\omega_s = \omega_X|_{X_s}$ for $s \in \Delta$. We set $\omega_{X/\Delta} = \{\omega_s\}_{s \in \Delta}$.

Recall that the Kähler form of $h_{\mathcal{X}}$ is assumed to be integral. Then ω_s and $\omega_s|_{X_s^\iota}$ are integral Kähler forms for $s \in \Delta^*$. By [25, Proposition 4.2], the volumes $\text{Vol}(X_s, \omega_s)$, $\text{Vol}(X_s^\iota, \omega_s|_{X_s^\iota})$ and the covolume $\text{Vol}_{L^2}(H^1(X_s^\iota, \mathbb{Z}), \omega_s|_{X_s^\iota})$ are rational numbers. Therefore these are constant functions on Δ and are denoted by $\text{Vol}(X_\infty, \omega_{\mathcal{X}}|_{X_\infty})$, $\text{Vol}(X_\infty^\iota, \omega_{\mathcal{X}}|_{X_\infty^\iota})$ and $\text{Vol}_{L^2}(H^1(X_\infty^\iota, \mathbb{Z}), \omega_{\mathcal{X}}|_{X_\infty^\iota})$, respectively.

We define a smooth function on Δ^* by

$$A(X/\Delta, h_{X/\Delta})(s) = A(X_s, \iota_s, h_s),$$

where $A(X_s, \iota_s, h_s)$ is a positive number defined in (5.2.1).

Consider the relative canonical bundle $K_{X/\Delta} = K_X \otimes f^*K_\Delta^{-1}$. Since $f_*K_{X/\Delta}$ is a torsion free sheaf on Δ , it is an invertible sheaf and we may regard it as a holomorphic line bundle on Δ .

Lemma 6.2.1. *There exists a holomorphic form $\xi \in H^0(X, K_X)$ such that $\text{div}(\xi) \subset X_0$.*

Proof. Since f_*K_X is a holomorphic line bundle on Δ , it is a trivial line bundle and there exists a nowhere-vanishing holomorphic section $\xi \in H^0(\Delta, f_*K_X) = H^0(X, K_X)$ such that $f_*K_X = \mathcal{O}_\Delta \cdot \xi$. The fiber of f_*K_X at $s \in \Delta^*$ is $H^0(X_s, K_X|_{X_s}) = \mathbb{C}\xi|_{X_s}$. Since $K_X|_{X_s} \cong K_{X_s} \cong \mathcal{O}_{X_s}$, $\xi|_{X_s}$ is a nowhere vanishing holomorphic section on X_s . Therefore we have $\text{div}(\xi) \subset X_0$. $\square$

By Lemma 6.2.1, we have a holomorphic form $\xi \in H^0(X, K_X)$ such that $\text{div}(\xi) \subset X_0$. We define a section $\eta_{X/\Delta} \in H^0(X, K_{X/\Delta})$ by

$$\eta_{X/\Delta} = \xi \otimes (f^*ds)^{-1}.$$

For $s \in \Delta^*$, we may identify $\eta_{X/\Delta}|_{X_s}$ with the Poincaré residue

$$\eta_s = \text{Res}_{X_s} \frac{\xi}{f - s} \in H^0(X_s, K_{X_s}).$$

Then we have $\xi|_{X_s} = \eta_s \wedge df$. We regard $\eta_{X/\Delta}$ as a family of holomorphic 4-forms and we regard it as a section of $H^0(\Delta, f_*K_{X/\Delta})$.

By [25, Theorem 4.4], we have

$$\log \|\eta_s\|_{L^2}^2 = \alpha^{4,0} \log |s|^2 + O(\log \log |s|^{-1}), \quad (6.2.1)$$

where $\alpha^{4,0}$ is a rational number defined by (cf. (6.1.12))

$$\alpha^{4,0} = \alpha_+^{4,0} + \alpha_-^{4,0}, \quad \alpha_\pm^{4,0} = -\frac{1}{2\pi i} \text{Tr}({}^l \log T_s|_{\text{Gr}_{F_\infty, \pm}^4 H^4(X_\infty)_\pm}).$$

Consider the projective bundle $\Pi' : \mathbb{P}(T\mathcal{X}_H^\iota)^\vee \rightarrow \mathcal{X}_H^\iota$. Let $\mu' : \mathcal{X}_H^\iota \setminus \Sigma_f \rightarrow \mathbb{P}(T\mathcal{X}_H^\iota)^\vee$ be the Gauss map defined by $\mu'(x) = [T_x X_{f(x)}^\iota]$. Let $A \in \mathbb{P}(T\mathcal{X}_H^\iota)^\vee$ and set $x = \Pi'(A)$. Since the $(+1)$ -eigenspace of the μ_2 -action on $T_x \mathcal{X}$ is identified with $T_x \mathcal{X}_H^\iota$ and since the (-1) -eigenspace is identified with $N_{\mathcal{X}_H^\iota/\mathcal{X},x}$, $A \oplus N_{\mathcal{X}_H^\iota/\mathcal{X},x}$ is a hyperplane of $T_x \mathcal{X} = T_x \mathcal{X}_H^\iota \oplus N_{\mathcal{X}_H^\iota/\mathcal{X},x}$. We define an injective holomorphic map $\phi : \mathbb{P}(T\mathcal{X}_H^\iota)^\vee \rightarrow \mathbb{P}(T\mathcal{X})^\vee$ by

$$\phi([A]) = [A \oplus N_{\mathcal{X}_H^\iota/\mathcal{X},x}].$$

Therefore $\mathbb{P}(T\mathcal{X}_H^\iota)^\vee$ is regarded as a closed submanifold of $\mathbb{P}(T\mathcal{X})^\vee$ via ϕ .

Recall that the Gauss maps ν and μ extend to meromorphic maps $\nu : \mathcal{X} \dashrightarrow \mathbb{P}(\Omega_{\mathcal{X}}^1)$ and $\mu : \mathcal{X} \dashrightarrow \mathbb{P}(T\mathcal{X})^\vee$. Then there exist a smooth projective manifold $\widetilde{\mathcal{X}}$, a normal crossing divisor E on $\widetilde{\mathcal{X}}$, a bimeromorphic morphism $q : \widetilde{\mathcal{X}} \rightarrow \mathcal{X}$, and two holomorphic maps $\widetilde{\nu} : \widetilde{\mathcal{X}} \rightarrow \mathbb{P}(\Omega_{\widetilde{\mathcal{X}}}^1)$, $\widetilde{\mu} : \widetilde{\mathcal{X}} \rightarrow \mathbb{P}(T\mathcal{X})^\vee$ such that

- $q^{-1}(\Sigma_f) = E$,
- the restriction $q|_{\widetilde{\mathcal{X}} \setminus E} : \widetilde{\mathcal{X}} \setminus E \rightarrow \mathcal{X} \setminus \Sigma_f$ is an isomorphism,
- on $\widetilde{\mathcal{X}} \setminus E$, $\widetilde{\nu} = \nu \circ q$ and $\widetilde{\mu} = \mu \circ q$.

The strict transform of $\mathcal{X}_H^\iota$ by $q : \widetilde{\mathcal{X}} \rightarrow \mathcal{X}$ is denoted by $\widetilde{\mathcal{X}}_H^\iota$. The restriction of $q : \widetilde{\mathcal{X}} \rightarrow \mathcal{X}$ is also denoted by $q : \widetilde{\mathcal{X}}_H^\iota \rightarrow \mathcal{X}_H^\iota$. Since the following diagram commutes

$$\begin{array}{ccccc} \widetilde{\mathcal{X}}_H^\iota \setminus E & \xrightarrow[\cong]{q} & \mathcal{X}_H^\iota \setminus \Sigma_f & \xrightarrow{\mu'} & \mathbb{P}(T\mathcal{X}_H^\iota)^\vee \\ \downarrow & & \downarrow & & \downarrow \phi \\ \widetilde{\mathcal{X}} \setminus E & \xrightarrow[\cong]{q} & \mathcal{X} \setminus \Sigma_f & \xrightarrow{\mu} & \mathbb{P}(T\mathcal{X})^\vee, \end{array}$$

we have $\widetilde{\mu}(\widetilde{\mathcal{X}}_H^\iota) \subset \mathbb{P}(T\mathcal{X}_H^\iota)^\vee$. Therefore there exists a holomorphic map $\widetilde{\mu}' : \widetilde{\mathcal{X}}_H^\iota \rightarrow \mathbb{P}(T\mathcal{X}_H^\iota)^\vee$ which is an extension of $\mu' \circ q : \widetilde{\mathcal{X}}_H^\iota \setminus E \rightarrow \mathbb{P}(T\mathcal{X}_H^\iota)^\vee$.

Let U and U' be the universal hyperplane bundle over $\mathbb{P}(T\mathcal{X})^\vee$ and $\mathbb{P}(T\mathcal{X}_H^\iota)^\vee$, respectively. These are equipped with hermitian metrics h_U and $h_{U'}$ respectively as in §6.1.1. We have an isometry of the holomorphic hermitian vector bundle over $\mathcal{X}_H^\iota \setminus \Sigma_f$:

$$(T\mathcal{X}_H^\iota/C, h_{\mathcal{X}_H^\iota/C}) = (\mu')^*(U', h_{U'}).$$

We define a ∂ and $\bar{\partial}$ -closed smooth form $\widetilde{\Omega}$ on $\widetilde{\mathcal{X}}_H^\iota$ by

$$\widetilde{\Omega} = \widetilde{\mu}'^* c_1(U', h_{U'})^2 - 8\widetilde{\mu}'^* c_2(U', h_{U'}) - \widetilde{\mu}^* c_1(U, h_U)|_{\widetilde{\mathcal{X}}_H^\iota}^2 + 3\widetilde{\mu}^* c_2(U, h_U)|_{\widetilde{\mathcal{X}}_H^\iota}.$$

On $\widetilde{\mathcal{X}}_H^\iota \setminus E$, we have

$$q^* \Omega = \widetilde{\Omega}.$$

Recall that E_0 was defined by $E_0 = E \cap q^{-1}(X_0)$. We define a rational number $\delta(X_0, \Omega_{X/\Delta}^1)$ by

$$\delta(X_0, \Omega_{X/\Delta}^1) = \frac{1}{48} \int_{E_0 \cap \widetilde{\mathcal{X}}_H^\iota} \tilde{\Omega} - \frac{1}{48} \int_{q^* \operatorname{div}(\xi) \cap \widetilde{\mathcal{X}}_H^\iota} \tilde{\Omega} - \frac{1}{16} \alpha^{4,0}(t^2 + 7). \quad (6.2.2)$$

Lemma 6.2.2. *The following identity holds:*

$$\log A(X/\Delta)(s) = \delta(X_0, \Omega_{X/\Delta}^1) \log |s|^2 + O(\log \log |s|^{-1}) \quad (s \rightarrow 0).$$

Proof. Note that $\omega_X^5/5!$ is the volume form on X and $\omega_{X/\Delta}^4/4!|_{X_s}$ is the volume form on X_s for each $s \in \Delta$. We have

$$\frac{\omega_X^5}{5!} = \frac{\omega_{X/\Delta}^4}{4!} \wedge \frac{idf \wedge \overline{df}}{\|df\|^2}.$$

Then we have

$$\frac{\eta_{X/\Delta} \wedge \overline{\eta}_{X/\Delta}}{\omega_{X/\Delta}^4/4!} = \frac{\eta_{X/\Delta} \wedge \overline{\eta}_{X/\Delta} \wedge df \wedge \overline{df}}{\omega_{X/\Delta}^4/4! \wedge df \wedge \overline{df}} = \frac{i}{\|df\|^2} \frac{\xi \wedge \overline{\xi}}{\omega_X^5/5!} = \frac{\|\xi\|^2}{\|df\|^2}.$$

We set $\tilde{f} = f \circ q : \widetilde{\mathcal{X}}_H^\iota \rightarrow \Delta$. By Lemma 5.3.3, we have

$$\begin{aligned} & 48 \log A(X/\Delta, h_{X/\Delta}) \\ &= f_* \left[\log \left\{ \frac{\omega_{X/\Delta}^4/4!}{\eta_{X/\Delta} \wedge \overline{\eta}_{X/\Delta}} \frac{\|\eta_{X/\Delta}\|_{L^2}^2}{\operatorname{Vol}(X_\infty, \omega_{\mathcal{X}}|_{X_\infty})} \right\} \Omega \right] \\ &= f_* \left[\log \frac{\|df\|^2}{\|\xi\|^2} \Omega \right] - 3(t^2 + 7) \log \frac{\|\eta_{X/\Delta}\|_{L^2}^2}{\operatorname{Vol}(X_\infty, \omega_{\mathcal{X}}|_{X_\infty})} \\ &= \tilde{f}_* \{ \log q^* \|df\|^2 \tilde{\Omega} \} - 3(t^2 + 7) \log \|\eta_{X/\Delta}\|_{L^2}^2 \\ &\quad - \tilde{f}_* \{ \log q^* \|\xi\|^2 \tilde{\Omega} \} + 3(t^2 + 7) \log \operatorname{Vol}(X_\infty, \omega_{\mathcal{X}}|_{X_\infty}). \end{aligned} \quad (6.2.3)$$

By [83, Lemma 4.4 and Corollary 4.6], we have

$$\tilde{f}_* \{ \log q^* \|df\|^2 \tilde{\Omega} \} = \left(\int_{E_0 \cap \widetilde{\mathcal{X}}_H^\iota} \tilde{\Omega} \right) \log |s|^2 + O(1) \quad (6.2.4)$$

and

$$\tilde{f}_* \{ \log q^* \|\xi\|^2 \tilde{\Omega} \} = \left(\int_{q^* \operatorname{div}(\xi) \cap \widetilde{\mathcal{X}}_H^\iota} \tilde{\Omega} \right) \log |s|^2 + O(1). \quad (6.2.5)$$

By the formula (6.2.1), (6.2.2), (6.2.3), (6.2.4) and (6.2.5), the proof is completed. $\square$

Consider the flat family $f : \mathcal{X}_H^\iota \rightarrow C$. Let us calculate the singularity of the analytic torsion $\tau(\overline{\mathcal{O}}_{\mathcal{X}_H^\iota})$ of $\overline{\mathcal{O}}_{\mathcal{X}_H^\iota}$, where $\mathcal{O}_{\mathcal{X}_H^\iota}$ is equipped with the canonical metric. Since $\mathcal{O}_{\mathcal{X}_H^\iota}$ is a holomorphic line bundle on $\mathcal{X}_H^\iota$, the determinant of the cohomology $\lambda(\mathcal{O}_{\mathcal{X}_H^\iota})$ is a holomorphic line bundle on C . We set

$$\hat{c}(X_0^\iota, \mathcal{O}_{X_0^\iota}) = \alpha(X_0^\iota, \mathcal{O}_{X_0^\iota}) - \sum_{q=0}^2 (-1)^q \alpha^{0,q}, \quad (6.2.6)$$

where we define

$$\alpha(X_0^\iota, \mathcal{O}_{X_0^\iota}) = \int_{E_0 \cap \widetilde{\mathcal{X}}_H^\iota} \widetilde{\mu}'^* \left\{ Td(U') \frac{Td(H) - 1}{c_1(H)} \right\} q^* ch(\mathcal{O}_{\mathcal{X}_H^\iota}), \quad (6.2.7)$$

and

$$\alpha^{0,q} = -\frac{1}{2\pi i} \text{Tr}({}^l \log T_s |_{\text{Gr}_{F_\infty}^0 H^q(X_\infty^\iota)}). \quad (6.2.8)$$

Here T is the monodromy operator and F_∞ is the Steenbrink's limiting Hodge filtration for the family $f : \mathcal{X}^\iota \rightarrow C$.

Lemma 6.2.3. *The following identity holds:*

$$\log \tau(\overline{\mathcal{O}}_{\mathcal{X}_H^\iota}) = \hat{c}(X_0^\iota, \mathcal{O}_{X_0^\iota}) \log |s|^2 + O(\log \log |s|^{-1}) \quad (s \rightarrow 0).$$

Proof. Let $\|\cdot\|_{Q, \lambda(\mathcal{O}_{\mathcal{X}_H^\iota})}^2$ and $\|\cdot\|_{L^2, \lambda(\mathcal{O}_{\mathcal{X}_H^\iota})}^2$ be the Quillen and L^2 -metrics on $\lambda(\mathcal{O}_{\mathcal{X}_H^\iota})$, respectively. Since $\lambda(\mathcal{O}_{\mathcal{X}_H^\iota})|_\Delta$ is a trivial line bundle on Δ , there exists a nowhere-vanishing holomorphic section σ of $\lambda(\mathcal{O}_{\mathcal{X}_H^\iota})|_\Delta$. By [83, Theorem 6.1] (See also [12]), we have

$$\log \|\sigma\|_{Q, \lambda(\mathcal{O}_{\mathcal{X}_H^\iota})}^2 = \alpha(X_0^\iota, \mathcal{O}_{X_0^\iota}) \log |s|^2 + O(1) \quad (s \rightarrow 0). \quad (6.2.9)$$

By [25, Theorem 4.4.(1)], we have

$$\log \|\sigma\|_{L^2, \lambda(\mathcal{O}_{\mathcal{X}_H^\iota})}^2 = \left(\sum_{q=0}^2 (-1)^q \alpha^{0,q} \right) \log |s|^2 + O(\log \log |s|^{-1}) \quad (s \rightarrow 0). \quad (6.2.10)$$

By (6.2.9) and (6.2.10), we obtain the desired result. $\square$

We set

$$\kappa(X_0, \Omega_{X/\Delta}^1, \mathcal{O}_{X^\iota}) = c(X_0, \Omega_{\mathcal{X}/C}^1) + \delta(X_0, \Omega_{X/\Delta}^1) - 2\hat{c}(X_0^\iota, \mathcal{O}_{X_0^\iota}) \quad (6.2.11)$$

Theorem 6.2.4. *The following identity holds:*

$$\log \tau_{M,K,\mathcal{X}/S} = \kappa(X_0, \Omega_{X/\Delta}^1, \mathcal{O}_{X^\iota}) \log |s|^2 + O(\log \log |s|^{-1}) \quad (s \rightarrow 0).$$

Proof. As we remarked above, the volumes $\text{Vol}(X_\infty, \omega_{\mathcal{X}}|_{X_\infty})$, $\text{Vol}(X_\infty^\iota, \omega_{\mathcal{X}}|_{X_\infty^\iota})$ and the covolume $\text{Vol}_{L^2}(H^1(X_\infty^\iota, \mathbb{Z}), \omega_{\mathcal{X}}|_{X_\infty^\iota})$ are constant functions on Δ . The result follows from Theorem 6.1.14, Definition 5.2.2, Lemmas 6.2.2 and 6.2.3. $\square$

6.2.2 Construction of a 1-parameter degeneration over a finite branched covering of a curve on the moduli space

The Baily-Borel compactification of the orthogonal modular variety $\mathcal{M}_{M,\mathcal{K}} = \Omega_{M^\perp}^+ / \Gamma_{M^\perp, \mathcal{K}}$ is denoted by $\mathcal{M}_{M,\mathcal{K}}^*$. Let $\Pi_{M^\perp, \mathcal{K}} : \Omega_{M^\perp}^+ \rightarrow \Omega_{M^\perp}^+ / \Gamma_{M^\perp, \mathcal{K}}$ be the natural projection.

Assumption 6.2.5. Let $C \subset \mathcal{M}_{M,\mathcal{K}}^*$ be an irreducible projective curve. We assume that C is not contained in $\bar{\mathcal{D}}_{M^\perp} \cup (\mathcal{M}_{M,\mathcal{K}}^* \setminus \mathcal{M}_{M,\mathcal{K}})$.

Proposition 6.2.6. *There exist a smooth projective curve B , a morphism $\varphi : B \rightarrow \mathcal{M}_{M,\mathcal{K}}^*$, a smooth projective variety $\mathcal{X}$ of complex dimension 5, an involution $\iota : \mathcal{X} \rightarrow \mathcal{X}$, and a surjective morphism $f : \mathcal{X} \rightarrow B$ such that*

- (1) $\varphi(B) = C$.
- (2) $f \circ \iota = f$.
- (3) *There exists a Zariski open subset $B^\circ \neq \emptyset$ of B such that $\varphi(B^\circ) \subset \mathcal{M}_{M,\mathcal{K}}^\circ$ and for each $b \in B^\circ$, the fiber (X_b, ι_b) is a manifold of $K3^{[2]}$ -type with involution of type $(M, \mathcal{K})$ whose period is $\varphi(b)$.*
- (4) $f : \mathcal{X} \rightarrow B$ is a normal crossing degeneration. Namely, its singular fibers are simple normal crossing divisor on $\mathcal{X}$.

Proof. We follow [82, Theorem 2.8] and [86, Theorem 3.1.(1)].

Step 1 By Assumption 6.2.5, there exists a point $x_0 \in C \cap \mathcal{M}_{M,\mathcal{K}}^\circ$ such that $x_0 \notin \text{Sing } C$. Let $\tilde{x}_0 \in \Pi_{M^\perp, \mathcal{K}}^{-1}(x_0)$ be a point. The stabilizer of $\Gamma_{M^\perp, \mathcal{K}}$ at $\tilde{x}_0$ is denoted by $\Gamma_{M^\perp, \mathcal{K}, \tilde{x}_0}$. Since $\Gamma_{M^\perp, \mathcal{K}}$ acts properly discontinuously on $\Omega_{M^\perp}^+$, there exists an open neighborhood $\widetilde{W} \subset \Omega_{M^\perp}^+ \setminus \bar{\mathcal{D}}_{M^\perp}$ of $\tilde{x}_0$ such that

$$\{\gamma \in \Gamma_{M^\perp, \mathcal{K}}; \gamma(\widetilde{W}) \cap \widetilde{W} \neq \emptyset\} = \Gamma_{M^\perp, \mathcal{K}, \tilde{x}_0}.$$

Note that $\Gamma_{M^\perp, \mathcal{K}, \tilde{x}_0}$ is a finite group. By shrinking $\widetilde{W}$, we assume that $\widetilde{W}$ is $\Gamma_{M^\perp, \mathcal{K}, \tilde{x}_0}$ -invariant. Then the restriction $\Pi_{M^\perp, \mathcal{K}} : \widetilde{W} \rightarrow \Pi_{M^\perp, \mathcal{K}}(\widetilde{W})$ can be identified with the quotient map $\widetilde{W} \rightarrow \widetilde{W} / \Gamma_{M^\perp, \mathcal{K}, \tilde{x}_0}$ by the finite group $\Gamma_{M^\perp, \mathcal{K}, \tilde{x}_0}$. Therefore there exists a smooth analytic curve U of $\Pi_{M^\perp, \mathcal{K}}^{-1}(C) \cap \widetilde{W} (\subset \Omega_{M^\perp}^+)$ such that $\Pi_{M^\perp, \mathcal{K}}(U)$ is a smooth analytic curve of $C \cap \Pi_{M^\perp, \mathcal{K}}(\widetilde{W})$ and the restriction $\Pi_{M^\perp, \mathcal{K}} : U \rightarrow \Pi_{M^\perp, \mathcal{K}}(U)$ is an isomorphism.

Let $\tilde{x} \in U \subset \Omega_{M^\perp}^+ \setminus \bar{\mathcal{D}}_{M^\perp}$ and set $x = \Pi_{M^\perp, \mathcal{K}}(\tilde{x}) \in \mathcal{M}_{M,\mathcal{K}}^\circ$. By Proposition 2.2.14, the image of the period map $P_{M,\mathcal{K}} : \tilde{\mathcal{M}}_{M,\mathcal{K}} \rightarrow \mathcal{M}_{M,\mathcal{K}}$ is $\mathcal{M}_{M,\mathcal{K}}^\circ$. There exists $(X, \iota) \in \tilde{\mathcal{M}}_{M,\mathcal{K}}$ such that $P_{M,\mathcal{K}}(X, \iota) = x$. Let $\pi : (\mathcal{X}, \iota) \rightarrow \text{Def}(X, \iota)$ be the Kuranishi family of (X, ι) . By the local Torelli theorem, we may regard $\text{Def}(X, \iota)$ as an open neighborhood of $\tilde{x}$ in $\Omega_{M^\perp}^+$. By shrinking U , we assume $U \subset \text{Def}(X, \iota)$. Let $\tilde{\gamma} : U \rightarrow \text{Def}(X, \iota)$ be the inclusion. Then there exists a family $P_{\mathcal{Z}} : (\mathcal{Z}, \iota_{\mathcal{Z}}) \rightarrow U$ of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$ with period $\tilde{\gamma} : U \rightarrow \text{Def}(X, \iota) (\subset \Omega_{M^\perp}^+)$.

By Lemma 5.3.10, we may assume that $P_Z : \mathcal{Z} \rightarrow U$ is a projective morphism. Since $\mathcal{Z}$ has a μ_2 -equivariant relatively ample line bundle, we may assume that there exist a sufficiently large positive integer $N > 0$ and an involution $\iota_{\mathbb{P}^N} : \mathbb{P}^N \rightarrow \mathbb{P}^N$ such that $\mathcal{Z} \subset \mathbb{P}^N \times U$, $P_Z = \text{pr}_2|_{\mathcal{Z}}$, and $\iota_Z = (\iota_{\mathbb{P}^N} \times \text{id}_U)|_{\mathcal{Z}}$, where $\text{pr}_2 : \mathbb{P}^N \times U \rightarrow U$ is the second projection.

The Hilbert polynomial of $Z_t = P_Z^{-1}(t)$ with respect to $\mathcal{O}_{\mathbb{P}^N}(1)$ is denoted by $P(m)$. Let $\text{Hilb}_P(\mathbb{P}^N)$ be the Hilbert scheme parametrizing the closed subschemes of $\mathbb{P}^N$ with Hilbert polynomial P . This is a projective variety and there is the universal projective flat family $\pi : \mathcal{X} \rightarrow \text{Hilb}_P(\mathbb{P}^N)$. The point of $\text{Hilb}_P(\mathbb{P}^N)$ corresponding to a subscheme Z of $\mathbb{P}^N$ with Hilbert polynomial P is denoted by $[Z]$. We define a morphism $c : U \rightarrow \text{Hilb}_P(\mathbb{P}^N)$ by $c(t) = [Z_t]$. The family $P_Z : \mathcal{Z} \rightarrow U$ is given by the pullback of $\pi : \mathcal{X} \rightarrow \text{Hilb}_P(\mathbb{P}^N)$ by $c : U \rightarrow \text{Hilb}_P(\mathbb{P}^N)$.

The involution $\iota_{\mathbb{P}^N} : \mathbb{P}^N \rightarrow \mathbb{P}^N$ induces an involution $I : \text{Hilb}_P(\mathbb{P}^N) \rightarrow \text{Hilb}_P(\mathbb{P}^N)$ given by $I([Z]) = [\iota_{\mathbb{P}^N}(Z)]$. We denote the fixed locus of I by $\text{Hilb}_P^I(\mathbb{P}^N)$, and denote the restriction of π by $\pi^I : \mathcal{X}^I = \pi^{-1}(\text{Hilb}_P^I(\mathbb{P}^N)) \rightarrow \text{Hilb}_P^I(\mathbb{P}^N)$. There exists an involution $\mathcal{I} : \mathcal{X}^I \rightarrow \mathcal{X}^I$ such that $\pi^I \circ \mathcal{I} = \pi^I$ and $\mathcal{I} = (\iota_{\mathbb{P}^N} \times \text{id}_{\text{Hilb}_P(\mathbb{P}^N)})|_{\mathcal{X}^I}$. Since $\iota_Z = (\iota_{\mathbb{P}^N} \times \text{id}_U)|_{\mathcal{Z}}$, we have $c(U) \subset \text{Hilb}_P^I(\mathbb{P}^N)$. Let T be the Zariski closure of $c(U)$ in $\text{Hilb}_P^I(\mathbb{P}^N)$. This is an irreducible reduced projective variety and $c(U) \cap (T \setminus \text{Sing } T) \neq \emptyset$. The family $P_Z : (\mathcal{Z}, \iota_Z) \rightarrow U$ is the pullback of $\pi^I|_T : (\mathcal{X}^I|_T, \mathcal{I}|_T) \rightarrow T$ by $c : U \rightarrow T$.

Step 2 Set $D = \{[V] \in T; \text{Sing } V \neq \emptyset\}$ and $T^\circ = T \setminus (\text{Sing } T \cup D)$. Then the restriction $\pi^I|_{T^\circ} : (\mathcal{X}^I|_{T^\circ}, \mathcal{I}|_{T^\circ}) \rightarrow T^\circ$ is a family of $K3^{[2]}$ -type manifolds with antisymplectic involution of type (M, K) . We denote its period map by $\mathcal{P}_{T^\circ} : T^\circ \rightarrow \mathcal{M}_{M,K}^\circ$. Since $c(U) \cap (T \setminus \text{Sing } T) \neq \emptyset$ and $c(U) \subset T$, we may assume that $c(U) \subset T^\circ$. Since the period map for $P_Z : (\mathcal{Z}, \iota_Z) \rightarrow U$ is $\gamma := \Pi_{M^\perp, K} \circ \tilde{\gamma} : U \rightarrow \mathcal{M}_{M,K}^\circ$, we have $\mathcal{P}_{T^\circ}(c(t)) = \gamma(t)$ for any $t \in U$.

Claim 1. *The period map $\mathcal{P}_{T^\circ} : T^\circ \rightarrow \mathcal{M}_{M,K}^\circ$ extends to a meromorphic map $\mathcal{P}_T : T \dashrightarrow \mathcal{M}_{M,K}^*$. Moreover, the Zariski closure of the graph of $\mathcal{P}_{T^\circ}$ in $T \times \mathcal{M}_{M,K}^*$ is contained in $T \times C$.*

Proof. Let $\mu : \tilde{T} \rightarrow T$ be a resolution of T with the following properties: $\tilde{T}$ is a projective variety, μ is a birational morphism, and there is a normal crossing divisor E on $\tilde{T}$ such that $\mu|_{\tilde{T} \setminus E} : \tilde{T} \setminus E \rightarrow T^\circ$ is an isomorphism. Note that $\mathcal{P}_{T^\circ} : T^\circ \rightarrow \mathcal{M}_{M,K}^\circ$ is locally liftable. Namely, for each point $t \in T^\circ$, there is an open neighborhood $V \subset T^\circ$, and a holomorphic map $\tilde{\mathcal{P}} : V \rightarrow \Omega_{M^\perp}^+$ such that $\Pi_{M,K} \circ \tilde{\mathcal{P}} = \mathcal{P}_{T^\circ}|_V$. Therefore $\mathcal{P}_{T^\circ} \circ \mu|_{\tilde{T} \setminus E} : \tilde{T} \setminus E \rightarrow \mathcal{M}_{M,K}^\circ$ is also locally liftable. By [23, 3.7. Theorem], $\mathcal{P}_{T^\circ} \circ \mu|_{\tilde{T} \setminus E}$ extends to a holomorphic map $\mathcal{P}_{\tilde{T}} : \tilde{T} \rightarrow \mathcal{M}_{M,K}^*$. Set $\mathcal{P}_T = \mathcal{P}_{\tilde{T}} \circ \mu^{-1} : T \dashrightarrow \mathcal{M}_{M,K}^*$. This is a meromorphic map and satisfies $\mathcal{P}_T|_{T^\circ} = \mathcal{P}_{T^\circ}$.

It suffices to check that $\mathcal{P}_{T^\circ}(T^\circ) \subset C$. Let T' be the Zariski closure of $(\mathcal{P}_{T^\circ})^{-1}(C)$ in T . This is a closed subvariety of T . Since $\mathcal{P}_{T^\circ}(c(U)) = \gamma(U) \subset C$, we have $c(U) \subset T'$. By the definition of T , we have $T \subset T'$. Since $T' \subset T$ by definition, we get $T = T'$. Thus we obtain $\mathcal{P}_{T^\circ}(T^\circ) \subset C$. $\square$

Let Γ be the Zariski closure of the graph of $\mathcal{P}_{T^\circ}$ in $T \times \mathcal{M}_{M,K}^*$. By the above claim, Γ is a closed subvariety of $T \times C$. Set $\Gamma^\circ = \Gamma \cap (T^\circ \times \mathcal{M}_{M,K}^\circ)$. For each $t \in U$, we have $(c(t), \gamma(t)) \in \Gamma^\circ$. Therefore, Γ° is a non-empty Zariski open subset of Γ . Let $p : \Gamma \rightarrow T$ and $q : \Gamma \rightarrow C$ be the natural projections. Since $p|_{\Gamma^\circ} : \Gamma^\circ \cong T^\circ$ is isomorphic, we have $\Gamma^\circ \subset \Gamma \setminus \text{Sing } \Gamma$.

Since $\mathcal{P}_{T^\circ}(c(t)) = \gamma(t) \in C$ ($t \in U$), the image of $q : \Gamma \rightarrow C$ contains an open subset of C . Therefore, $q : \Gamma \rightarrow C$ is dominant. Fix a projective embedding $\Gamma \subset \mathbb{P}^{N'}$. There is a hyperplane $H_1 \subset \mathbb{P}^{N'}$ such that $q|_{\Gamma \cap H_1} : \Gamma \cap H_1 \rightarrow C$ is dominant and $\dim(\Gamma \cap H_1) = \dim \Gamma - 1$. Repeating this process, we have hyperplanes $H_1, \dots, H_r \subset \mathbb{P}^{N'}$ such that $q|_{\Gamma \cap H_1 \cap \dots \cap H_r} : \Gamma \cap H_1 \cap \dots \cap H_r \rightarrow C$ is dominant and $\Gamma \cap H_1 \cap \dots \cap H_r$ is a curve. Let B' be an irreducible component of $\Gamma \cap H_1 \cap \dots \cap H_r$ such that $q|_{B'} : B' \rightarrow C$ is dominant. We denote by B the normalization of B' , and denote by h the composition of the normalization morphism $B \rightarrow B'$ and the embedding $B' \rightarrow \Gamma$. Set $\varphi = q \circ h : B \rightarrow C$. Then B is a smooth projective curve and $\varphi : B \rightarrow C$ is a surjective morphism.

Consider the family of $K3^{[2]}$ -type manifolds $\pi_B : \mathcal{X}^I|_T \times_T B \rightarrow B$ induced from $\pi^I|_T : (\mathcal{X}^I|_T, \mathcal{I}|_T) \rightarrow T$ and the involution $\mathcal{I} \times \text{id}_B : \mathcal{X}^I|_T \times_T B \rightarrow \mathcal{X}^I|_T \times_T B$ induced from $\mathcal{I} : \mathcal{X}^I \rightarrow \mathcal{X}^I$ by $p \circ h : B \rightarrow T$. Its period map is given by $\mathcal{P}_T \circ p \circ h = q \circ h = \varphi$. Set $B^\circ = B \cap h^{-1}(\Gamma^\circ)$. Then B° is a non-empty Zariski open subset of B .

Step 3 By [9, Theorem 13.4], there exist a resolution of singularities $\Pi : \mathcal{W} \rightarrow \mathcal{X}^I|_T \times_T B$, and a holomorphic involution $\theta : \mathcal{W} \rightarrow \mathcal{W}$ such that $\mathcal{W}$ is a smooth projective manifold and such that $\Pi : \mathcal{W} \rightarrow \mathcal{X}^I|_T \times_T B$ is μ_2 -equivariant with respect to the μ_2 -actions induced from θ and $\mathcal{I} \times \text{id}_B$. We set $\tilde{\pi} = \pi_B \circ \Pi : (\mathcal{W}, \theta) \rightarrow B$.

By [50, Theorem 3.35], there exists a blowup sequence functor $\mathcal{BP}$ defined on all triples (X, I, E) , where X is a smooth scheme of finite type over $\mathbb{C}$, $I \subset \mathcal{O}_X$ is an ideal sheaf that is not zero on any irreducible component of X , E is a simple normal crossing divisor on X , satisfying the following properties:

- (1) In the blowup sequence $\mathcal{BP}(X, I, E) =$

$$\Pi : X_r \xrightarrow{\pi_{r-1}} X_{r-1} \xrightarrow{\pi_{r-2}} \dots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X_0 = X,$$

all centers of blowups are smooth.

- (2) The pullback $\Pi^* I \subset \mathcal{O}_{X_r}$ is the ideal sheaf of a simple normal crossing divisor.
- (3) $\mathcal{BP}$ commutes with smooth morphism. In particular, for any automorphism $\tau : X \rightarrow X$,

$$\mathcal{BP}(X, \tau^* I, \tau^{-1}(E)) = \tau^* \mathcal{BP}(X, I, E).$$

Here $\tau^* \mathcal{BP}(X, I, E)$ is induced from the pullback

$$\begin{array}{ccccccc} \Pi' : X'_r & \xrightarrow{\tau_{r-1}^* \pi_{r-1}} & X'_{r-1} & \xrightarrow{\tau_{r-2}^* \pi_{r-2}} & \dots & \xrightarrow{\tau_1^* \pi_1} & X'_1 \xrightarrow{\tau^* \pi_0} X_0 = X \\ \downarrow \tau_r & & \downarrow \tau_{r-1} & & & & \downarrow \tau_1 & \downarrow \tau \\ \Pi : X_r & \xrightarrow{\pi_{r-1}} & X_{r-1} & \xrightarrow{\pi_{r-2}} & \dots & \xrightarrow{\pi_1} & X_1 \xrightarrow{\pi_0} X_0 = X \end{array}$$

by deleting $\tau_i^* \pi_i$ whose center is empty and reindexing the resulting blowup sequence. Let $\hat{W}$ be the union of the singular fibers of $\tilde{\pi}$ and let $I_{\hat{W}}$ be its ideal sheaf. Therefore the blowup sequence $\mathcal{BP}(\mathcal{W}, I_{\hat{W}}, \emptyset)$ gives a smooth projective variety $\mathcal{X}$, a projective birational morphism $\Pi' : \mathcal{X} \rightarrow \mathcal{W}$, a simple normal crossing divisor $\hat{X} \subset \mathcal{X}$ with $\hat{X} = (\Pi')^* \hat{W}$. Moreover, $\mathcal{BP}(\mathcal{W}, I_{\hat{W}}, \emptyset) = \mathcal{BP}(\mathcal{W}, \theta^* I_{\hat{W}}, \emptyset) = \theta^* \mathcal{BP}(\mathcal{W}, I_{\hat{W}}, \emptyset)$ induces an involution $\iota : \mathcal{X} \rightarrow \mathcal{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\iota} & \mathcal{X} \\ \Pi' \downarrow & & \downarrow \Pi' \\ \mathcal{W} & \xrightarrow{\theta} & \mathcal{W}. \end{array}$$

Thus $\Pi' : (\mathcal{X}, \iota) \rightarrow (\mathcal{W}, \theta)$ is μ_2 -equivariant and the singular fibers of $f := \tilde{\pi} \circ \Pi'$ are simple normal crossing and the data $(B, \phi, \mathcal{X}, \iota, f)$ satisfies the conditions (1), (2), (3), (4) in the statement. This completes the proof. $\square$

Theorem 6.2.7. *Let $C \subset \mathcal{M}_{M, \mathcal{K}}^*$ be an irreducible projective curve satisfying Assumption 6.2.5. Let $p \in C \cap \tilde{\mathcal{D}}_{M^\perp}$ be a smooth point of C . Choose a coordinate (Δ, s) on C centered at p such that $\Delta \cap \tilde{\mathcal{D}}_{M^\perp} = \{p\}$ and $s(\Delta)$ is the unit disk. There exists a constant $a \in \mathbb{Q}$ such that*

$$\tau_{M, \mathcal{K}}(s) = a \log |s|^2 + O(\log \log |s|^{-1}) \quad (s \rightarrow 0).$$

Proof. Applying Theorem 6.2.4 to the flat family $f : (\mathcal{X}, \iota) \rightarrow B$ obtained in Proposition 6.2.6, we obtain the desired result. $\square$

6.3 Natural involutions

In this section, we study the curvature formula for $\tau_{M, \mathcal{K}}$ in more detail when $\mathcal{K}$ is natural.

6.3.1 2-elementary K3 surfaces and their Hilbert schemes

Let M_0 be a primitive hyperbolic 2-elementary sublattice of $L_{K3} = U^{\oplus 3} \oplus E_8^{\oplus 2}$. Set $r = \text{rk } M_0$, $l = \dim_{\mathbb{Z}_2}(M_0^\vee/M_0)$. Moreover when $M_0 \not\cong U(2) \oplus E_8(2)$, we set

$$g = 11 - \frac{r+l}{2}, \quad k = \frac{r-l}{2} \tag{6.3.1}$$

Recall that L_2 is given by $L_2 = L_{K3} \oplus \mathbb{Z}e$. We define a sublattice $\widetilde{M}_0$ of L_2 by

$$\widetilde{M}_0 = M_0 \oplus \mathbb{Z}e.$$

Then $\widetilde{M}_0$ is an admissible sublattice of L_2 .

Definition 6.3.1. Let $\mathcal{K} \in \text{KT}(\widetilde{M}_0)$. A hyperplane H of $\widetilde{M}_{0,\mathbb{R}}$ is a face of $\mathcal{K}$ if $H \cap \partial\mathcal{K}$ contains an open subset of H .

Definition 6.3.2. A Kähler-type chamber $\mathcal{K} \in \text{KT}(\widetilde{M}_0)$ is natural if the hyperplane $M_{0,\mathbb{R}}$ is a face of $\mathcal{K}$.

Let $\mathcal{K} \in \text{KT}(\widetilde{M}_0)$ be a natural Kähler-type chamber. Then $(X, \iota) = (Y^{[2]}, \sigma^{[2]})$ is a $K3^{[2]}$ -type manifold with involution of type $(\widetilde{M}_0, \mathcal{K})$.

Let $f : (\mathcal{X}, \iota) \rightarrow S$ be a family of $K3^{[2]}$ -type manifolds with involution of type $(\widetilde{M}_0, \mathcal{K})$, and let $g : (\mathcal{Y}, \sigma) \rightarrow S$ be a family of 2-elementary K3 surfaces of type M_0 such that $X_s = Y_s^{[2]}$ and $\iota_s = \sigma_s^{[2]}$ for each $s \in S$. Let $h_{\mathcal{X}/S}$ be an ι -invariant fiberwise Kähler metric on $T\mathcal{X}/S$ and $h_{\mathcal{Y}/S}$ be a σ -invariant fiberwise Kähler metric on $T\mathcal{Y}/S$. Recall that $t = \text{Tr}(\iota^*|_{H^{1,1}(X)})$. Recall that the differential form $\omega_{H^1(\mathcal{X}^\iota/S)}$ was defined by (5.3.18). The aim of this section is to show that

$$\omega_{H^1(\mathcal{X}^\iota/S)} = (t+1)c_1(g_*\Omega_{\mathcal{Y}\sigma/S}^1, h_{L^2}),$$

and to prove that the function $\tau_{\widetilde{M}_0, \mathcal{K}, \mathcal{X}/S}$ satisfies the same $\partial\bar{\partial}$ -equation as the equation satisfied by the invariant τ_{M_0} of 2-elementary K3 surfaces of type M_0 .

If $M_0 = E_8(2) \oplus U(2)$, then the fixed locus X^ι of the fiber (X, ι) is the Enriques surface Y/σ . Therefore we have $\omega_{H^1(\mathcal{X}^\iota/S)} = c_1(g_*\Omega_{\mathcal{Y}\sigma/S}^1, h_{L^2}) = 0$ and this is the trivial case. In the following, we assume that $M_0 \neq E_8(2) \oplus U(2)$.

6.3.2 The Hodge form for natural involutions

Recall that the direct image sheaves $g_*\Omega_{\mathcal{Y}\sigma/S}^1$, $R^1g_*\mathcal{O}_{\mathcal{Y}\sigma}$, $f_*\Omega_{\mathcal{X}^\iota/S}^1$, and $R^1f_*\mathcal{O}_{\mathcal{X}^\iota}$ are locally free $\mathcal{O}_S$ -modules and we may identify them with the corresponding holomorphic vector bundles over S . By the Hodge decomposition, we have

$$\overline{H^0(Y_s^\sigma, \Omega_{Y_s^\sigma}^1)} = H^1(Y_s^\sigma, \mathcal{O}_{Y_s^\sigma}) \quad \text{and} \quad \overline{H^0(X_s^\iota, \Omega_{X_s^\iota}^1)} = H^1(X_s^\iota, \mathcal{O}_{X_s^\iota})$$

for each $s \in S$ and they induce $\mathbb{C}$ -antilinear homomorphisms of complex vector bundles

$$c : g_*\Omega_{\mathcal{Y}\sigma/S}^1 \rightarrow R^1g_*\mathcal{O}_{\mathcal{Y}\sigma} \quad \text{and} \quad c : f_*\Omega_{\mathcal{X}^\iota/S}^1 \rightarrow R^1f_*\mathcal{O}_{\mathcal{X}^\iota}.$$

Lemma 6.3.3. *There exist isomorphisms of holomorphic vector bundles on S*

$$\phi : (g_*\Omega_{\mathcal{Y}\sigma/S}^1)^{\oplus k+1} \rightarrow f_*\Omega_{\mathcal{X}^\iota/S}^1 \quad \text{and} \quad \psi : (R^1g_*\mathcal{O}_{\mathcal{Y}\sigma})^{\oplus k+1} \rightarrow R^1f_*\mathcal{O}_{\mathcal{X}^\iota}$$

such that the following diagram commutes:

$$\begin{array}{ccc} (g_*\Omega_{\mathcal{Y}\sigma/S}^1)^{\oplus k+1} & \xrightarrow{\phi} & f_*\Omega_{\mathcal{X}^\iota/S}^1 \\ \downarrow c & & \downarrow c \\ (R^1g_*\mathcal{O}_{\mathcal{Y}\sigma})^{\oplus k+1} & \xrightarrow{\psi} & R^1f_*\mathcal{O}_{\mathcal{X}^\iota} \end{array}$$

Proof. Let (Y, σ) be a 2-elementary K3 surface of type M_0 and set $(X, \iota) = (Y^{[2]}, \sigma^{[2]})$. By Proposition 4.1.4, the fixed locus of a 2-elementary K3 surface (Y, σ) of type M_0 is

$$Y^\sigma = \begin{cases} \emptyset & (M_0 = E_8(2) \oplus U(2)), \\ E \sqcup F & (M_0 = E_8(2) \oplus U), \\ C \sqcup E_1 \sqcup E_2 \sqcup \cdots \sqcup E_k & (\text{otherwise}), \end{cases} \quad (6.3.2)$$

where E, F are elliptic curves on Y , C is a curve of genus g on Y , and E_i ($i = 1, \dots, k$) are (-2) -curves on Y . The fixed locus X^ι of (X, ι) is given by

$$X^\iota = \begin{cases} Y/\sigma & (M_0 = E_8(2) \oplus U(2)), \\ E^{(2)} \sqcup F^{(2)} \sqcup (E \times F) \sqcup Y/\sigma & (M_0 = E_8(2) \oplus U), \\ C^{(2)} \sqcup (\sqcup_i E_i^{(2)}) \sqcup (\sqcup_i C \times E_i) \sqcup (\sqcup_{i < j} E_i \times E_j) \sqcup Y/\sigma & (\text{otherwise}), \end{cases} \quad (6.3.3)$$

where $E^{(2)}$, $F^{(2)}$, $C^{(2)}$ and $E_i^{(2)}$ are the symmetric product of the curves E , F , C and E_i , respectively. Note that $E^{(2)}$, $F^{(2)}$ are ruled surfaces and that Y/σ , $E_i^{(2)}$, $E_i \times E_j$ are rational surfaces.

If $M_0 \neq E_8(2) \oplus U$, $E_8(2) \oplus U(2)$, then we have by (6.3.2), (6.3.3)

$$\begin{aligned} H^0(Y_s^\sigma, \Omega_{Y_s^\sigma}^1) &= H^0(C, \Omega_C^1), \\ H^0(X_s^\iota, \Omega_{X_s^\iota}^1) &= H^0(C^{(2)}, \Omega_{C^{(2)}}^1) \oplus \oplus_i H^0(C \times E_i, \Omega_{C \times E_i}^1). \end{aligned} \quad (6.3.4)$$

Let $p, q : C \times C \rightarrow C$ be the first and second projection. For a holomorphic 1-form α on C ,

$$p^* \alpha + q^* \alpha$$

induces a holomorphic 1-form β on $C^{(2)}$. By [55, (6.3)], the following homomorphism is an isomorphism

$$H^0(C, \Omega_C^1) \cong H^0(C^{(2)}, \Omega_{C^{(2)}}^1), \quad \alpha \mapsto \beta. \quad (6.3.5)$$

By the Künneth formula, we have

$$H^0(C \times E_i, \Omega_{C \times E_i}^1) \cong H^0(C, \Omega_C^1). \quad (6.3.6)$$

By (6.3.4), (6.3.5) and (6.3.6), we have the canonical isomorphism

$$\begin{aligned} H^0(X_s^\iota, \Omega_{X_s^\iota}^1) &= H^0(C^{(2)}, \Omega_{C^{(2)}}^1) \oplus \oplus_i H^0(C \times E_i, \Omega_{C \times E_i}^1) \\ &\cong H^0(C, \Omega_C^1) \oplus \oplus_i H^0(C, \Omega_C^1) \\ &= H^0(Y_s^\sigma, \Omega_{Y_s^\sigma}^1)^{\oplus k+1}. \end{aligned} \quad (6.3.7)$$

If $M_0 = E_8(2) \oplus U$, then we have by (6.3.2), (6.3.3)

$$\begin{aligned} H^0(Y^\sigma, \Omega_{Y^\sigma}^1) &= H^0(E, \Omega_E^1) \oplus H^0(F, \Omega_F^1), \\ H^0(X^\iota, \Omega_{X^\iota}^1) &= H^0(E^{(2)}, \Omega_{E^{(2)}}^1) \oplus H^0(F^{(2)}, \Omega_{F^{(2)}}^1) \oplus H^0(E \times F, \Omega_{E \times F}^1). \end{aligned}$$

By (6.3.5) and by the Künneth formula, we have the canonical isomorphism

$$\begin{aligned} H^0(X_s^\iota, \Omega_{X_s^\iota}^1) &= H^0(E^{(2)}, \Omega_{E^{(2)}}^1) \oplus H^0(F^{(2)}, \Omega_{F^{(2)}}^1) \oplus H^0(E \times F, \Omega_{E \times F}^1) \\ &\cong H^0(E, \Omega_E^1) \oplus H^0(F, \Omega_F^1) \oplus H^0(E, \Omega_E^1) \oplus H^0(F, \Omega_F^1) \\ &= H^0(Y_s^\sigma, \Omega_{Y_s^\sigma}^1)^{\oplus 2}. \end{aligned} \quad (6.3.8)$$

By (6.3.7), (6.3.8), we have an isomorphism of holomorphic vector bundles

$$\phi : (g_* \Omega_{\mathcal{Y}^\sigma/S}^1)^{\oplus k+1} \cong f_* \Omega_{\mathcal{X}^\iota/S}^1.$$

Similarly, we have an isomorphism of holomorphic vector bundles

$$\psi : (R^1 g_* \mathcal{O}_{\mathcal{Y}^\sigma})^{\oplus k+1} \cong R^1 f_* \mathcal{O}_{\mathcal{X}^\iota}.$$

By their constructions, the following diagram commutes:

$$\begin{array}{ccc} H^0(Y_s^\sigma, \Omega_{Y_s^\sigma}^1)^{\oplus k+1} & \xrightarrow{\phi_s} & H^0(X_s^\iota, \Omega_{X_s^\iota}^1) \\ \downarrow c & & \downarrow c \\ H^1(Y_s^\sigma, \mathcal{O}_{Y_s^\sigma})^{\oplus k+1} & \xrightarrow{\psi_s} & H^1(X_s^\iota, \mathcal{O}_{X_s^\iota}). \end{array}$$

This completes the proof. $\square$

Let $\alpha_1, \dots, \alpha_m$ be a local holomorphic frame of $(g_* \Omega_{\mathcal{Y}^\sigma/S}^1)^{\oplus k+1}$, and let $\beta_1, \dots, \beta_m$ be a local holomorphic frame of $(R^1 g_* \mathcal{O}_{\mathcal{Y}^\sigma})^{\oplus k+1}$. We define real-valued functions $\tilde{c}_1(\phi)$ and $\tilde{c}_1(\psi)$ by

$$\tilde{c}_1(\phi) = -\log \frac{\det(\langle \phi(\alpha_i), \phi(\alpha_j) \rangle_{L^2})}{\det(\langle \alpha_i, \alpha_j \rangle_{L^2})}, \quad \tilde{c}_1(\psi) = -\log \frac{\det(\langle \psi(\beta_i), \psi(\beta_j) \rangle_{L^2})}{\det(\langle \beta_i, \beta_j \rangle_{L^2})} \quad (6.3.9)$$

Note that $\tilde{c}_1(\phi)$ and $\tilde{c}_1(\psi)$ are independent of the choice of holomorphic frames and they are globally defined on S .

Lemma 6.3.4. *We have $\tilde{c}_1(\phi) = \tilde{c}_1(\psi)$.*

Proof. Note that $\bar{\alpha}_1, \dots, \bar{\alpha}_m$ is smooth frame of $R^1 g_* \mathcal{O}_{\mathcal{Y}^\sigma}^{\oplus k+1}$ which is not holomorphic, and that $\psi(\bar{\alpha}_1) = \overline{\phi(\alpha_1)}, \dots, \psi(\bar{\alpha}_m) = \overline{\phi(\alpha_m)}$ is smooth frame of $R^1 f_* \mathcal{O}_{\mathcal{X}^\iota}^{\oplus k+1}$ which is not holomorphic. Let $C = (c_{ij})$ be a $m \times m$ -matrix defined by $\bar{\alpha}_i = \sum_j c_{ij} \beta_j$. Then we have $\overline{\phi(\alpha_i)} = \psi(\bar{\alpha}_i) = \psi(\sum_j c_{ij} \beta_j) = \sum_j c_{ij} \psi(\beta_j)$, and we have

$$\begin{aligned} \frac{\det(\langle \overline{\phi(\alpha_i)}, \overline{\phi(\alpha_j)} \rangle_{L^2})}{\det(\langle \bar{\alpha}_i, \bar{\alpha}_j \rangle_{L^2})} &= \frac{\det(C \cdot (\langle \psi(\beta_i), \psi(\beta_j) \rangle_{L^2}) \cdot {}^t \bar{C})}{\det(C \cdot (\langle \beta_i, \beta_j \rangle_{L^2}) \cdot {}^t \bar{C})} \\ &= \frac{\det(\langle \psi(\beta_i), \psi(\beta_j) \rangle_{L^2})}{\det(\langle \beta_i, \beta_j \rangle_{L^2})}. \end{aligned} \quad (6.3.10)$$

Let γ and δ be smooth $(1, 0)$ -forms. Then δ is primitive, and by [80, Proposition 6.29], we have

$$\langle \bar{\gamma}, \bar{\delta} \rangle_{L^2} = \overline{\langle \gamma, \delta \rangle_{L^2}} = \langle \delta, \gamma \rangle_{L^2}. \quad (6.3.11)$$

The same equality holds where γ and δ are smooth $(0, 1)$ -forms. By (6.3.9), (6.3.10) and (6.3.11), we have

$$\begin{aligned} \tilde{c}_1(\phi) &= -\log \frac{\det(\langle \phi(\alpha_i), \phi(\alpha_j) \rangle_{L^2})}{\det(\langle \alpha_i, \alpha_j \rangle_{L^2})} \\ &= -\log \left\{ \frac{\det(\langle \phi(\alpha_i), \phi(\alpha_j) \rangle_{L^2})}{\det(\langle \bar{\phi}(\alpha_i), \bar{\phi}(\alpha_j) \rangle_{L^2})} \frac{\det(\langle \bar{\phi}(\alpha_i), \bar{\phi}(\alpha_j) \rangle_{L^2})}{\det(\langle \bar{\alpha}_i, \bar{\alpha}_j \rangle_{L^2})} \frac{\det(\langle \bar{\alpha}_i, \bar{\alpha}_j \rangle_{L^2})}{\det(\langle \alpha_i, \alpha_j \rangle_{L^2})} \right\} \\ &= -\log \left\{ \frac{\det(\langle \phi(\alpha_i), \phi(\alpha_j) \rangle_{L^2})}{\det^t(\langle \phi(\alpha_i), \phi(\alpha_j) \rangle_{L^2})} \frac{\det(\langle \psi(\beta_i), \psi(\beta_j) \rangle_{L^2})}{\det(\langle \beta_i, \beta_j \rangle_{L^2})} \frac{\det^t(\langle \alpha_i, \alpha_j \rangle_{L^2})}{\det(\langle \alpha_i, \alpha_j \rangle_{L^2})} \right\} \\ &= \tilde{c}_1(\psi), \end{aligned}$$

which completes the proof. $\square$

Proposition 6.3.5. *The following identity holds:*

$$c_1(f_*\Omega_{\mathcal{X}^\iota/S}^1, h_{L^2}) - c_1(R^1f_*\mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) = 2(k+1)c_1(g_*\Omega_{\mathcal{Y}^\sigma/S}^1, h_{L^2}).$$

Proof. By the definition of $\tilde{c}_1(\phi)$ and $\tilde{c}_1(\psi)$, we have

$$\begin{aligned} c_1(f_*\Omega_{\mathcal{X}^\iota/S}^1, h_{L^2}) - c_1((g_*\Omega_{\mathcal{Y}^\sigma/S}^1)^{\oplus k+1}, h_{L^2}) &= dd^c \tilde{c}_1(\phi), \\ c_1(R^1f_*\mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) - c_1((R^1g_*\mathcal{O}_{\mathcal{Y}^\sigma})^{\oplus k+1}, h_{L^2}) &= dd^c \tilde{c}_1(\psi). \end{aligned} \quad (6.3.12)$$

By the Serre duality (See Lemma 5.4.1), we have

$$c_1(R^1g_*\mathcal{O}_{\mathcal{Y}^\sigma}, h_{L^2}) = -c_1(g_*\Omega_{\mathcal{Y}^\sigma/S}^1, h_{L^2}). \quad (6.3.13)$$

By (6.3.12) and (6.3.13) and Lemma 6.3.4, we have the desired result. $\square$

Lemma 6.3.6. *Let (E, h) be a hermitian holomorphic vector bundle on S of rank $g \geq 1$. Then we have*

$$c_1(\wedge^2 E, h) = (g-1)c_1(E, h).$$

Proof. It suffices to show that there exists a canonical isometry

$$(\det \wedge^2 E, h) \cong ((\det E)^{\otimes g-1}, h).$$

Let $e_1, \dots, e_g$ be a local holomorphic frame of E . Set $e_{ij} = e_i \wedge e_j$. Then $\{e_{ij}; i < j\}$ forms a local holomorphic frame of $\wedge^2 E$ and

$$e_{12} \wedge e_{13} \wedge \dots \wedge e_{1g} \wedge e_{23} \wedge \dots \wedge e_{(g-1)g} \quad (6.3.14)$$

is a nowhere vanishing local holomorphic section of $\det \wedge^2 E$. On the other hand,

$$(e_1 \wedge \cdots \wedge e_g) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_g) \quad (6.3.15)$$

is a nowhere vanishing local holomorphic section of $(\det E)^{\otimes g-1}$.

Let $e'_1, \dots, e'_g$ be another local holomorphic frame of E and let $A = (a_{ij})$ be the $g \times g$ -matrix defined by

$$e'_i = \sum_j a_{ij} e_j.$$

Then we have

$$e'_{12} \wedge e'_{13} \wedge \cdots \wedge e'_{1g} \wedge e'_{23} \wedge \cdots \wedge e'_{(g-1),g} = (\det A)^{g-1} e_{12} \wedge e_{13} \wedge \cdots \wedge e_{1g} \wedge e_{23} \wedge \cdots \wedge e_{(g-1),g}$$

and

$$(e'_1 \wedge \cdots \wedge e'_g) \otimes \cdots \otimes (e'_1 \wedge \cdots \wedge e'_g) = (\det A)^{g-1} (e_1 \wedge \cdots \wedge e_g) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_g).$$

Therefore the correspondence from the section (6.3.14) to the section (6.3.15) is independent of the choice of $e_1, \dots, e_g$ and induces an isomorphism of holomorphic line bundles

$$\det \wedge^2 E \cong (\det E)^{\otimes g-1}. \quad (6.3.16)$$

Moreover, if $e_1(s), \dots, e_g(s)$ is an orthonormal basis of (E_s, h_s) at $s \in S$, then the norm of the sections (6.3.14) and (6.3.15) at s are 1. Since $s \in S$ is arbitrary, the isomorphism (6.3.16) is an isometry. $\square$

Proposition 6.3.7. *The following identity holds:*

$$c_1(f_* \Omega_{\mathcal{X}^\iota/S}^2, h_{L^2}) = (g-1)c_1(g_* \Omega_{\mathcal{Y}^\sigma/S}^1, h_{L^2}).$$

Proof. If $M_0 = E_8(2) \oplus U$, then we have by (6.3.2), (6.3.3)

$$H^0(Y_s^\sigma, \Omega_{Y_s^\sigma}^1) = H^0(E, \Omega_E^1) \oplus H^0(F, \Omega_F^1), \quad H^0(X_s^\iota, \Omega_{X_s^\iota}^2) = H^0(E \times F, \Omega_{E \times F}^2),$$

where E and F are elliptic curves. By the Künneth formula, we have the canonical isomorphism

$$H^0(X_s^\iota, \Omega_{X_s^\iota}^2) = H^0(E \times F, \Omega_{E \times F}^2) \cong H^0(E, \Omega_E^1) \otimes H^0(F, \Omega_F^1) = \wedge^2 H^0(Y_s^\sigma, \Omega_{Y_s^\sigma}^1). \quad (6.3.17)$$

If $M_0 \neq E_8(2) \oplus U$, $E_8(2) \oplus U(2)$, then we have by (6.3.2), (6.3.3)

$$H^0(Y_s^\sigma, \Omega_{Y_s^\sigma}^1) = H^0(C, \Omega_C^1), \quad H^0(X_s^\iota, \Omega_{X_s^\iota}^2) = H^0(C^{(2)}, \Omega_{C^{(2)}}^2),$$

where C is a curve of genus g . Let $p, q : C \times C \rightarrow C$ be the first and second projection. For a holomorphic 1-form α, β on C ,

$$p^* \alpha \wedge q^* \beta - p^* \beta \wedge q^* \alpha$$

induces a holomorphic 2-form γ on $C^{(2)}$. By [55, (6.3)], the following homomorphism is an isomorphism

$$\wedge^2 H^0(C, \Omega_C^1) \cong H^0(C^{(2)}, \Omega_{C^{(2)}}^2), \quad \alpha \wedge \beta \mapsto \gamma.$$

Therefore, we have the canonical isomorphism

$$H^0(X_s^\iota, \Omega_{X_s^\iota}^2) = H^0(C^{(2)}, \Omega_{C^{(2)}}^2) \cong \wedge^2 H^0(C, \Omega_C^1) = \wedge^2 H^0(Y_s^\sigma, \Omega_{Y_s^\sigma}^1). \quad (6.3.18)$$

By (6.3.17), (6.3.18), we have the isomorphism of holomorphic vector bundles

$$\zeta : \wedge^2 g_* \Omega_{\mathcal{Y}^\sigma/S}^1 \cong f_* \Omega_{\mathcal{X}^\iota/S}^2.$$

Similarly, we have an isomorphism of holomorphic vector bundles

$$\xi : \wedge^2 R^1 g_* \mathcal{O}_{\mathcal{Y}^\sigma} \cong R^2 f_* \mathcal{O}_{\mathcal{X}^\iota}.$$

By [55, (6.3)] and the Künneth formula, the following diagram commutes:

$$\begin{array}{ccc} \wedge^2 g_* \Omega_{\mathcal{Y}^\sigma/S}^1 & \xrightarrow{\zeta} & f_* \Omega_{\mathcal{X}^\iota/S}^2 \\ \downarrow c & & \downarrow c \\ \wedge^2 R^1 g_* \mathcal{O}_{\mathcal{Y}^\sigma} & \xrightarrow{\xi} & R^2 f_* \mathcal{O}_{\mathcal{X}^\iota} \end{array} \quad (6.3.19)$$

Let $\gamma_1, \dots, \gamma_{m'}$ be a local holomorphic frame of $\wedge^2 g_* \Omega_{\mathcal{Y}^\sigma/S}^1$ and let $\delta_1, \dots, \delta_{m'}$ be a local holomorphic frame of $\wedge^2 R^1 g_* \mathcal{O}_{\mathcal{Y}^\sigma}$. We define real-valued functions $\tilde{c}_1(\zeta)$ and $\tilde{c}_1(\xi)$ by

$$\tilde{c}_1(\zeta) = -\log \frac{\det(\langle \zeta(\gamma_i), \zeta(\gamma_j) \rangle_{L^2})}{\det(\langle \gamma_i, \gamma_j \rangle_{L^2})}, \quad \tilde{c}_1(\xi) = -\log \frac{\det(\langle \xi(\delta_i), \xi(\delta_j) \rangle_{L^2})}{\det(\langle \delta_i, \delta_j \rangle_{L^2})} \quad (6.3.20)$$

Note that $\tilde{c}_1(\zeta)$ and $\tilde{c}_1(\xi)$ are independent of the choice of holomorphic frames and they are globally defined on S .

Let $C = (c_{ij})$ be a $m \times m$ -matrix defined by $\bar{\gamma}_i = \sum_j c_{ij} \delta_j$. By (6.3.19), we have

$$\begin{aligned} \frac{\det(\langle \overline{\zeta(\gamma_i)}, \overline{\zeta(\gamma_j)} \rangle_{L^2})}{\det(\langle \bar{\gamma}_i, \bar{\gamma}_j \rangle_{L^2})} &= \frac{\det(C \cdot (\langle \xi(\delta_i), \xi(\delta_j) \rangle_{L^2}) \cdot {}^t \bar{C})}{\det(C \cdot (\langle \delta_i, \delta_j \rangle_{L^2}) \cdot {}^t \bar{C})} \\ &= \frac{\det(\langle \xi(\delta_i), \xi(\delta_j) \rangle_{L^2})}{\det(\langle \delta_i, \delta_j \rangle_{L^2})}. \end{aligned} \quad (6.3.21)$$

By (6.3.20) and (6.3.21), we have

$$\begin{aligned} \tilde{c}_1(\zeta) &= -\log \frac{\det(\langle \zeta(\gamma_i), \zeta(\gamma_j) \rangle_{L^2})}{\det(\langle \gamma_i, \gamma_j \rangle_{L^2})} \\ &= -\log \left\{ \frac{\det(\langle \zeta(\gamma_i), \zeta(\gamma_j) \rangle_{L^2})}{\det(\langle \overline{\zeta(\gamma_i)}, \overline{\zeta(\gamma_j)} \rangle_{L^2})} \frac{\det(\langle \overline{\zeta(\gamma_i)}, \overline{\zeta(\gamma_j)} \rangle_{L^2})}{\det(\langle \bar{\gamma}_i, \bar{\gamma}_j \rangle_{L^2})} \frac{\det(\langle \bar{\gamma}_i, \bar{\gamma}_j \rangle_{L^2})}{\det(\langle \gamma_i, \gamma_j \rangle_{L^2})} \right\} \\ &= -\log \left\{ \frac{\det(\langle \zeta(\gamma_i), \zeta(\gamma_j) \rangle_{L^2})}{\det({}^t(\langle \zeta(\gamma_i), \zeta(\gamma_j) \rangle_{L^2}))} \frac{\det(\langle \xi(\delta_i), \xi(\delta_j) \rangle_{L^2})}{\det(\langle \delta_i, \delta_j \rangle_{L^2})} \frac{\det({}^t(\langle \gamma_i, \gamma_j \rangle_{L^2}))}{\det(\langle \gamma_i, \gamma_j \rangle_{L^2})} \right\} \\ &= \tilde{c}_1(\xi). \end{aligned} \quad (6.3.22)$$

By the definition of $\tilde{c}_1(\zeta)$ and $\tilde{c}_1(\xi)$, we have

$$\begin{aligned} c_1(f_*\Omega_{\mathcal{X}^\iota/S}^2, h_{L^2}) - c_1(\wedge^2 g_*\Omega_{\mathcal{Y}^\sigma/S}^1, h_{L^2}) &= dd^c \tilde{c}_1(\zeta), \\ c_1(R^2 f_*\mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) - c_1(\wedge^2 R^1 g_*\mathcal{O}_{\mathcal{Y}^\sigma}, h_{L^2}) &= dd^c \tilde{c}_1(\xi). \end{aligned} \quad (6.3.23)$$

By Lemma 6.3.6 and by the Serre duality (See Lemma 5.4.1), we have

$$\begin{aligned} c_1(\wedge^2 g_*\Omega_{\mathcal{Y}^\sigma/S}^1, h_{L^2}) &= (g-1)c_1(g_*\Omega_{\mathcal{Y}^\sigma/S}^1, h_{L^2}), \\ c_1(\wedge^2 R^1 g_*\mathcal{O}_{\mathcal{Y}^\sigma}, h_{L^2}) &= (g-1)c_1(R^1 g_*\mathcal{O}_{\mathcal{Y}^\sigma}, h_{L^2}) = -(g-1)c_1(g_*\Omega_{\mathcal{Y}^\sigma/S}^1, h_{L^2}). \end{aligned} \quad (6.3.24)$$

By (6.3.22), (6.3.23) and (6.3.24), we have

$$2c_1(f_*\Omega_{\mathcal{X}^\iota/S}^2, h_{L^2}) = c_1(f_*\Omega_{\mathcal{X}^\iota/S}^2, h_{L^2}) - c_1(R^2 f_*\mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) = 2(g-1)c_1(g_*\Omega_{\mathcal{Y}^\sigma/S}^1, h_{L^2}),$$

which completes the proof. $\square$

Corollary 6.3.8. *The following identity holds:*

$$\omega_{H^\cdot(\mathcal{X}^\iota/S)} = (t+1)c_1(g_*\Omega_{\mathcal{Y}^\sigma/S}^1, h_{L^2}).$$

Proof. By Definition 5.3.7 and Propositions 6.3.5 and 6.3.7, we have

$$\omega_{H^\cdot(\mathcal{X}^\iota/S)} = 2(k-g+2)c_1(g_*\Omega_{\mathcal{Y}^\sigma/S}^1, h_{L^2}). \quad (6.3.25)$$

Let (Y, σ) be a 2-elementary K3 surface of type M_0 and set $(X, \iota) = (Y^{[2]}, \sigma^{[2]})$. By [6, 3.1], we have $e(Y^\sigma) = t-1$. Since the fixed locus Y^σ of (Y, σ) is given by (6.3.2), we have

$$2(k-g+2) = t+1. \quad (6.3.26)$$

By (6.3.25) and (6.3.26), we have the desired result. $\square$

6.3.3 An invariant of $K3^{[2]}$ -type manifolds with natural involution

Recall that the period domain for 2-elementary K3 surfaces of type M_0 is defined by

$$\Omega_{M_0^\perp} = \{[\eta] \in \mathbb{P}(M_0^\perp \otimes \mathbb{C}); (\eta, \eta) = 0, (\eta, \bar{\eta}) > 0\}.$$

and

$$\mathcal{M}_{M_0} = \Omega_{M_0^\perp} / O(M_0^\perp).$$

This is a quasi-projective variety. (See [2, Theorems 10.4 and 10.11])

Recall that the period of a 2-elementary K3 surface (Y, σ) is defined by

$$\bar{\pi}_{M_0}(Y, \sigma) = [\alpha(H^{2,0}(Y))] \in \mathcal{M}_{M_0},$$

where α is a marking of (Y, σ) . For the family $g : (\mathcal{Y}, \sigma) \rightarrow S$ of 2-elementary K3 surfaces of type M_0 , we define the period map by

$$\bar{\pi}_{M_0} : S \rightarrow \mathcal{M}_{M_0}, \quad s \mapsto \bar{\pi}_{M_0}(Y_s, \sigma_s).$$

Fix a vector $l_0 \in M_{0, \mathbb{R}}^\perp$ satisfying $(l_0, l_0) \geq 0$. Since M_0 is hyperbolic and hence $\text{sign } M_0^\perp = (2, 20 - r(M_0))$, $\Omega_{M_0^\perp}$ consists of two connected components, both of which are isomorphic to a bounded symmetric domain of type IV of dimension $20 - r(M_0)$. The Bergman metric $\omega_{M_0^\perp}$ on $\Omega_{M_0^\perp}$ is defined by

$$\omega_{M_0^\perp}([\eta]) = -dd^c \log B_{M_0^\perp}([\eta]) \quad ([\eta] \in \Omega_{M_0^\perp})$$

where

$$B_{M_0^\perp}([\eta]) = \frac{(\eta, \bar{\eta})}{|(\eta, l_0)|^2} \quad ([\eta] \in \Omega_{M_0^\perp}).$$

Since $\omega_{M_0^\perp}$ is $O(M_0^\perp)$ -invariant, it induces a Kähler form $\omega_{\mathcal{M}_{M_0}}$ on the orbifold $\mathcal{M}_{M_0} = \Omega_{M_0^\perp}/O(M_0^\perp)$. Since the period maps of $f : (\mathcal{X}, \iota) \rightarrow S$ and $g : (\mathcal{Y}, \sigma) \rightarrow S$ coincide, we have

$$2\bar{\pi}_{M_0}^* \omega_{\mathcal{M}_{M_0}} = 2P_{M, \mathcal{K}}^* \omega_{\mathcal{M}_{M, \mathcal{K}}} = c_1(f_* K_{\mathcal{X}/S}, h_{L^2}). \quad (6.3.27)$$

Recall that $Sp(2g, \mathbb{Z})$ is the Siegel modular group, that $\mathfrak{S}_g$ is the Siegel upper half space given by

$$\mathfrak{S}_g = \{\tau \in M_g(\mathbb{C}); {}^t\tau = \tau, \text{Im } \tau > 0\},$$

and that $\mathcal{A}_g$ is the Siegel modular variety of degree g defined by

$$\mathcal{A}_g = \mathfrak{S}_g/Sp(2g, \mathbb{Z}),$$

where the $Sp(2g, \mathbb{Z})$ -action on $\mathfrak{S}_g$ is given by

$$\gamma \cdot \tau = (A\tau + B)(C\tau + D)^{-1}, \quad \tau \in \mathfrak{S}_g, \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}).$$

(See Section 3.2.1.)

The period map for the family of curves $g : \mathcal{Y}^\sigma \rightarrow S$ is denoted by

$$J_{M_0, \mathcal{Y}^\sigma/S} : S \rightarrow \mathcal{A}_g, \quad s \mapsto [\Omega(Y_s^\sigma)],$$

where $\Omega(Y_s^\sigma)$ is the period matrix of Y_s^σ . The Bergman metric $\omega_{\mathfrak{S}_g}$ on $\mathfrak{S}_g$ is defined by

$$\omega_{\mathfrak{S}_g}(\tau) = -dd^c \log \det \text{Im } \tau, \quad \tau \in \mathfrak{S}_g.$$

Since $\omega_{\mathfrak{S}_g}$ is $Sp(2g, \mathbb{Z})$ -invariant, it induces a Kähler form $\omega_{\mathcal{A}_g}$ on the Siegel modular variety $\mathcal{A}_g = \mathfrak{S}_g/Sp(2g, \mathbb{Z})$. We have

$$c_1(g_* \Omega_{\mathcal{Y}^\sigma/S}^1, h_{L^2}) = -dd^c \log \det \langle \psi_i, \psi_j \rangle_{L^2} = J_{M_0, \mathcal{Y}^\sigma/S}^* \omega_{\mathcal{A}_g}, \quad (6.3.28)$$

where $\psi_1, \dots, \psi_g$ is a local frame of $g_*\Omega_{\mathcal{Y}/S}^1$.

By the formulas (6.3.1) and (6.3.26), we have

$$2r - 19 = t, \quad (6.3.29)$$

where $r = r(M_0)$ is the rank of M_0 .

Theorem 6.3.9. *Let $g : (\mathcal{Y}, \sigma) \rightarrow S$ be a family of 2-elementary K3 surfaces of type M_0 and let $f : (\mathcal{X}, \iota) \rightarrow S$ be the family of $K3^{[2]}$ -type manifolds with antisymplectic involution of type $(\widetilde{M}_0, K)$ induced from $g : (\mathcal{Y}, \sigma) \rightarrow S$. Namely,*

$$X_s = Y_s^{[2]}, \quad \iota_s = \sigma_s^{[2]}$$

for all $s \in S$. Then the following equality holds:

$$-dd^c \log \tau_{\widetilde{M}_0, K, \mathcal{X}/S} = 2(r-9) \left\{ \frac{r-6}{4} \bar{\pi}_{M_0}^* \omega_{\mathcal{M}_{M_0}} + J_{M_0, \mathcal{Y}^\sigma/S}^* \omega_{\mathcal{A}_g} \right\}.$$

Proof. By Theorem 5.3.11, by Corollary 6.3.8, and by (6.3.27), (6.3.28) and (6.3.29), we have

$$\begin{aligned} -dd^c \log \tau_{\widetilde{M}_0, K, \mathcal{X}/S} &= \frac{(t+1)(t+7)}{16} c_1(f_* K_{\mathcal{X}/S}, h_{L^2}) + \omega_{H^1(\mathcal{X}^\iota/S)} \\ &= \frac{(t+1)(t+7)}{8} \bar{\pi}_{M_0}^* \omega_{\mathcal{M}_{M_0}} + (t+1) J_{M_0, \mathcal{Y}^\sigma/S}^* \omega_{\mathcal{A}_g} \\ &= \frac{(r-6)(r-9)}{2} \bar{\pi}_{M_0}^* \omega_{\mathcal{M}_{M_0}} + 2(r-9) J_{M_0, \mathcal{Y}^\sigma/S}^* \omega_{\mathcal{A}_g} \\ &= 2(r-9) \left\{ \frac{r-6}{4} \bar{\pi}_{M_0}^* \omega_{\mathcal{M}_{M_0}} + J_{M_0, \mathcal{Y}^\sigma/S}^* \omega_{\mathcal{A}_g} \right\}, \end{aligned}$$

and we obtain the result. $\square$

Recall that τ_{M_0} is the invariant of 2-elementary K3 surface of type M_0 constructed in Definition 4.1.5. Let $g : (\mathcal{Y}, \sigma) \rightarrow S$ be a family of 2-elementary K3 surfaces of type M_0 . We define the function $\tau_{M_0, \mathcal{Y}/S}$ on S by

$$\tau_{M_0, \mathcal{Y}/S}(s) = \tau_{M_0}(Y_s, \sigma_s) \quad (s \in S).$$

By [82, Theorem 5.6], the following equality holds:

$$dd^c \log \tau_{M_0, \mathcal{Y}/S} = \frac{r-6}{4} \bar{\pi}_{M_0}^* \omega_{\mathcal{M}_{M_0}} + J_{M_0, \mathcal{Y}^\sigma/S}^* \omega_{\mathcal{A}_g}.$$

Therefore, if $r \neq 9$, then the functions $\tau_{\widetilde{M}_0, \mathcal{X}/S}^{\frac{1}{2r-18}}$ and $\tau_{M_0, \mathcal{Y}/S}$ satisfies the same curvature equation.

6.4 Comparison of the invariants

In this section, we compare the invariants $\tau_{\widetilde{M}_0, \mathcal{K}}$ and τ_{M_0} when $\mathcal{K}$ is natural. An example with non-natural $\mathcal{K}$ is also discussed.

6.4.1 Relations of orthogonal modular varieties

In this subsection, we compare Joumaah's orthogonal modular variety constructed in Subsection 2.2.3 and Yoshikawa's orthogonal modular variety constructed in Subsection 4.1.1.

Let M_0 be a primitive hyperbolic 2-elementary sublattice of L_{K3} . Recall that $\iota_{M_0} \in O(L_{K3})$ is an involution of L_{K3} which is an extension of

$$M_0 \oplus M_0^\perp \rightarrow M_0 \oplus M_0^\perp, \quad (m, n) \mapsto (m, -n)$$

Set $\Delta(M_0) = \{d \in M_0; d^2 = -2\}$, and $\tilde{\mathcal{C}}_{M_0} = \{x \in M_{0, \mathbb{R}}; x^2 > 0\}$. The set of connected components of $\tilde{\mathcal{C}}_{M_0} \setminus \bigcup_{d \in \Delta(M_0)} d^\perp$ is denoted by $\text{KT}(M_0)$.

Let $\mathcal{K}_0 \in \text{KT}(M_0)$. Set

$$\Gamma(\mathcal{K}_0) = \{\sigma \in \text{Mon}^2(L_{K3}); \sigma \circ \iota_{M_0} = \iota_{M_0} \circ \sigma, \text{ and } \sigma(\mathcal{K}_0) = \mathcal{K}_0\},$$

and

$$\Gamma_{M_0^\perp, \mathcal{K}_0} = \{\sigma|_{M_0^\perp} \in O(M_0^\perp); \sigma \in \Gamma(\mathcal{K}_0)\}.$$

Since $\Gamma(\mathcal{K}_0) \subset \text{Mon}^2(L_{K3})$, we have $\Gamma_{M_0^\perp, \mathcal{K}_0} \subset O^+(M_0^\perp)$.

Lemma 6.4.1. *For any $\mathcal{K}_0 \in \text{KT}(M_0)$, we have $\Gamma_{M_0^\perp, \mathcal{K}_0} = O^+(M_0^\perp)$.*

Proof. Choose $g \in O^+(M_0^\perp)$. By [87, Proposition 11.2], there exists an isometry $\sigma \in O(L_{K3})$ such that $\sigma \circ \iota_{M_0} = \iota_{M_0} \circ \sigma$ and $\sigma|_{M_0^\perp} = g$. By [62, Corollary 1.5.2], there exists an involution $\xi \in O(L_{K3})$ such that $\xi|_{M_0^\perp} = \text{id}$ and $sn_{\mathbb{R}}(\xi|_{M_0}) = -1$. By replacing σ with $\xi \circ \sigma$, if necessary, we may assume that $\sigma \in O^+(L_{K3})$ and that it preserves the connected components of $\tilde{\mathcal{C}}_{M_0}$.

Let W be the subgroup of $\text{Mon}^2(L_{K3})$ generated by the reflections s_d for $d \in \Delta(M_0)$. Note that $s_d|_{M_0^\perp} = \text{id}_{M_0^\perp}$ for each $d \in \Delta(M_0)$. Let $\mathcal{C}_{M_0}$ be the connected component of $\tilde{\mathcal{C}}_{M_0}$ containing $\mathcal{K}_0$, and let $\text{KT}(M_0)_+$ be the set of Kähler-type chambers contained in $\mathcal{C}_{M_0}$. Then $\sigma(\mathcal{K}_0) \in \text{KT}(M_0)_+$ and W acts on $\text{KT}(M_0)_+$ transitively (cf. [51, Theorem 2.9]). Therefore there exists an element $w \in W$ such that $(w\sigma)(\mathcal{K}_0) = \mathcal{K}_0$. Since $w \in W \subset \text{Mon}^2(L_{K3})$ and $\sigma \in O^+(L_{K3}) = \text{Mon}^2(L_{K3})$, we have $w\sigma \in \Gamma(\mathcal{K}_0)$ and this implies $g = (w\sigma)|_{M_0^\perp} \in \Gamma_{M_0^\perp, \mathcal{K}_0}$. $\square$

Recall that L_2 is given by $L_2 = L_{K3} \oplus \mathbb{Z}e$ and that $\widetilde{M}_0$ is an admissible sublattice of L_2 defined by

$$\widetilde{M}_0 = M_0 \oplus \mathbb{Z}e.$$

We define an involution $\iota_{\widetilde{M}_0}$ of L_2 by

$$\iota_{\widetilde{M}_0}(x_0 + ae) = \iota_{M_0}(x_0) + ae.$$

Recall that a Kähler-type chamber $\mathcal{K} \in \text{KT}(\widetilde{M}_0)$ is natural if the hyperplane $M_{0,\mathbb{R}}$ is a face of $\mathcal{K}$. Namely, $M_{0,\mathbb{R}} \cap \partial\mathcal{K}$ contains an open subset of $M_{0,\mathbb{R}}$.

Lemma 6.4.2. *If $\delta = d + ae \in \Delta(\widetilde{M}_0)$ ($d \in M_0, a \in \mathbb{Z}$), then one of the following holds*

$$(1) \quad d^2 \geq 0, \quad (2) \quad d \in \Delta(M_0), \quad \text{or} \quad (3) \quad \frac{d}{2} \in \Delta(M_0).$$

Proof. If $\delta^2 = -2$, then $d^2 = 2a^2 - 2 \geq -2$. Therefore we have (1) $d^2 \geq 0$ or (2) $d \in \Delta(M_0)$.

Assume that $\delta^2 = -10$ and $(\delta, L_2) = 2\mathbb{Z}$. Set $d' = \frac{d}{2} \in L_{K3,\mathbb{Q}}$. Since $L_{K3} \subset L_2$ and $2\mathbb{Z} \supset (\delta, L_{K3}) = (d, L_{K3})$, we have $(d', L_{K3}) \subset \mathbb{Z}$. Since L_{K3} is unimodular, we have $d' \in L_{K3}^\vee = L_{K3}$. Since $2d' = d \in M_0$ and M_0 is primitive, we have $d' \in M_0$. Therefore $d^2 = 4(d')^2 \in 8\mathbb{Z}$. If $d^2 \geq 0$, then (1) holds. So we may assume that $d^2 \leq -8$. Since $\delta^2 = -10$, we have $a^2 = \frac{d^2}{2} + 5 \leq 1$. If $a = 0$, then $d^2 = -10 \notin 8\mathbb{Z}$, which is impossible. If $a^2 = 1$, then $d^2 = -8$ and $(d')^2 = -2$. Therefore, we have (3) $\frac{d}{2} \in \Delta(M_0)$. $\square$

Since $M_0^\perp = \widetilde{M}_0^\perp$, we may identify $O(M_0^\perp) = O(\widetilde{M}_0^\perp)$.

Theorem 6.4.3. *If $\mathcal{K} \in \text{KT}(\widetilde{M}_0)$ is natural, then $\Gamma_{\widetilde{M}_0^\perp, \mathcal{K}} = O^+(M_0^\perp)$.*

Proof. It is clear that $\Gamma_{\widetilde{M}_0^\perp, \mathcal{K}} \subset O^+(\widetilde{M}_0^\perp) = O^+(M_0^\perp)$. Since $e \in \Delta(\widetilde{M}_0)$, either $(\mathcal{K}, e) < 0$ or $(\mathcal{K}, e) > 0$ holds. We assume that

$$(\mathcal{K}, e) < 0. \tag{6.4.1}$$

Step 1 Let $p : \widetilde{M}_{0,\mathbb{R}} \rightarrow M_{0,\mathbb{R}}$ be the orthogonal projection. Since $p(\mathcal{K})$ is a connected open subset of $\mathcal{E}_{M_0} \setminus \bigcup_{d \in \Delta(M_0)} d^\perp$, there exists a unique chamber $\mathcal{K}_0 \in \text{KT}(M_0)$ such that $p(\mathcal{K}) \subset \mathcal{K}_0$. By the continuity of p , we have $\bar{\mathcal{K}} \cap M_{0,\mathbb{R}} \subset \overline{p(\mathcal{K})}$. Since $\mathcal{K} \in \text{KT}(\widetilde{M}_0)$ is natural, $\bar{\mathcal{K}} \cap M_{0,\mathbb{R}}$ contains an open subset of $M_{0,\mathbb{R}}$, and $\bar{\mathcal{K}} \cap M_{0,\mathbb{R}} \cap p(\mathcal{K}) \neq \emptyset$. We fix an element $\omega_0 \in \bar{\mathcal{K}} \cap M_{0,\mathbb{R}}$ with $\omega_0 \in p(\mathcal{K}) (\subset \mathcal{K}_0)$. Then there exists a real number $a \in \mathbb{R}$ such that $\omega_0 + ae \in \mathcal{K}$. We set $\omega = \omega_0 + ae$. By (6.4.1), we have $a > 0$.

We claim that $\omega_0 + be \in \mathcal{K}$ for any $b \in (0, a)$. Let $\delta \in \Delta(\widetilde{M}_0)$. If $(\mathcal{K}, \delta) > 0$, then $(\bar{\mathcal{K}}, \delta) \geq 0$. Since $\omega_0 \in \bar{\mathcal{K}}$ and $\omega \in \mathcal{K}$, we have $(\omega_0, \delta) \geq 0$ and $(\omega, \delta) > 0$. For any $b \in (0, a)$, we have

$$\omega_0 + be = \frac{a-b}{a}\omega_0 + \frac{b}{a}\omega,$$

and $(\omega_0 + be, \delta) > 0$. Similarly, if $(\mathcal{K}, \delta) < 0$, then $(\omega_0 + be, \delta) < 0$ for any $b \in (0, a)$. Since $\mathcal{K}$ is a connected component of $\mathcal{E}_{\widetilde{M}_0} \setminus \bigcup_{\delta \in \Delta(\widetilde{M}_0)} \delta^\perp$, we have

$$\omega_0 + be \in \mathcal{K}$$

for any $b \in (0, a)$.

Step 2 Let $g \in O^+(M_0^\perp)$. By Lemma 6.4.1, there exists an element $\tau \in \Gamma(\mathcal{K}_0)$ such that $\tau|_{M_0^\perp} = g$. We define $\sigma \in O(L_2)$ by

$$\sigma(x_0 + ae) = \tau(x_0) + ae, \quad (x_0 \in L_{K3}, a \in \mathbb{Z}).$$

By the construction of σ and $\iota_{\widetilde{M}_0}$, we have $\sigma \circ \iota_{\widetilde{M}_0} = \iota_{\widetilde{M}_0} \circ \sigma$. If we write $\tau = s_{v_1} \dots s_{v_m}$ ($v_i \in L_{K3, \mathbb{R}}$), then $v_i \in L_2$ and $\sigma = s_{v_1} \dots s_{v_m}$. Therefore $sn_{\mathbb{R}}(\sigma) = sn_{\mathbb{R}}(\tau) = +1$ and we have $\sigma \in O^+(L_2) = \text{Mon}^2(L_2)$. Therefore we have $\sigma \in \Gamma(\widetilde{M}_0)$.

Suppose that $\mathcal{K} \cap \sigma(\mathcal{K}) = \emptyset$. Then there exists $\delta \in \Delta(\widetilde{M}_0)$ such that

$$(\mathcal{K}, \delta) > 0 \quad \text{and} \quad (\sigma(\mathcal{K}), \delta) < 0. \quad (6.4.2)$$

Write $\delta = d + \lambda e$, where $d \in M_{0, \mathbb{R}}$ and $\lambda \in \mathbb{Z}$. We define a real-valued function $f(t)$ on $[0, 1]$ by

$$f(t) = (t\tau(\omega_0) + (1-t)\omega_0, d), \quad (t \in [0, 1]).$$

Since $\tau \in \Gamma(\mathcal{K}_0)$, we have $\tau(\omega_0) \in \mathcal{K}_0$. Since $\mathcal{K}_0$ is a convex set, the line segment $\{t\tau(\omega_0) + (1-t)\omega_0; 0 \leq t \leq 1\}$ is contained in $\mathcal{K}_0$. By Lemma 6.4.2, the hyperplane $d^\perp$ in $M_{0, \mathbb{R}}$ does not intersect $\mathcal{C}_{M_0} \setminus \bigcup_{\delta \in \Delta(M_0)} \delta^\perp$. Therefore, we have $(x_0, d) \neq 0$ for any $x_0 \in \mathcal{K}_0$. Hence $f(t) \neq 0$ for any $t \in [0, 1]$. Therefore, there exists a positive constant $C > 0$ such that $|f(t)| \geq C$ for any $0 < t < 1$. Let $a' > 0$ be a positive number that satisfies

$$a' < \begin{cases} \min \left\{ a, \frac{C}{2|\lambda|} \right\} & \text{if } \lambda \neq 0 \\ a & \text{if } \lambda = 0 \end{cases}$$

and set $\omega' = \omega_0 + a'e$. By Step 1, $\omega_0 + be \in \mathcal{K}$ for any $b \in (0, a)$. In particular, $\omega' \in \mathcal{K}$. Hence $\sigma(\omega') \in \sigma(\mathcal{K})$. By (6.4.2), there is a real number $0 < t_0 < 1$ such that

$$0 = (t_0\sigma(\omega') + (1-t_0)\omega', \delta) = f(t_0) - 2a'\lambda.$$

If $\lambda = 0$, we have $0 = |f(t_0)| \geq C > 0$, which is impossible. Suppose that $\lambda \neq 0$. Since $|f(t_0)| \geq C$ and $a' < \frac{C}{2|\lambda|}$, we get

$$C \leq |f(t_0)| = 2a'|\lambda| < C,$$

which is also impossible. Thus we have $\mathcal{K} \cap \sigma(\mathcal{K}) \neq \emptyset$. Since $\mathcal{K}$ and $\sigma(\mathcal{K})$ are connected components of $\mathcal{C}_{\widetilde{M}_0} \setminus \bigcup_{\delta \in \Delta(\widetilde{M}_0)} \delta^\perp$, we have $\mathcal{K} = \sigma(\mathcal{K})$, and $\sigma \in \Gamma(\mathcal{K})$. Therefore $g = \tau|_{M_0^\perp} = \sigma|_{\widetilde{M}_0^\perp} \in \Gamma_{\widetilde{M}_0^\perp, \mathcal{K}}$. Since $g \in O^+(M_0^\perp)$ is arbitrary, we have $\Gamma_{\widetilde{M}_0^\perp, \mathcal{K}} = O^+(M_0^\perp)$. If $(\mathcal{K}, e) > 0$, then we can prove the statement in the same manner. $\square$

Corollary 6.4.4. *Let M_0 be a primitive hyperbolic 2-elementary sublattice of L_{K3} and set $\widetilde{M}_0 = M_0 \oplus \mathbb{Z}e$. Let $\mathcal{K} \in \text{KT}(\widetilde{M}_0)$ be a natural chamber. The identity map $\Omega_{M_0^\perp} \rightarrow \Omega_{\widetilde{M}_0^\perp}$ induces an isomorphism of orthogonal modular varieties $\mathcal{M}_{M_0} \cong \mathcal{M}_{\widetilde{M}_0, \mathcal{K}}$.*

Recall that $\tilde{\mathcal{M}}_{M_0}$ is the set of isomorphism classes of 2-elementary K3 surfaces of type M_0 , and that $\bar{\pi}_{M_0} : \tilde{\mathcal{M}}_{M_0} \rightarrow \mathcal{M}_{M_0}$ is the period map defined by

$$\bar{\pi}_{M_0}(Y, \sigma) = [\alpha(H^{2,0}(Y))] \quad ((Y, \sigma) \in \tilde{\mathcal{M}}_{M_0}),$$

where $\alpha : H^2(Y, \mathbb{Z}) \rightarrow L_{K3}$ is a marking of (Y, σ) . Recall that $\tilde{\mathcal{M}}_{\widetilde{M_0}, \mathcal{K}}$ is the set of isomorphism classes of $K3^{[2]}$ -type manifolds with involution of type $(\widetilde{M_0}, \mathcal{K})$, and $P_{\widetilde{M_0}, \mathcal{K}} : \tilde{\mathcal{M}}_{\widetilde{M_0}, \mathcal{K}} \rightarrow \mathcal{M}_{\widetilde{M_0}, \mathcal{K}}$ is the period map. We have a natural map $\Phi : \tilde{\mathcal{M}}_{M_0} \rightarrow \tilde{\mathcal{M}}_{\widetilde{M_0}, \mathcal{K}}$ defined by

$$\Phi(Y, \sigma) = (Y^{[2]}, \sigma^{[2]}) \quad ((Y, \sigma) \in \tilde{\mathcal{M}}_{M_0}).$$

Then the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathcal{M}}_{M_0} & \xrightarrow{\Phi} & \tilde{\mathcal{M}}_{\widetilde{M_0}, \mathcal{K}} \\ \bar{\pi}_{M_0} \downarrow & & \downarrow P_{\widetilde{M_0}, \mathcal{K}} \\ \mathcal{M}_{M_0} & \xrightarrow{\cong} & \mathcal{M}_{\widetilde{M_0}, \mathcal{K}} \end{array}$$

Example 6.4.5. Let C be a smooth sextic in $\mathbb{P}^2$ and let $Y \rightarrow \mathbb{P}^2$ be the double cover branched over C . The covering involution of $Y \rightarrow \mathbb{P}^2$ is denoted by $\sigma : Y \rightarrow Y$. By Example 2.2.2, $(Y^{[2]}, \sigma^{[2]})$ and $(\text{elm}_{Y/\sigma}(Y^{[2]}), \text{elm}_{Y/\sigma}(\sigma^{[2]}))$ are manifolds of $K3^{[2]}$ -type with antisymplectic involution. The invariant lattice $H^2(Y, \mathbb{Z})^\sigma$ is generated by an element h_0 with $h_0^2 = +2$. By Example 2.2.6, we have $H^2(Y^{[2]}, \mathbb{Z})^{\sigma^{[2]}} \cong \mathbb{Z}h_0 \oplus \mathbb{Z}\varepsilon$. By [36, Proposition 25.14], we also have $H^2(\text{elm}_{Y/\sigma}(Y^{[2]}), \mathbb{Z})^{\text{elm}_{Y/\sigma}(\sigma^{[2]})} \cong \mathbb{Z}h_0 \oplus \mathbb{Z}\varepsilon$.

We define a sublattice M_0 of L_{K3} by $M_0 = \mathbb{Z}h$, where $h \in L_{K3}$ satisfies $h^2 = 2$. It is a primitive hyperbolic 2-elementary sublattice of L_{K3} . We set $\widetilde{M_0} = M_0 \oplus \mathbb{Z}\varepsilon$. Then $\widetilde{M_0}$ is an admissible sublattice of L_2 . By [45, Example 9.12], we have

$$\Delta(\widetilde{M_0}) = \pm\{e, 2h + 3e, 2h - 3e\} \quad \text{and} \quad \text{KT}(\widetilde{M_0})/\Gamma_{\widetilde{M_0}} = \{[\mathcal{K}], [\mathcal{K}']\},$$

where $\mathcal{K}$ and $\mathcal{K}'$ are Kähler-type chambers defined by $\mathcal{K} = \mathbb{R}_{>0}h + \mathbb{R}_{>0}(3h + 2e)$ and $\mathcal{K}' = \mathbb{R}_{>0}(3h + 2e) + \mathbb{R}_{>0}(h + e)$. By [45, Example 9.12], $\mathcal{K}$ is natural, $\rho(Y^{[2]}, \sigma^{[2]}) = [\mathcal{K}]$ and $\rho(\text{elm}_{Y/\sigma}(Y^{[2]}), \text{elm}_{Y/\sigma}(\sigma^{[2]})) = [\mathcal{K}']$.

By Theorem 6.4.3 and Corollary 6.4.4, we have $\Gamma_{\widetilde{M_0}^\perp, \mathcal{K}} = O^+(M_0^\perp)$ and the identity map $\Omega_{M_0^\perp} \rightarrow \Omega_{\widetilde{M_0}^\perp}$ gives an isomorphism of $\mathcal{M}_{M_0} \cong \mathcal{M}_{\widetilde{M_0}, \mathcal{K}}$.

We prove that the same statement holds for the non-natural Kähler-type chamber $[\mathcal{K}']$.

Theorem 6.4.6. *With the same notation as in Example 6.4.5, we have $\Gamma_{\widetilde{M_0}^\perp, \mathcal{K}'} = O^+(M_0^\perp)$. In particular, the identity map $\Omega_{M_0^\perp} \rightarrow \Omega_{\widetilde{M_0}^\perp}$ induces an isomorphism of orthogonal modular varieties $\mathcal{M}_{M_0} \cong \mathcal{M}_{\widetilde{M_0}, \mathcal{K}'}$.*

Proof. It suffices to show that $\Gamma_{\widetilde{M}_0^\perp, \mathcal{K}'} = \Gamma_{\widetilde{M}_0^\perp, \mathcal{K}}$. Let $g \in \Gamma_{\widetilde{M}_0^\perp, \mathcal{K}'}$. There exists an element $\sigma \in \Gamma(\mathcal{K}')$ such that $\sigma|_{\widetilde{M}_0^\perp} = g$. Since the boundary $\partial\mathcal{K}'$ consists of the rays $\mathbb{R}_{\geq 0}(3h + 2e)$ and $\mathbb{R}_{\geq 0}(h + e)$ and since $\sigma|_{\widetilde{M}_0}$ preserves $\partial\mathcal{K}'$, we have

$$(\sigma|_{\widetilde{M}_0})(3h + 2e) = 3h + 2e \quad \text{and} \quad (\sigma|_{\widetilde{M}_0})(h + e) = h + e.$$

Hence $\sigma|_{\widetilde{M}_0} = \text{id}_{\widetilde{M}_0}$. Therefore $\sigma \in \Gamma(\mathcal{K})$ and $g = \sigma|_{\widetilde{M}_0^\perp} \in \Gamma_{\widetilde{M}_0^\perp, \mathcal{K}}$. Similarly we have $\Gamma_{\widetilde{M}_0^\perp, \mathcal{K}'} \supset \Gamma_{\widetilde{M}_0^\perp, \mathcal{K}}$, which completes the proof. $\square$

Note that a birational transformation of an irreducible holomorphic symplectic manifold preserves its period ([36, Proposition 25.14]). We have a natural map $\Psi : \tilde{\mathcal{M}}_{M_0} \rightarrow \tilde{\mathcal{M}}_{\widetilde{M}_0, \mathcal{K}'}$ defined by

$$\Psi(Y, \sigma) = (\text{elm}_{Y/\sigma}(Y^{[2]}), \text{elm}_{Y/\sigma}(\sigma^{[2]})) \quad ((Y, \sigma) \in \tilde{\mathcal{M}}_{M_0}).$$

Then the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathcal{M}}_{M_0} & \xrightarrow{\Psi} & \tilde{\mathcal{M}}_{\widetilde{M}_0, \mathcal{K}'} \\ \pi_{M_0} \downarrow & & \downarrow P_{\widetilde{M}_0, \mathcal{K}'} \\ \mathcal{M}_{M_0} & \xrightarrow{\cong} & \mathcal{M}_{\widetilde{M}_0, \mathcal{K}'} \end{array}$$

6.4.2 An equation of currents on $\mathcal{M}_{M_0}$

Let M_0 be a primitive hyperbolic 2-elementary sublattice of L_{K3} . Let $\Pi_{M_0^\perp} : \Omega_{M_0^\perp} \rightarrow \mathcal{M}_{M_0}$ be the natural projection. For $\delta \in \Delta(M_0^\perp) = \{\delta \in M_0^\perp; \delta^2 = -2\}$, we set

$$H_\delta^\circ = \{[\eta] \in \Omega_{M_0^\perp}; O(M_0^\perp)_{[\eta]} = \{\pm 1, \pm s_\delta\}\},$$

where $O(M_0^\perp)_{[\eta]}$ is the stabilizer of $[\eta]$. We define an open subset $\mathcal{D}_{M_0^\perp}^\circ$ of $\mathcal{D}_{M_0^\perp}$ by

$$\mathcal{D}_{M_0^\perp}^\circ = \bigcup_{\delta \in \Delta(M_0^\perp)} H_\delta^\circ.$$

The isometry group $O(M_0^\perp)$ preserves $\mathcal{D}_{M_0^\perp}^\circ$ and we set

$$\overline{\mathcal{D}}_{M_0^\perp}^\circ = \mathcal{D}_{M_0^\perp}^\circ / O(M_0^\perp).$$

It is an open subset of $\mathcal{D}_{M_0^\perp}$.

Lemma 6.4.7. *Assume that $r(M_0) \leq 18$. Then the following hold.*

- (1) $\mathcal{D}_{M_0^\perp} \setminus \mathcal{D}_{M_0^\perp}^\circ$ is an analytic subset of $\Omega_{M_0^\perp}$ with $\text{codim}_{\Omega_{M_0^\perp}}(\mathcal{D}_{M_0^\perp} \setminus \mathcal{D}_{M_0^\perp}^\circ) \geq 2$. In particular, $\Omega_{M_0^\perp} \cup \mathcal{D}_{M_0^\perp}^\circ = \Omega_{M_0^\perp} \setminus (\mathcal{D}_{M_0^\perp} \setminus \mathcal{D}_{M_0^\perp}^\circ)$ is an $O(M_0^\perp)$ -invariant Zariski open subset of $\Omega_{M_0^\perp}$.

- (2) $\mathcal{D}_{M_0^\perp}^\circ$ is a smooth hypersurface of $\Omega_{M_0^\perp}^\circ \cup \mathcal{D}_{M_0^\perp}^\circ$.
- (3) $\overline{\mathcal{D}}_{M_0^\perp}^\circ \subset \mathcal{M}_{M_0} \setminus \text{Sing } \mathcal{M}_{M_0}$, and $\overline{\mathcal{D}}_{M_0^\perp}^\circ \subset \overline{\mathcal{D}}_{M_0^\perp} \setminus \text{Sing } \overline{\mathcal{D}}_{M_0^\perp}$.

Proof. See [82, Proposition 1.9]. $\square$

Set $\widetilde{M}_0 = M_0 \oplus \mathbb{Z}e$. Then $\widetilde{M}_0$ is an admissible sublattice of $L_2 = L_{K3} \oplus \mathbb{Z}e$.

Lemma 6.4.8. *We have $\Delta(M_0^{\perp L_{K3}}) = \Delta(\widetilde{M}_0^{\perp L_2})$.*

Proof. Suppose that there exists an element $\delta \in \widetilde{M}_0^\perp$ such that $\delta^2 = -10, (\delta, L_2) = 2\mathbb{Z}$. Since $\delta \in \widetilde{M}_0^\perp = M_0^\perp \subset L_{K3}$, we have $\delta' := \frac{\delta}{2} \in L_{K3, \mathbb{Q}}$. Since $(\delta, L_2) = 2\mathbb{Z}$, we have $(\delta', L_{K3}) \subset \mathbb{Z}$. Since L_{K3} is unimodular, we have $\delta' \in L_{K3}^\vee = L_{K3}$. Therefore $\mathbb{Z} \ni (\delta')^2 = \frac{1}{4}\delta^2 = -\frac{5}{2} \notin \mathbb{Z}$, which is a contradiction. $\square$

Therefore the isomorphisms

$$\phi : \mathcal{M}_{M_0} \cong \mathcal{M}_{\widetilde{M}_0, \mathcal{K}} \quad \text{and} \quad \psi : \mathcal{M}_{M_0} \cong \mathcal{M}_{\widetilde{M}_0, \mathcal{K}'}$$

constructed in Corollary 6.4.4 and Theorem 6.4.6 satisfy

$$\phi(\overline{\mathcal{D}}_{M_0^\perp}) = \overline{\mathcal{D}}_{\widetilde{M}_0^\perp} \quad \text{and} \quad \psi(\overline{\mathcal{D}}_{M_0^\perp}) = \overline{\mathcal{D}}_{\widetilde{M}_0^\perp}.$$

In the following, we identify $\mathcal{M}_{\widetilde{M}_0, \mathcal{K}}$ and $\mathcal{M}_{\widetilde{M}_0, \mathcal{K}'}$ with $\mathcal{M}_{M_0}$ and identify $\overline{\mathcal{D}}_{\widetilde{M}_0^\perp}$ with $\overline{\mathcal{D}}_{M_0^\perp}$.

Let u be a pluriharmonic function on $\mathcal{M}_{M_0}^\circ$. We assume the following: Let C be an irreducible projective curve on the Baily-Borel compactification $\mathcal{M}_{M_0}^*$ satisfying Assumption 6.2.5 and let $p \in C \cap \overline{\mathcal{D}}_{M_0^\perp}$ be a smooth point of C . Choose a coordinate (Δ, s) on C centered at p such that $\Delta \cap \overline{\mathcal{D}}_{M_0^\perp} = \{p\}$ and $s(\Delta)$ is the unit disk. Then there exists a rational number $a \in \mathbb{Q}$ such that

$$(u|_\Delta)(s) = a \log |s|^2 + O(\log \log |s|^{-1}). \quad (6.4.3)$$

Lemma 6.4.9. *The equation of currents*

$$dd^c u = \sum_i a_i \delta_{\Gamma_i}$$

holds on $\mathcal{M}_{M_0}$, where $\overline{\mathcal{D}}_{M_0^\perp} = \sum_i \Gamma_i$ is the irreducible decomposition.

Proof. Let $x \in \mathcal{D}_{M_0^\perp}^\circ$. By [82, Section 7 (Step 1)], there exists a system of local coordinates $(U \cong \Delta^m, t, z_1, \dots, z_{m-1})$ on $\Omega_{M_0^\perp}^\circ \cup \mathcal{D}_{M_0^\perp}^\circ$ that satisfies

- (1) $\mathcal{D}_{M_0^\perp}^\circ \cap U = \{(t, z) \in U; t = 0\}$.
- (2) For any $z \in \Delta^{m-1}$, there exists an irreducible curve C_z on $\mathcal{M}_{M_0}^*$ such that C_z contains $\Pi_{M_0^\perp}(\Delta \times \{z\})$ as an open set.

By the assumption of u , for each $z \in \Delta^{m-1}$, $u|_{\Pi_{M_0^\perp}(\Delta \times \{z\})}$ satisfies

$$u|_{\Pi_{M_0^\perp}(\Delta \times \{z\})} = a_z \log |t|^2 + O(\log \log |t|^{-1}) \quad (t \rightarrow 0),$$

where $a_z \in \mathbb{Q}$. Set $H = \mathcal{D}_{M_0^\perp}^\circ \cap U$ and $\Delta_z = \Pi_{M_0^\perp}(\Delta \times \{z\})$. It suffices to show that $a = a_z$ is independent of the choice of $z \in \Delta^{m-1}$, and the equation of currents

$$dd^c u = a \delta_H$$

holds on U .

Since $\bar{\partial}\partial u = 0$, ∂u is a holomorphic 1-form on $\mathcal{M}_{M_0}^\circ$. Let $z \in \Delta^{m-1}$. Since $(u|_{\Delta_z})(t) - a_z \log |t|^2$ is a harmonic function on $\Delta_z \setminus \{z\}$ and it satisfies

$$(u|_{\Delta_z})(t) - a_z \log |t|^2 = O(\log \log |t|^{-1}), \quad (6.4.4)$$

it extends to a harmonic function on Δ_z . Moreover a_z is the residue of $\partial u|_{\Delta_z}$ at $t = 0$. Since $a_z \in \mathbb{Q}$ and a_z depends continuously on $z \in \Delta^{m-1}$, $a = a_z$ is independent of the choice of $z \in \Delta^{m-1}$. By (6.4.4), there exists a non-negative function $C(z)$ on Δ^{m-1} which is not necessarily continuous such that

$$|u(t, z) - a \log |t|^2| \leq C(z) \log \log |t|^{-1}$$

holds on U . By [82, Proposition 3.11.], we have $dd^c u = a \delta_H$, and the proof is completed. $\square$

6.4.3 The invariants for the case where $r(M_0) \leq 17$ and $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible

Set $\widetilde{M}_0 = M_0 \oplus \mathbb{Z}e$. Then $\widetilde{M}_0$ is an admissible sublattice of $L_2 = L_{K3} \oplus \mathbb{Z}e$. Let $\mathcal{K} \in \text{KT}(\widetilde{M}_0)$ be a natural Kähler-type chamber.

Assumption 6.4.10. We assume that $r(M_0) \leq 17$, and $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible.

By [87, 11.3] and [28, FIGURE 1], there exist 23 types of primitive hyperbolic 2-elementary sublattice of L_{K3} which satisfy the above assumption.

By Corollary 6.4.4, we have an isomorphism

$$\phi : \mathcal{M}_{M_0} \rightarrow \mathcal{M}_{\widetilde{M}_0, \mathcal{K}}.$$

We define a real-valued smooth function $\sigma_{\widetilde{M}_0, \mathcal{K}}$ on $\mathcal{M}_{M_0}^\circ$ by $\sigma_{\widetilde{M}_0, \mathcal{K}} = \tau_{\widetilde{M}_0, \mathcal{K}} \circ \phi$. Namely, it is defined by

$$\sigma_{\widetilde{M}_0, \mathcal{K}}(Y, \sigma) = \tau_{\widetilde{M}_0, \mathcal{K}}(Y^{[2]}, \sigma^{[2]})$$

for any 2-elementary K3 surface (Y, σ) of type M_0 . By Theorem 6.3.9 and by [82, Theorem 5.6.], the function $\log \left(\sigma_{\widetilde{M}_0, \mathcal{K}} \cdot \tau_{M_0}^{2(r-9)} \right)$ is a pluriharmonic function on $\mathcal{M}_{M_0}^\circ$.

Lemma 6.4.11. *The function $\log \left(\sigma_{\widetilde{M}_0, K} \cdot \tau_{M_0}^{2(r-9)} \right)$ satisfies (6.4.3).*

Proof. The proof is straightforward from [82, Theorem 6.5] and Theorem 6.2.7. $\square$

Theorem 6.4.12. *If $r(M_0) \leq 17$ and $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible, then there exists a constant C depending only on M_0 such that*

$$\tau_{\widetilde{M}_0, K}(Y^{[2]}, \sigma^{[2]}) = C \tau_{M_0}(Y, \sigma)^{-2(r-9)}$$

for any 2-elementary K3 surface (Y, σ) of type M_0 .

Proof. We show that $u = \log \left(\sigma_{\widetilde{M}_0, K} \cdot \tau_{M_0}^{2(r-9)} \right)$ is a constant function on $\mathcal{M}_{M_0}^\circ$. By Lemma 6.4.9, the equation of currents

$$dd^c u = a \delta_{\bar{\mathcal{D}}_{M_0^\perp}} \tag{6.4.5}$$

holds on $\mathcal{M}_{M_0}$. We choose an irreducible projective curve C on $\mathcal{M}_{M_0}^*$ that satisfies the following conditions:

- (1) $C \subset \mathcal{M}_{M_0}$ (2) $C \cap \bar{\mathcal{D}}_{M_0^\perp} \neq \emptyset$ (3) $\text{Sing } C \cap \bar{\mathcal{D}}_{M_0^\perp} = \emptyset$
- (4) C intersects $\bar{\mathcal{D}}_{M_0^\perp}$ transversally.

Let $\eta : \tilde{C} \rightarrow C$ be the normalization. Note that $\eta^* \partial u$ is a meromorphic 1-form on $\tilde{C}$ and that its residue at $p \in \eta^{-1}(C \cap \bar{\mathcal{D}}_{M_0^\perp})$ is equal to a by (6.4.5) and (4). By the residue theorem, we have

$$\sum_{p \in \eta^{-1}(C \cap \bar{\mathcal{D}}_{M_0^\perp})} a = 0.$$

Thus we obtain $a = 0$, and u extends to a pluriharmonic function on $\mathcal{M}_{M_0}$. Since $r(M_0) \leq 17$, we have $\text{codim } \mathcal{M}_{M_0}^* \setminus \mathcal{M}_{M_0} \geq 2$. Therefore u extends to a pluriharmonic function on $\mathcal{M}_{M_0}^*$ by [34, Satz 4]. Since $\mathcal{M}_{M_0}^*$ is compact, u is constant. $\square$

6.4.4 The invariants for the case where $M_0 = \langle 2 \rangle$

In Subsection 6.4.4, we always suppose that $r = r(M_0) = 1$ and $t = -17$. Hence $M_0 = \mathbb{Z}h$ with $h^2 = 2$. Set $\widetilde{M}_0 = M_0 \oplus \mathbb{Z}e$. By Example 6.4.5, $\text{KT}(\widetilde{M}_0)/\Gamma_{\widetilde{M}_0}$ consists of two classes, one natural and the other non-natural. Let $K \in \text{KT}(\widetilde{M}_0)$ be a natural Kähler-type chamber and let $K' \in \text{KT}(\widetilde{M}_0)$ be a non-natural Kähler-type chamber.

Lemma 6.4.13. *Let $f : (\mathcal{X}, \iota) \rightarrow S$ be a family of K3^[2]-type manifolds with antisymplectic involution of type $(\widetilde{M}_0, K')$, and let $g : (\mathcal{Y}, \sigma) \rightarrow S$ be a family of 2-elementary K3 surfaces of type M_0 such that*

$$X_s = \text{elm}_{Y_s/\sigma_s}(Y_s^{[2]}), \quad \text{and} \quad \iota_s = \text{elm}_{Y_s/\sigma_s}(\sigma_s^{[2]})$$

for each $s \in S$. For the definitions of $\text{elm}_{Y_s/\sigma_s}(Y_s^{[2]})$ and $\text{elm}_{Y_s/\sigma_s}(\sigma_s^{[2]})$, see Example 2.2.2. The following equality of $(1, 1)$ -forms on S holds:

$$-dd^c \log \tau_{\widetilde{M}_0, \mathcal{K}', \mathcal{X}/S} = 2(r-9) \left\{ \frac{r-6}{4} \bar{\pi}_{M_0}^* \omega_{\mathcal{M}_{M_0}} + J_{M_0, \mathcal{Y}^\sigma/S}^* \omega_{\mathcal{A}_g} \right\}.$$

Proof. By Theorem 5.3.11, we have

$$-dd^c \log \tau_{\widetilde{M}_0, \mathcal{K}, \mathcal{X}/S} = \frac{(t+1)(t+7)}{16} c_1(f_* K_{\mathcal{X}/S}, h_{L^2}) + \omega_{H^\cdot(\mathcal{X}^\iota/S)}. \quad (6.4.6)$$

Let $h : (\mathcal{X}, \theta) \rightarrow S$ be a family of $K3^{[2]}$ -type manifolds with antisymplectic involution of type $(\widetilde{M}_0, \mathcal{K})$ such that

$$Z_s = Y_s^{[2]}, \quad \text{and} \quad \theta_s = \sigma_s^{[2]}$$

for each $s \in S$. Then we have

$$X_s = \text{elm}_{Y_s/\sigma_s}(Z_s), \quad \text{and} \quad \iota_s = \text{elm}_{Y_s/\sigma_s}(\theta_s)$$

for each $s \in S$. Since the birational transform $Z_s \dashrightarrow X_s$ does not change the period ([36, Proposition 25.14]), we have

$$c_1(f_* K_{\mathcal{X}/S}, h_{L^2}) = c_1(h_* K_{\mathcal{X}/S}, h_{L^2}) = 2\bar{\pi}_{M_0}^* \omega_{\mathcal{M}_{M_0}}. \quad (6.4.7)$$

On the other hand, since $Y_s/\sigma_s = \mathbb{P}^2$ for all $s \in S$, we have

$$\omega_{H^\cdot(\mathcal{X}^\iota/S)} = \omega_{H^\cdot(\mathcal{X}^\theta/S)} = 2(r-9) J_{M_0, \mathcal{Y}^\sigma/S}^* \omega_{\mathcal{A}_g}. \quad (6.4.8)$$

Combining (6.4.6), (6.4.7) and (6.4.8), we obtain the desired result. $\square$

We set

$$\begin{aligned} \Delta^+ &= \{d \in \Delta(M_0^\perp); d/2 \in (M_0^\perp)^\vee\}, & \Delta^- &= \{d \in \Delta(M_0^\perp); d/2 \notin (M_0^\perp)^\vee\}, \\ \mathcal{D}_{M_0^\perp}^\pm &= \sum_{d \in \Delta^\pm} d^\perp, & \bar{\mathcal{D}}_{M_0^\perp}^\pm &= \mathcal{D}_{M_0^\perp}^\pm / O(M_0^\perp). \end{aligned}$$

By [87, 11.3.] and [28, FIGURE 1.], $\bar{\mathcal{D}}_{M_0^\perp}$ consists of two connected components $\bar{\mathcal{D}}_{M_0^\perp}^+$ and $\bar{\mathcal{D}}_{M_0^\perp}^-$.

By Corollary 6.4.4 and Theorem 6.4.6, we have the isomorphisms

$$\phi : \mathcal{M}_{M_0} \rightarrow \mathcal{M}_{\widetilde{M}_0, \mathcal{K}} \quad \text{and} \quad \psi : \mathcal{M}_{M_0} \rightarrow \mathcal{M}_{\widetilde{M}_0, \mathcal{K}'}.$$

We define real-valued smooth functions $\sigma_{\widetilde{M}_0, \mathcal{K}}$ and $\sigma_{\widetilde{M}_0, \mathcal{K}'}$ on $\mathcal{M}_{M_0}^\circ$ by $\sigma_{\widetilde{M}_0, \mathcal{K}} = \tau_{\widetilde{M}_0, \mathcal{K}} \circ \phi$ and $\sigma_{\widetilde{M}_0, \mathcal{K}'} = \tau_{\widetilde{M}_0, \mathcal{K}'} \circ \psi$. Namely, they are defined by

$$\sigma_{\widetilde{M}_0, \mathcal{K}}(Y, \sigma) = \tau_{\widetilde{M}_0, \mathcal{K}}(Y^{[2]}, \sigma^{[2]}), \quad \sigma_{\widetilde{M}_0, \mathcal{K}'}(Y, \sigma) = \tau_{\widetilde{M}_0, \mathcal{K}'}(\text{elm}_{Y/\sigma}(Y^{[2]}), \text{elm}_{Y/\sigma}(\sigma^{[2]}))$$

for any 2-elementary K3 surface (Y, σ) of type M_0 . By Theorem 6.3.9, Lemma 6.4.13 and [82, Theorem 5.6.], the functions $\log \left(\sigma_{\widetilde{M}_0, \mathcal{K}} \cdot \tau_{M_0}^{2(r-9)} \right)$ and $\log \left(\sigma_{\widetilde{M}_0, \mathcal{K}'} \cdot \tau_{M_0}^{2(r-9)} \right)$ are pluriharmonic functions on $\mathcal{M}_{M_0}^\circ$.

Lemma 6.4.14. *The functions $\log \left(\sigma_{\widetilde{M}_0, \mathcal{K}} \cdot \tau_{M_0}^{2(r-9)} \right)$ and $\log \left(\sigma_{\widetilde{M}_0, \mathcal{K}'} \cdot \tau_{M_0}^{2(r-9)} \right)$ satisfy (6.4.3).*

Proof. The proof is straightforward from [82, Theorem 6.5.] and Theorem 6.2.7. $\square$

Theorem 6.4.15. *There exist positive constants $C_1, C_2 > 0$ such that for any 2-elementary K3 surface (Y, σ) of type $M_0 = \langle 2 \rangle$, the following identities hold:*

$$\begin{aligned} \tau_{\widetilde{M}_0, \mathcal{K}}(Y^{[2]}, \sigma^{[2]}) &= C_1 \tau_{M_0}(Y, \sigma)^{-2(r-9)}, \\ \tau_{\widetilde{M}_0, \mathcal{K}'}(\text{elm}_{Y/\sigma}(Y^{[2]}), \text{elm}_{Y/\sigma}(\sigma^{[2]})) &= C_2 \tau_{M_0}(Y, \sigma)^{-2(r-9)}. \end{aligned}$$

Proof. We follow [52, Lemma 8.7]. We set $u = \log \left(\sigma_{\widetilde{M}_0, \mathcal{K}} \cdot \tau_{M_0}^{2(r-9)} \right)$ or $\log \left(\sigma_{\widetilde{M}_0, \mathcal{K}'} \cdot \tau_{M_0}^{2(r-9)} \right)$. It suffices to show that u is a constant function on $\mathcal{M}_{M_0}^\circ$. By Lemma 6.4.9, the equation of currents

$$dd^c u = a \delta_{\bar{\mathcal{D}}_{M_0^\perp}^+} + b \delta_{\bar{\mathcal{D}}_{M_0^\perp}^-}$$

holds on $\mathcal{M}_{M_0}$, where $a, b \in \mathbb{Q}$. Therefore we have

$$dd^c u = b \delta_{\bar{\mathcal{D}}_{M_0^\perp}^-}$$

as currents on $\mathcal{M}_{M_0} \setminus \bar{\mathcal{D}}_{M_0^\perp}^+$.

Let $U \subset |\mathcal{O}_{\mathbb{P}^2}(6)|$ be the space of smooth plane sextics, and let $V \subset |\mathcal{O}_{\mathbb{P}^2}(6)|$ be that of sextics with at most one node. By [75], the geometric quotient U/PGL_3 is contained in V/PGL_3 as an open subset, there is an isomorphism $U/\text{PGL}_3 \cong \mathcal{M}_{M_0}^\circ$, and it extends to an open embedding $V/\text{PGL}_3 \rightarrow \mathcal{M}_{M_0} \setminus \bar{\mathcal{D}}_{M_0^\perp}^+$.

Let $f : V \rightarrow \mathcal{M}_{M_0} \setminus \bar{\mathcal{D}}_{M_0^\perp}^+$ be the composition of the quotient map $V \rightarrow V/\text{PGL}_3$ and the open embedding $V/\text{PGL}_3 \rightarrow \mathcal{M}_{M_0} \setminus \bar{\mathcal{D}}_{M_0^\perp}^+$. We have the equation of currents

$$dd^c f^* u = \nu b \delta_{f^* \bar{\mathcal{D}}_{M_0^\perp}^-} \quad (6.4.9)$$

on V , where $\nu \in \mathbb{Z}_{>0}$.

Note that the codimension of $|\mathcal{O}_{\mathbb{P}^2}(6)| \setminus V$ is 2. There exists an irreducible projective curve C in $|\mathcal{O}_{\mathbb{P}^2}(6)|$ such that

- (1) $C \subset V$ (2) $C \cap f^* \bar{\mathcal{D}}_{M_0^\perp}^- \neq \emptyset$ (3) $\text{Sing } C \cap f^* \bar{\mathcal{D}}_{M_0^\perp}^- = \emptyset$
- (4) C intersects $f^* \bar{\mathcal{D}}_{M_0^\perp}^-$ transversally.

Let $\eta : \tilde{C} \rightarrow C$ be the normalization. Note that $\eta^* f^* \partial u$ is a meromorphic 1-form on $\tilde{C}$ whose residue at $p \in \eta^{-1}(C \cap f^* \bar{\mathcal{D}}_{M_0^\perp}^-)$ is a positive multiple of νb , say $m_p \nu b$, $m_p > 0$. By the residue theorem, we have

$$\sum_{p \in \eta^{-1}(C \cap f^* \bar{\mathcal{D}}_{M_0^\perp}^-)} m_p \nu b = 0. \quad (6.4.10)$$

By (6.4.10), we have $b = 0$ so that f^*u is a pluriharmonic function on V by (6.4.9). Since $\text{codim}_{|\mathcal{O}(6)|}(|\mathcal{O}_{\mathbb{P}^2}(6)| \setminus V) = 2$, f^*u extends to a pluriharmonic function on $|\mathcal{O}_{\mathbb{P}^2}(6)|$ by [34, Satz 4]. Thus f^*u is a constant function on $|\mathcal{O}(6)|$, and u is also constant. $\square$

6.5 The singularity of μ_2 -equivariant Quillen metrics

In this section, we determine the singularity of μ_2 -equivariant Quillen metrics for one-parameter degenerations of projective manifolds, and we prove Theorem 6.5.4, which was used in the proof of Proposition 6.1.4.

This section is based on the idea of Ken-Ichi Yoshikawa, and we follow the appendix of [43]. (See also Sections 3 and 4 of [85] for the case of general compact group actions.)

6.5.1 Smoothness of μ_2 -equivariant Quillen metrics

Let $\mathcal{M}$ be a complex manifold endowed with a holomorphic μ_2 -action and let B be a complex manifold endowed with a trivial μ_2 -action. Let $\pi : \mathcal{M} \rightarrow B$ be a μ_2 -equivariant proper holomorphic submersion. Assume that there is a μ_2 -equivariant relatively ample line bundle $\mathcal{O}_{\mathcal{M}}(1)$ on $\mathcal{M}$. Let $F \rightarrow \mathcal{M}$ be a μ_2 -equivariant holomorphic vector bundle. We set $M_b = \pi^{-1}(b)$ and $F_b = F|_{M_b}$ for $b \in B$. Then μ_2 preserves the fibers M_b and F_b , and $\pi : \mathcal{M} \rightarrow B$ is a family of projective manifolds endowed with μ_2 -action. We define $(R^q\pi_*F)_{\pm}$ as the sheaf on B associated to the presheaf

$$U \mapsto H^q(\pi^{-1}(U), F|_{\pi^{-1}(U)})_{\pm},$$

where $H^q(\pi^{-1}(U), F|_{\pi^{-1}(U)})_{\pm}$ is the (± 1) -eigenspace of $H^q(\pi^{-1}(U), F|_{\pi^{-1}(U)})$ with respect to the ι -action. Then $R^q\pi_*F = (R^q\pi_*F)_+ \oplus (R^q\pi_*F)_-$.

Lemma 6.5.1. *Let $K \subset B$ be a compact subset. Then there exist a complex of μ_2 -equivariant holomorphic vector bundles $F_{\bullet} : 0 \rightarrow F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_m \rightarrow 0$ on a neighborhood of $\pi^{-1}(K)$ and a homomorphism $i : F|_{\pi^{-1}(K)} \rightarrow F_0$ of μ_2 -equivariant vector bundles satisfying the following conditions:*

- (1) *The sequence of vector bundles $0 \rightarrow F|_{\pi^{-1}(K)} \rightarrow F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_m \rightarrow 0$ is exact.*
- (2) *$H^q(M_b, F_i|_{M_b}) = 0$ for all $q > 0$, $i \geq 0$ and $b \in K$.*

Proof. We follow [70, Sect. 7.2.7]. For $\nu \in \mathbb{N}$, we set $E_{\nu} = \pi_*\mathcal{O}_{\mathcal{M}}(\nu)$. Since $\mathcal{O}_{\mathcal{M}}(1)$ is relatively ample, there exists an open neighborhood U of K and $m_0 \in \mathbb{N}$ such that E_{ν} is a μ_2 -equivariant vector bundle over U and such that the evaluation homomorphism $\pi^*E_{\nu} \rightarrow \mathcal{O}_{\mathcal{M}}(\nu)|_{\pi^{-1}(U)} \rightarrow 0$ is surjective and μ_2 -equivariant for $\nu \geq m_0$. Considering its dual, we get a μ_2 -equivariant injective homomorphism of vector bundles

$0 \rightarrow \mathcal{O}_{\mathcal{M}}|_{\pi^{-1}(U)} \rightarrow (\pi^* E_\nu^\vee)(\nu)$. Tensoring with F , we get a μ_2 -equivariant exact sequence of vector bundles $0 \rightarrow F|_{\pi^{-1}(U)} \rightarrow (\pi^* E_\nu^\vee) \otimes F(\nu)|_{\pi^{-1}(U)}$ over $\pi^{-1}(U)$, where $F(\nu) = F \otimes \mathcal{O}_{\mathcal{M}}(\nu)$. We set $F_0 = (\pi^* E_\nu^\vee) \otimes F(\nu)|_{\pi^{-1}(U)}$. Then there exist an open neighborhood U_1 of K and $m_1 \in \mathbb{N}$ such that for $\nu \geq m_1$, $H^q(M_b, F(\nu)|_{M_b}) = 0$ ($\forall q > 0, \forall b \in U_1$). Applying the same construction to F_0/F , we obtain an open neighborhood U_2 of K with $K \subset U_2 \subset U_1$ a μ_2 -equivariant vector bundle F_1 on $\pi^{-1}(U_2)$ and an exact sequence of μ_2 -equivariant vector bundles $0 \rightarrow F|_{\pi^{-1}(U_2)} \rightarrow F_0|_{\pi^{-1}(U_2)} \rightarrow F_1$ on $\pi^{-1}(U_2)$ satisfying (2) for F_0 and F_1 . Repeating this $m-1$ times, we get the exact sequence $0 \rightarrow F|_{\pi^{-1}(U_m)} \rightarrow F_0|_{\pi^{-1}(U_m)} \rightarrow F_1|_{\pi^{-1}(U_m)} \rightarrow \cdots \rightarrow F_{m-1}|_{\pi^{-1}(U_m)}$ defined on an open neighborhood U_m of K with $K \subset U_m \subset U_{m-1} \subset \cdots \subset U_1 \subset U$, where $m = \dim M_b$. Then we set $F_m = F_{m-1}/F_{m-2}$ and we get the exact sequence of μ_2 -equivariant holomorphic vector bundles $0 \rightarrow F \rightarrow F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_m \rightarrow 0$ on $\pi^{-1}(U_m)$. By construction, $H^q(M_b, F_i|_{M_b}) = 0$ for all $q > 0$, $0 \leq i \leq m-1$ and $b \in U_m$. Then $H^q(M_b, F_m|_{M_b}) = 0$ for all $q > 0$ and $b \in U_m$. $\square$

By Lemma 6.5.1, $\pi_* F_\bullet : 0 \rightarrow \pi_* F_0 \rightarrow \pi_* F_1 \rightarrow \cdots \rightarrow \pi_* F_m \rightarrow 0$ is a complex of μ_2 -equivariant locally free sheaves of finite rank over K such that for all $q \geq 0$,

$$R^q \pi_* F|_K \cong \mathcal{H}^q(\pi_* F_\bullet) = \text{Ker} \{ \pi_* F_q \rightarrow \pi_* F_{q+1} \} / \text{Im} \{ \pi_* F_{q-1} \rightarrow \pi_* F_q \}. \quad (6.5.1)$$

Let $(\pi_* F_i)_\pm$ be the (± 1) -eigensubbundle of $\pi_* F_i$ with respect to the μ_2 -action. Since (6.5.1) is an isomorphism of μ_2 -equivariant sheaves on K , we have the following canonical isomorphism of coherent sheaves for all $q \geq 0$

$$(R^q \pi_* F)_\pm|_K \cong \mathcal{H}^q((\pi_* F_\bullet)_\pm). \quad (6.5.2)$$

For any coherent sheaf $\mathcal{F}$ on B , one can associate the invertible sheaf $\det \mathcal{F}$ on B ([47], [15]). We define

$$\lambda_\pm(F) = \otimes_{q \geq 0} (\det(R^q \pi_* F)_\pm)^{(-1)^q}, \quad \lambda_{\mu_2}(F) = \lambda_+(F) \otimes \lambda_-(F).$$

Since there is a canonical isomorphism $\det(\pi_* F_\bullet)_\pm|_K \cong \det \mathcal{H}^\bullet((\pi_* F_\bullet)_\pm)$ on K (cf. [47, p. 43 b])), we deduce from (6.5.2) the canonical isomorphism of line bundles on K (see also [15, p.337-p.338])

$$\lambda_\pm(F) \cong \otimes_{i \geq 0} \lambda_\pm(F_i)^{(-1)^i}.$$

Let $\mathcal{I}_b \subset \mathcal{O}_B$ be the ideal sheaf defining $b \in B$. Since $(\pi_* F_i)_\pm \otimes (\mathcal{O}_B/\mathcal{I}_b) \cong H^0(M_b, F_i)_\pm$ for all $b \in K$ and $i \geq 0$ by Lemma 6.5.1 (2), we have the following canonical isomorphism of lines for all $b \in K$:

$$\begin{aligned} \lambda_\pm(F)_b &= \lambda_\pm(F) \otimes (\mathcal{O}_B/\mathcal{I}_b) \\ &\cong \otimes_{i \geq 0} \lambda_\pm(F_i)^{(-1)^i} \otimes (\mathcal{O}_B/\mathcal{I}_b) \\ &\cong \otimes_{i \geq 0} (\det(\pi_* F_i)_\pm)^{(-1)^i} \otimes (\mathcal{O}_B/\mathcal{I}_b) \\ &\cong \otimes_{i \geq 0} (\det H^0(M_b, F_i|_{M_b})_\pm)^{(-1)^i} \\ &\cong \otimes_{q \geq 0} (\det H^q(M_b, F|_{M_b})_\pm)^{(-1)^q} \\ &= \lambda_\pm(F|_{M_b}). \end{aligned} \quad (6.5.3)$$

Let $h_{\mathcal{M}}$ be a μ_2 -invariant Kähler metric on $\mathcal{M}$ and let h_F be a μ_2 -invariant hermitian metric on F . Via the canonical isomorphism (6.5.3), $\lambda_{\mu_2}(F)$ is endowed with the equivariant Quillen metric $\|\cdot\|_{Q, \lambda_{\mu_2}(F)}(\iota)$ with respect to $h_{\mathcal{M}}|_{T\mathcal{M}/B}$, h_F such that for all $b \in B$,

$$\|\cdot\|_{Q, \lambda_{\mu_2}(F)}(\iota)(b) = \|\cdot\|_{Q, \lambda_{\mu_2}(F_b)}(\iota).$$

For an open subset $U \subset B$, a holomorphic section $\sigma = (\sigma_+, \sigma_-)$ is called admissible if both σ_+ and σ_- are nowhere vanishing on U .

Proposition 6.5.2. *Let $U \subset B$ be a small open subset such that $\lambda_{\mu_2}(F)|_U$ admits an admissible holomorphic section $\sigma = (\sigma_+, \sigma_-)$. Then $\log \|\sigma\|_{Q, \lambda_{\mu_2}(F)}^2(\iota)$ is a smooth function on U .*

Proof. Since U is small, we may assume by Lemma 6.5.1 that there exists a complex of μ_2 -equivariant holomorphic vector bundles $F_{\bullet}$ and a μ_2 -equivariant injective homomorphism $i : F|_{\pi^{-1}(U)} \rightarrow F_0$ satisfying Lemma 6.5.1 (1), (2). We have the canonical isomorphism of line bundles on U

$$\varphi_{\pm} : \lambda_{\pm}(F) \cong \otimes_{q \geq 0} \det \mathcal{H}^q((\pi_* F_{\bullet})_{\pm})^{(-1)^q} \cong \otimes_{i \geq 0} \lambda_{\pm}(F_i)^{(-1)^i}.$$

Shrinking U if necessary, we may assume that there exist admissible holomorphic sections $\sigma_i = ((\sigma_i)_+, (\sigma_i)_-)$ of $\lambda_{\mu_2}(F_i)$ defined on U such that $\varphi_{\pm}(\sigma_{\pm}) = \otimes_{i \geq 0} (\sigma_i)_{\pm}^{(-1)^i}$.

Let h_{F_i} be a μ_2 -invariant hermitian metric on F_i and let $\|\cdot\|_{Q, \lambda_{\mu_2}(F_i)}(\iota)$ be the equivariant Quillen metric on $\lambda_{\mu_2}(F_i)$ with respect to the μ_2 -invariant metrics $h_{\mathcal{M}/B} = h_{\mathcal{M}}|_{T\mathcal{M}/B}$ and h_{F_i} . Let $\tilde{ch}_{\iota}(\overline{F}, \overline{F}_{\bullet}) \in \tilde{A}(\mathcal{M}^{\iota})$ be the Bott-Chern secondary class (Theorem 1.2.5) such that

$$dd^c \tilde{ch}_{\iota}(\overline{F}, \overline{F}_{\bullet}) = \sum_{i \geq 0} (-1)^i ch_{\iota}(F_i, h_{F_i}) - ch_{\iota}(F, h_F).$$

By Theorem 1.2.8, we get the following equation of functions on U

$$\begin{aligned} \log \|\sigma\|_{Q, \lambda_{\mu_2}(F)}^2(\iota) &= \sum_{i \geq 0} (-1)^i \log \|\sigma_i\|_{Q, \lambda_{\mu_2}(F_i)}^2(\iota) + \left[\pi_* \{Td_{\iota}(\overline{T\mathcal{M}/B}) \tilde{ch}_{\iota}(\overline{F}, \overline{F}_{\bullet})\} \right]^{(0)} \\ &\equiv \sum_{i \geq 0} (-1)^i \log \|\sigma_i\|_{Q, \lambda_{\mu_2}(F_i)}^2(\iota) \pmod{C^{\infty}(U)}, \end{aligned} \tag{6.5.4}$$

where $\overline{T\mathcal{M}/B} = (T\mathcal{M}/B, h_{\mathcal{M}/B})$, $\overline{F} = (F, h_F)$ and $[\alpha]^{(2d)}$ denotes the component of degree $2d$ of a differential form α . Since $\pi : \pi^{-1}(U) \rightarrow U$ is a proper holomorphic submersion and since $b \mapsto h^0(M_b, F_i|_{M_b})$ is a constant function on U , it follows from [15, Th.3.5] that $\log \|\sigma_i\|_{Q, \lambda_{\mu_2}(F_i)}^2(\iota)$ is a smooth function on U . This, together with (6.5.4), implies the result. $\square$

6.5.2 Singularity of equivariant Quillen metrics

Let ι be the generator of μ_2 . Let $\mathcal{X}$ be a projective manifold endowed with a holomorphic μ_2 -action. Let C be a smooth projective curve endowed with a trivial μ_2 -action. Let $\pi : \mathcal{X} \rightarrow C$ be a μ_2 -equivariant surjective holomorphic map. Hence μ_2 preserves the fibers of π . Let $\Sigma_\pi = \{x \in \mathcal{X}; d\pi(x) = 0\}$ be the critical locus of π and let $\Delta_\pi = \pi(\Sigma_\pi) \subset C$ be the discriminant locus of π . Assume that there exists a μ_2 -equivariant ample line bundle on $\mathcal{X}$. We set $X_s = \pi^{-1}(s)$ for $s \in C$ and

$$C^\circ = C \setminus \Delta_\pi, \quad \mathcal{X}^\circ = \pi^{-1}(C^\circ), \quad \pi^\circ = \pi|_{\mathcal{X}^\circ}.$$

Then $\pi^\circ : \mathcal{X}^\circ \rightarrow C^\circ$ is a family of projective algebraic manifolds with μ_2 -action.

Let $T\mathcal{X}/C$ be the μ_2 -equivariant subbundle of $T\mathcal{X}|_{\mathcal{X} \setminus \Sigma_\pi}$ defined by $T\mathcal{X}/C = \text{Ker } \pi_*|_{\mathcal{X} \setminus \Sigma_\pi}$. Let $h_{\mathcal{X}}$ be a μ_2 -invariant Kähler metric on $\mathcal{X}$ and set $h_{\mathcal{X}/C} = h_{\mathcal{X}}|_{T\mathcal{X}/C}$. Let $\xi \rightarrow \mathcal{X}$ be a μ_2 -equivariant holomorphic vector bundle on $\mathcal{X}$ endowed with a μ_2 -invariant hermitian metric h_ξ . We set $\xi_s = \xi|_{X_s}$ for $s \in C$.

Let $0 \in \Delta_\pi$ be a critical value of π . Let (S, s) be a coordinate neighborhood of C centered at 0 such that $S \cap \Delta_\pi = \{0\}$. We set $X = \pi^{-1}(S)$ and $S^\circ = S \setminus \{0\}$.

For $s \in S^\circ$, let $\tau_{\mu_2}(X_s, \xi_s)(\iota)$ be the equivariant analytic torsion of (X_s, ξ_s) with respect to $h_{X_s} = h_{\mathcal{X}}|_{X_s}$ and $h_{\xi_s} = h_\xi|_{X_s}$. Let $\lambda_{\mu_2}(\xi)$ be the equivariant determinant of the cohomology of ξ . Then $\lambda_{\mu_2}(\xi)|_{C^\circ}$ is equipped with the equivariant Quillen metric $\|\cdot\|_{Q, \lambda_{\mu_2}(\xi)}(\iota)$ with respect to the μ_2 -invariant metrics $h_{\mathcal{X}/C}$, h_ξ . Let σ be an admissible holomorphic section of $\lambda_{\mu_2}(\xi)|_S$. By Proposition 6.5.2, $\log \|\sigma(s)\|_{Q, \lambda_{\mu_2}(\xi)}^2(\iota)$ is a smooth function on S° . In this subsection, following Bismut [12] and Yoshikawa [83], we determine the behavior of $\log \|\sigma(s)\|_{Q, \lambda_{\mu_2}(\xi)}^2(\iota)$ as $s \rightarrow 0$.

Let $\Gamma \subset \mathcal{X} \times C$ be the graph of π . Then Γ is a smooth divisor on $\mathcal{X} \times C$ preserved by the μ_2 -action on $\mathcal{X} \times C$. Let $[\Gamma]$ be the μ_2 -equivariant holomorphic line bundle on $\mathcal{X} \times C$ associated to Γ . Let $\varsigma_\Gamma \in H^0(\mathcal{X} \times C, [\Gamma])$ be the canonical section of $[\Gamma]$ such that $\text{div}(\varsigma_\Gamma) = \Gamma$. We identify $\mathcal{X}$ with Γ via the projection $\Gamma \rightarrow \mathcal{X}$.

Let $i : \Gamma \hookrightarrow \mathcal{X} \times C$ be the inclusion. Let $p_1 : \mathcal{X} \times C \rightarrow \mathcal{X}$ and $p_2 : \mathcal{X} \times C \rightarrow C$ be the projections. By the μ_2 -equivariances of i, p_1, p_2 , we have the exact sequence of μ_2 -equivariant coherent sheaves on $\mathcal{X} \times C$,

$$0 \rightarrow \mathcal{O}_{\mathcal{X} \times C}([\Gamma]^{-1} \otimes p_1^* \xi) \xrightarrow{\otimes \varsigma_\Gamma} \mathcal{O}_{\mathcal{X} \times C}(p_1^* \xi) \rightarrow i_* \mathcal{O}_\Gamma(p_1^* \xi) \rightarrow 0. \quad (6.5.5)$$

Let $\lambda_{\mu_2}(p_1^* \xi)$, $\lambda_{\mu_2}([\Gamma]^{-1} \otimes p_1^* \xi)$ and $\lambda_{\mu_2}(\xi)$ be the equivariant determinants of the direct images $R(p_2)_* \mathcal{O}(p_1^* \xi)$, $R(p_2)_* \mathcal{O}_{\mathcal{X} \times C}([\Gamma]^{-1} \otimes p_1^* \xi)$, $R\pi_* \mathcal{O}_{\mathcal{X}}(\xi)$, respectively. Under the isomorphism $p_1^* \xi|_\Gamma \cong \xi$ induced from the identification $p_1 : \Gamma \rightarrow \mathcal{X}$, we define the holomorphic vector bundles λ_{μ_2} , $\lambda_\pm$ on C by

$$\begin{aligned} \lambda_{\mu_2} &= \lambda_{\mu_2}([\Gamma]^{-1} \otimes p_1^* \xi) \otimes \lambda_{\mu_2}(p_1^* \xi)^{-1} \otimes \lambda_{\mu_2}(\xi) = \lambda_+ \oplus \lambda_-, \\ \lambda_\pm &= \lambda_\pm([\Gamma]^{-1} \otimes p_1^* \xi) \otimes \lambda_\pm(p_1^* \xi)^{-1} \otimes \lambda_\pm(\xi). \end{aligned}$$

Since $\lambda_\pm$ is the determinant of the acyclic complex of coherent sheaves on C obtained as the (± 1) -component of the long exact sequence of direct image sheaves on C associated

to (6.5.5), $\lambda_{\pm}$ is canonically isomorphic to $\mathcal{O}_C$ ([11], [17], [47]). The canonical nowhere vanishing section of $\lambda_{\pm}$ is denoted by $(\sigma_{KM})_{\pm}$. Then we set $\sigma_{KM} = ((\sigma_{KM})_+, (\sigma_{KM})_-)$.

Let $U \subset S$ be a relatively compact neighborhood of $0 \in \Delta_{\pi}$ and set $U^{\circ} = U \setminus \{0\}$. On $X = \pi^{-1}(S)$, we identify π (resp. $d\pi$) with $s \circ \pi$ (resp. $d(s \circ \pi)$). Hence $\pi \in \mathcal{O}(X)$ and $d\pi \in H^0(X, \Omega_X^1)$ in what follows.

Let $h_{[\Gamma]}$ be a μ_2 -invariant C^{∞} hermitian metric on $[\Gamma]$ such that

$$h_{[\Gamma]}(\varsigma_{\Gamma}, \varsigma_{\Gamma})(w, t) = \begin{cases} |\pi(w) - t|^2 & \text{if } (w, t) \in \pi^{-1}(U) \times U, \\ 1 & \text{if } (w, t) \in (\mathcal{X} \setminus X) \times U \end{cases} \quad (6.5.6)$$

and let $h_{[\Gamma]^{-1}}$ be the metric on $[\Gamma]^{-1}$ induced from $h_{[\Gamma]}$.

Let $\|\cdot\|_{Q, \lambda_{\mu_2}(\xi)}(\iota)$ be the equivariant Quillen metric on $\lambda_{\mu_2}(\xi)$ with respect to $h_{\mathcal{X}/C}$, h_{ξ} . Let $\|\cdot\|_{Q, \lambda_{\mu_2}([\Gamma]^{-1} \otimes p_1^* \xi)}(\iota)$ (resp. $\|\cdot\|_{Q, \lambda_{\mu_2}(p_1^* \xi)}(\iota)$) be the equivariant Quillen metric on $\lambda_{\mu_2}([\Gamma]^{-1} \otimes p_1^* \xi)$ (resp. $\lambda_{\mu_2}(p_1^* \xi)$) with respect to $h_{\mathcal{X}}$, $h_{[\Gamma]^{-1}} \otimes h_{\xi}$ (resp. $h_{\mathcal{X}}$, h_{ξ}). Let $\|\cdot\|_{Q, \lambda_{\mu_2}}(\iota)$ be the equivariant Quillen metric on λ_{μ_2} defined by the tensor product of those on $\lambda_{\mu_2}([\Gamma]^{-1} \otimes p_1^* \xi)$, $\lambda_{\mu_2}(p_1^* \xi)$, $\lambda_{\mu_2}(\xi)$.

Let $\Pi : \mathbb{P}(T\mathcal{X})^{\vee} \rightarrow \mathcal{X}$ be the projective space bundle over $\mathcal{X}$ with fibers $\mathbb{P}(T\mathcal{X})_x^{\vee} = \mathbb{P}(T_x\mathcal{X})^{\vee}$. Let $\mathcal{U} \rightarrow \mathbb{P}(T\mathcal{X})^{\vee}$ be the universal hyperplane bundle and set $\mathcal{H} = \Pi^*T\mathcal{X}/\mathcal{U}$. As before, we have the following exact sequence of holomorphic vector bundles on $\mathbb{P}(T\mathcal{X})^{\vee}$:

$$\mathcal{S} : 0 \rightarrow \mathcal{U} \rightarrow \Pi^*T\mathcal{X} \rightarrow \mathcal{H} \rightarrow 0.$$

We define the Gauss map $\nu : \mathcal{X} \setminus \Sigma_{\pi} \rightarrow \mathbb{P}(T\mathcal{X})^{\vee}$ by $\nu(x) = [T_x X_{\pi(x)}]$. Then we have the equality of vector bundles $T\mathcal{X}/C = \nu^*\mathcal{U}$ on $\mathcal{X} \setminus \Sigma_{\pi}$. Let

$$q : (\widetilde{\mathcal{X}}, E) \rightarrow (\mathcal{X}, \Sigma_{\pi})$$

be a resolution of the indeterminacy of ν such that $\tilde{\nu} = \nu \circ q$ extends to a morphism from $\widetilde{\mathcal{X}}$ to $\mathbb{P}(T\mathcal{X})^{\vee}$. Here $E = q^{-1}(\Sigma_{\pi})$ is the exceptional divisor of q , whose ideal sheaf is defined by $\mathcal{I}_E = q^*\mathcal{I}_{\Sigma_{\pi}}$. In this way, $\tilde{\nu}^*\mathcal{U}$ is viewed as an extension of $T\mathcal{X}/C|_{\mathcal{X} \setminus \Sigma_{\pi}}$ to a vector bundle on $\widetilde{\mathcal{X}}$. Set

$$E_0 = E \cap q^{-1}(X_0).$$

Let $\mathcal{X}^{\iota} \subset \mathcal{X}$ be the set of fixed points of ι . Then we have the decomposition

$$\mathcal{X}^{\iota} = \mathcal{X}_H^{\iota} \sqcup \mathcal{X}_V^{\iota},$$

where $\mathcal{X}_H^{\iota}$ is the horizontal component and $\mathcal{X}_V^{\iota}$ is the vertical component. Namely, any connected component of $\mathcal{X}_H^{\iota}$ is flat over C and $\mathcal{X}_V^{\iota} \subset \pi^{-1}(\Delta_{\pi})$. Let $\widetilde{\mathcal{X}}_H^{\iota} \subset \widetilde{\mathcal{X}}$ be the proper transform of $\mathcal{X}_H^{\iota}$.

We define a topological invariant of the germ $(\pi : (\mathcal{X}, X_0) \rightarrow (S, 0), \xi)$ by

$$\begin{aligned} \alpha_{\iota}(X_0, \xi) = & \int_{E_0 \cap \widetilde{\mathcal{X}}_H^{\iota}} \tilde{\nu}^* \left\{ \frac{1 - Td(\mathcal{H})^{-1}}{c_1(\mathcal{H})} \right\} q^* \{Td_{\iota}(T\mathcal{X})ch_{\iota}(\xi)\} \\ & - \int_{\mathcal{X}_V^{\iota} \cap X_0} Td_{\iota}(T\mathcal{X})ch_{\iota}(\xi). \end{aligned}$$

In the next theorem, for functions f, g on U° , we write $f \equiv g$ if $f - g \in C^0(U)$.

Theorem 6.5.3. *The following identity of functions on S° holds:*

$$\log \|\sigma_{KM}\|_{Q,\lambda_{\mu_2}}^2(\iota) \equiv \alpha_\iota(X_0, \xi) \log |s|^2.$$

Proof. We follow [12, Sect.5], [82, Theorem 6.3], [83, Theorem 5.1]. The proof is quite parallel to that of [83, Theorem 5.1]. The major differences stem from the fact that $\mathcal{X}^\iota$ consists of the horizontal component $\mathcal{X}_H^\iota$ and vertical component $\mathcal{X}_V^\iota$ and these components give different contributions to the singularity of $\log \|\sigma_{KM}\|_{Q,\lambda_{\mu_2}}^2(\iota)$.

Step 1 Let $[X_s] = [\Gamma]|_{X_s}$ be the holomorphic line bundle on $\mathcal{X}$ associated to the divisor X_s . The canonical section of $[X_s]$ is defined by $\varsigma_s = \varsigma_\Gamma|_{\mathcal{X} \times \{s\}} \in H^0(\mathcal{X}, [X_s])$. Then $\text{div}(\varsigma_s) = X_s$. Let $i_s : X_s \hookrightarrow \mathcal{X}$ be the natural embedding. By (6.5.5), we get the exact sequence of μ_2 -equivariant coherent sheaves on $\mathcal{X}$,

$$0 \rightarrow \mathcal{O}_{\mathcal{X}}([X_s]^{-1} \otimes \xi) \xrightarrow{\otimes \varsigma_s} \mathcal{O}_{\mathcal{X}}(\xi) \rightarrow (i_s)_* \mathcal{O}_{X_s}(\xi) \rightarrow 0 \quad (6.5.7)$$

and the canonical isomorphism $(\lambda_{\mu_2})_s \cong \lambda_{\mu_2}([X_s]^{-1} \otimes \xi) \otimes \lambda_{\mu_2}(\xi)^{-1} \otimes \lambda_{\mu_2}(\xi_s)$.

Set $h_{[X_s]} = h_{[\Gamma]}|_{\mathcal{X} \times \{s\}}$, which is a μ_2 -invariant hermitian metric on $[X_s]$. Let $h_{[X_s]^{-1}}$ be the μ_2 -invariant hermitian metric on $[X_s]^{-1}$ induced from $h_{[X_s]}$. Let $N_s = N_{X_s/\mathcal{X}}$ (resp. $N_s^\vee = N_{X_s/\mathcal{X}}^\vee$) be the normal (resp. conormal) bundle of X_s in $\mathcal{X}$. Then $d\pi|_{X_s} \in H^0(X_s, N_s^\vee)$ generates $N_s^\vee$ for $s \in S^\circ$ and $d\pi|_{X_s}$ is μ_2 -invariant (cf. [11, (2.2)]). Let $a_{N_s^\vee}$ be the μ_2 -invariant hermitian metric on $N_s^\vee$ defined by $a_{N_s^\vee}(d\pi|_{X_s}, d\pi|_{X_s}) = 1$. Let a_{N_s} be the μ_2 -invariant hermitian metric on N_s induced from $a_{N_s^\vee}$. We have the equality $c_1(N_s, a_{N_s}) = 0$ for $s \in U^\circ$. By [83, Proof of Theorem 5.1 Step 1], the μ_2 -invariant metrics $h_{[X_s]^{-1}} \otimes h_\xi$ and h_ξ verify assumption (A) of Bismut [10, Definition 1.5] with respect to a_{N_s} and $h_\xi|_{X_s}$.

Step 2 Let $\mathcal{E}_s$ be the exact sequence of μ_2 -equivariant holomorphic vector bundles on X_s defined by

$$\mathcal{E}_s : 0 \rightarrow TX_s \rightarrow T\mathcal{X}|_{X_s} \rightarrow N_s \rightarrow 0.$$

By Theorem 1.2.4, one has the Bott-Chern class $\widetilde{Td}_\iota(\mathcal{E}_s; h_{X_s}, h_{\mathcal{X}}, a_{N_s}) \in \tilde{A}_{X_s^\iota}$ such that

$$dd^c \widetilde{Td}_\iota(\mathcal{E}_s; h_{X_s}, h_{\mathcal{X}}, a_{N_s}) = Td_\iota(TX_s, h_{X_s})Td_\iota(N_s, a_{N_s})|_{X_s^\iota} - Td_\iota(T\mathcal{X}, h_{\mathcal{X}}).$$

Here, to get the equality $Td_\iota(N_s, a_{N_s}) = Td(N_s, a_{N_s})|_{X_s^\iota}$, we used the triviality of the μ_2 -action on $N_s|_{X_s^\iota}$. Set

$$(X_H^\iota)_s = \mathcal{X}_H^\iota \cap X_s.$$

Applying the embedding formula of Bismut [11] (see also [12, Theorem 5.6]) to the μ_2 -equivariant embedding $i_s : X_s \hookrightarrow \mathcal{X}$ and to the exact sequence (6.5.7), we get for all $s \in U^\circ$

$$\begin{aligned} \log \|\sigma_{KM}(s)\|_{Q,\lambda_{\mu_2}}^2(\iota) &= \int_{(\mathcal{X}_V^\iota \times \{s\}) \sqcup (\mathcal{X}_H^\iota \times \{s\})} - \frac{Td_\iota(T\mathcal{X}, h_{\mathcal{X}})ch_\iota(\xi, h_\xi)}{Td([\Gamma], h_{[\Gamma]})} \log h_{[\Gamma]}(\varsigma_\Gamma, \varsigma_\Gamma) \\ &\quad - \int_{(X_H^\iota)_s} \frac{\widetilde{Td}_\iota(\mathcal{E}_s; h_{X_s}, h_{\mathcal{X}}, a_{N_s})ch_\iota(\xi, h_\xi)}{Td(N_s, a_{N_s})} + C, \end{aligned} \quad (6.5.8)$$

where C is a topological constant independent of $s \in U^\circ$ (see Theorem 1.2.7 and its proof). Substituting (6.5.6) and $c_1(N_s, a_{N_s}) = 0$ into (6.5.8), we get for $s \in U^\circ$

$$\begin{aligned} \log \|\sigma_{KM}(s)\|_{Q, \lambda_{\mu_2}}^2(\iota) &\equiv - \int_{\mathcal{X}_H^\iota \times \{s\}} Td_\iota(T\mathcal{X}, h_{\mathcal{X}}) ch_\iota(\xi, h_\xi) \log |\pi - s|^2 \\ &- \left\{ \int_{(\mathcal{X}_V^\iota \cap X_0) \times \{s\}} Td_\iota(T\mathcal{X}) ch_\iota(\xi) \right\} \log |s|^2 - \int_{(X_H^\iota)_s} \widetilde{Td}_\iota(\mathcal{E}_s; h_{X_s}, h_{\mathcal{X}}, a_{N_s}) ch_\iota(\xi, h_\xi) \\ &\equiv - \left\{ \int_{\mathcal{X}_V^\iota \cap X_0} Td_\iota(T\mathcal{X}) ch_\iota(\xi) \right\} \log |s|^2 - \int_{(X_H^\iota)_s} \widetilde{Td}_\iota(\mathcal{E}_s; h_{X_s}, h_{\mathcal{X}}, a_{N_s}) ch_\iota(\xi, h_\xi), \end{aligned} \quad (6.5.9)$$

where we used the equalities $h_{[\Gamma]}(\varsigma_\Gamma, \varsigma_\Gamma)|_{X_0 \times \{s\}} = |s|^2$ and $\mathcal{X}_V^\iota \cap \pi^{-1}(U) = \mathcal{X}_V^\iota \cap X_0$ to get the first equality and [83, Theorem 9.1] to get the second equality.

Step 3 Let $\mathcal{L}$ be the tautological line bundle on $\mathbb{P}(\Omega_{\mathcal{X}}^1 \otimes \pi^*TC)$, which is canonically identified with a line bundle on $\mathbb{P}(T\mathcal{X})^\vee$. The vector bundles $\mathcal{U}$ and $\mathcal{L}$ are endowed with the hermitian metrics induced from the one on $\Pi^*T\mathcal{X}$ via the inclusions $\mathcal{U} \hookrightarrow \Pi^*T\mathcal{X}$ and $\mathcal{L} \hookrightarrow \Pi^*(\Omega_{\mathcal{X}}^1 \otimes \pi^*TC)$, respectively. Namely, $h_{\mathcal{U}} = (\Pi^*h_{\mathcal{X}})|_{\mathcal{U}}$ and $h_{\mathcal{L}} = \Pi^*(h_{\mathcal{X}}^\vee \otimes \pi^*h_C)|_{\mathcal{L}}$. Write $\overline{\mathcal{U}} = (\mathcal{U}, h_{\mathcal{U}})$ and $\overline{\mathcal{L}} = (\mathcal{L}, h_{\mathcal{L}})$.

Let h_{N_s} be the hermitian metric on N_s induced from $h_{\mathcal{X}}$ by the C^∞ isomorphism $N_s \cong (TX_s)^\perp$. Let $\widetilde{Td}(N_s; a_{N_s}, h_{N_s}) \in \tilde{A}_{X_s}$ be the Bott-Chern secondary class such that

$$dd^c \widetilde{Td}(N_s; a_{N_s}, h_{N_s}) = Td(N_s, a_{N_s}) - Td(N_s, h_{N_s}).$$

Since μ_2 acts trivially on $N_s|_{X_s^\iota}$, we deduce from [32, Propositions 1.3.2 and 1.3.4] that

$$\begin{aligned} \widetilde{Td}_\iota(\mathcal{E}_s; h_{X_s}, h_{\mathcal{X}}, a_{N_s}) &= \widetilde{Td}_\iota(\mathcal{E}_s; h_{X_s}, h_{\mathcal{X}}, h_{N_s}) + Td_\iota(TX_s, h_{X_s}) \widetilde{Td}(N_s; a_{N_s}, h_{N_s}) \\ &= \widetilde{Td}_\iota(\mathcal{E}_s; h_{X_s}, h_{\mathcal{X}}, h_{N_s}) + \nu^* Td_\iota(\overline{\mathcal{U}}) \nu^* \left\{ \frac{1 - Td(-c_1(\overline{\mathcal{L}}))}{-c_1(\overline{\mathcal{L}})} \right\} \log \|d\pi\|^2|_{(X_H^\iota)_s} \\ &= \left[\nu^* \widetilde{Td}_\iota(\mathcal{S}; h_{\mathcal{U}}, \Pi^*h_{\mathcal{X}}, h_{\mathcal{H}}) + \nu^* \left\{ Td_\iota(\overline{\mathcal{U}}) \frac{1 - Td(-c_1(\overline{\mathcal{L}}))}{-c_1(\overline{\mathcal{L}})} \right\} \log \|d\pi\|^2 \right] |_{(X_H^\iota)_s}. \end{aligned} \quad (6.5.10)$$

We used [83, (13)] and $(TX_s, h_{X_s}) = \nu^*\overline{\mathcal{U}}|_{X_s}$ to get the second equality and we used the relation $(\mathcal{E}_s; h_{X_s}, h_{\mathcal{X}}, h_{N_s}) = \nu^*(\mathcal{S}; h_{\mathcal{U}}, \Pi^*h_{\mathcal{X}}, h_{\mathcal{H}})|_{X_s}$ and the functoriality of the Bott-Chern class (Theorem 1.2.4) to get the third equality. Set $\tilde{\pi} = \pi \circ q$. Substituting (6.5.10) into (6.5.9) and using the fact that $[(\pi|_{\widetilde{\mathcal{X}}_H^\iota})_* \omega]^{(0)} \equiv 0$ for any smooth differential form ω on $\widetilde{\mathcal{X}}_H^\iota$ (cf. [4, Theorem 1]), we deduce from the same argument as in [83, p.74 1.1-1.13] that

$$\begin{aligned} \log \|\sigma_{KM}(s)\|_{Q, \lambda_{\mu_2}}^2(\iota) &\equiv - \left\{ \int_{\mathcal{X}_V^\iota \cap X_0} Td_\iota(T\mathcal{X}) ch_\iota(\xi) \right\} \log |s|^2 \\ &+ (\tilde{\pi}|_{\widetilde{\mathcal{X}}_H^\iota})^* \left[\tilde{\nu}^* \left\{ Td_\iota(\overline{\mathcal{U}}) \frac{Td(-c_1(\overline{\mathcal{L}})) - 1}{-c_1(\overline{\mathcal{L}})} \right\} q^* ch_\iota(\xi, h_\xi) (q^* \log \|d\pi\|^2) \right]^{(0)}. \end{aligned} \quad (6.5.11)$$

By [83, Corollary 4.6] applied to the second term of the right hand side of (6.5.11), we get

$$\begin{aligned} \log \|\sigma_{KM}(s)\|_{Q,\lambda_{\mu_2}}^2(\iota) &\equiv - \left\{ \int_{\mathcal{X}_V^\iota \cap X_0} Td_\iota(T\mathcal{X}) ch_\iota(\xi) \right\} \log |s|^2 \\ &\quad + \left[\int_{\widetilde{\mathcal{X}_H^\iota} \cap E_0} \tilde{\nu}^* \left\{ Td_\iota(\mathcal{U}) \frac{Td(\mathcal{H}) - 1}{c_1(\mathcal{H})} \right\} q^* ch_\iota(\xi) \right] \log |s|^2 \\ &= \alpha_\iota(X_0, \xi) \log |s|^2, \end{aligned}$$

where the last equality follows from the identity $Td_\iota(\mathcal{U})Td(\mathcal{H}) = \Pi^*Td_\iota(T\mathcal{X})$. This completes the proof. $\square$

Theorem 6.5.4. *The following identity of functions on S° holds:*

$$\log \|\sigma\|_{Q,\lambda_{\mu_2}(\xi)}^2(\iota) \equiv \alpha_\iota(X_0, \xi) \log |s|^2.$$

Proof. There exist admissible holomorphic sections

$$\alpha = (\alpha_+, \alpha_-) \in \Gamma(U, \lambda_{\mu_2}(p_1^*\xi)), \quad \beta = (\beta_+, \beta_-) \in \Gamma(U, \lambda_{\mu_2}([\Gamma]^{-1} \otimes p_1^*\xi))$$

such that $\sigma_{KM} = \beta \otimes \alpha^{-1} \otimes \sigma$ on U , i.e., $(\sigma_{KM})_\pm = \beta_\pm \otimes \alpha_\pm^{-1} \otimes \sigma_\pm$. Then we have

$$\begin{aligned} \log \|\sigma\|_{Q,\lambda_{\mu_2}(\xi)}^2(\iota) &= \log \|\sigma_{KM}\|_{Q,\lambda_{\mu_2}}^2(\iota) + \log \|\alpha\|_{Q,\lambda_{\mu_2}(p_1^*\xi)}^2(\iota) - \log \|\beta\|_{Q,\lambda_{\mu_2}([\Gamma]^{-1} \otimes p_1^*\xi)}^2(\iota) \\ &\equiv \alpha_\iota(X_0, \xi) \log |s|^2 \end{aligned}$$

by Proposition 6.5.2 and Theorem 6.5.3. This proves the theorem. $\square$

Chapter 7

The invariant $\tau_{M,\mathcal{K}}$ and the BCOV invariant

In this chapter, we compare our invariant $\tau_{M,\mathcal{K}}$ with the BCOV invariant of Calabi-Yau 4-folds of Camere-Garbagnati-Mongardi [24]. In some cases, we give an explicit formula for the BCOV invariant of these manifolds.

7.1 Comparison between $\tau_{M,\mathcal{K}}$ and τ_{BCOV}

7.1.1 Calabi-Yau 4-folds of Camere-Garbagnati-Mongardi and their BCOV invariants

Throughout this section, we fix an admissible sublattice M of L_2 and a Kähler-type chamber $\mathcal{K} \in \text{KT}(M)$.

Let (X, ι) be a manifold of $K3^{[2]}$ -type with involution of type $(M, \mathcal{K})$. The fixed locus is denoted by X^ι . The blowup of X along the fixed locus X^ι is denoted by $\tilde{X}$. The involution $\iota : X \rightarrow X$ induces an involution $\tilde{\iota} : \tilde{X} \rightarrow \tilde{X}$ of $\tilde{X}$. By Camere-Garbagnati-Mongardi [24, Theorem 3.6], the quotient $\tilde{X}/\tilde{\iota}$ is a Calabi-Yau 4-fold and we set

$$Z = \tilde{X}/\tilde{\iota}.$$

Definition 7.1.1. *The manifold Z is called a Calabi-Yau manifold of Camere-Garbagnati-Mongardi.*

Let $f : (\mathcal{X}, \iota) \rightarrow S$ be a family of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$. Let $X_s = f^{-1}(s)$ be the fiber at $s \in S$. The fixed locus of ι is denoted by $\mathcal{X}^\iota$ and the blowup of $\mathcal{X}$ along $\mathcal{X}^\iota$ is denoted by $\tau : \tilde{\mathcal{X}} = \text{Bl}_{\mathcal{X}^\iota}(\mathcal{X}) \rightarrow \mathcal{X}$. The involution ι induces an involution $\tilde{\iota}$ of $\tilde{\mathcal{X}}$ and its quotient is denoted by $\mathcal{Z} = \tilde{\mathcal{X}}/\tilde{\iota}$. The morphism induced from $\tilde{\mathcal{X}} \xrightarrow{\tau} \mathcal{X} \xrightarrow{f} S$ is denoted by $h : \mathcal{Z} \rightarrow S$. By [24, Theorem 3.6], each fiber $Z_s = h^{-1}(s)$ is a Calabi-Yau 4-fold and $h : \mathcal{Z} \rightarrow S$ is a family of Calabi-Yau 4-folds.

Let $h_{\mathcal{X}/S}$ be an ι -invariant hermitian metric on the relative tangent bundle $T\mathcal{X}/S$ which is fiberwise Kähler. Since $\dim H^q(X_s, \Omega_{X_s}^p)$ is independent of $s \in S$, $R^q f_* \Omega_{\mathcal{X}/S}^p$ is a locally free sheaf and we regard it as a holomorphic vector bundle on S . By Hodge theory, $R^q f_* \Omega_{\mathcal{X}/S}^p$ is equipped with the ι -invariant hermitian metric. This metric is called the L^2 -metric and is denoted by h_{L^2} .

Let $h_{\mathcal{Z}/S}$ be a hermitian metric on $T\mathcal{Z}/S$ which is fiberwise Kähler. In the same manner as above, the holomorphic vector bundle $R^q h_* \Omega_{\mathcal{Z}/S}^p$ is equipped with the L^2 metric, which is denoted by h_{L^2} .

Define a function $\tau_{BCOV, \mathcal{Z}/S}$ on S by

$$\tau_{BCOV, \mathcal{Z}/S}(s) = \tau_{BCOV}(Z_s) \quad (s \in S).$$

By Theorem 4.2.4, $\tau_{BCOV, \mathcal{Z}/S}$ satisfies the following curvature equation on S :

$$dd^c \log \tau_{BCOV, \mathcal{Z}/S} = \sum_{k=0}^8 (-1)^k \omega_{H^k} - \frac{\chi}{12} \omega_{WP}, \quad (7.1.1)$$

where

$$\omega_{H^k} = \sum_{p+q=k} pc_1(R^q h_* \Omega_{\mathcal{Z}/S}^p, h_{L^2}) \quad \text{and} \quad \omega_{WP} = c_1(h_* K_{\mathcal{Z}/S}, h_{L^2}). \quad (7.1.2)$$

In the next subsection, we show that $\tau_{M, \mathcal{K}, \mathcal{Z}/S}$ satisfies the same equation (7.1.1).

Theorem 7.1.2. *The following equation on S holds:*

$$dd^c \log \tau_{M, \mathcal{K}, \mathcal{Z}/S} = \sum_{k=0}^8 (-1)^k \omega_{H^k} - \frac{\chi}{12} \omega_{WP}.$$

7.1.2 Proof of Theorem 7.1.2

Lemma 7.1.3. *The following identity holds:*

$$\sum_{k=0}^8 (-1)^k \omega_{H^k} = 2 \sum_{q=1}^3 (-1)^q c_1(R^q h_* \Omega_{\mathcal{Z}/S}^1, h_{L^2}) - 4 \{c_1(h_* \mathcal{O}_{\mathcal{Z}}, h_{L^2}) + c_1(R^4 h_* \mathcal{O}_{\mathcal{Z}}, h_{L^2})\}.$$

Proof. Since each fiber Z_s is a Calabi-Yau 4-fold, we have

$$R^q h_* \Omega_{\mathcal{Z}/S}^p = 0 \quad (7.1.3)$$

for $(p, q) = (1, 0), (2, 0), (3, 0), (0, 1), (0, 2), (0, 3), (1, 4), (2, 4), (3, 4), (4, 1), (4, 2), (4, 3)$. By (7.1.2) and (7.1.3), we have

$$\begin{aligned} \omega_{H^0} &= 0, \quad \omega_{H^1} = 0, \quad \omega_{H^2} = c_1(R^1 h_* \Omega_{\mathcal{Z}/S}^1, h_{L^2}), \\ \omega_{H^3} &= c_1(R^2 h_* \Omega_{\mathcal{Z}/S}^1, h_{L^2}) + 2c_1(R^1 h_* \Omega_{\mathcal{Z}/S}^2, h_{L^2}). \end{aligned} \quad (7.1.4)$$

By the Serre duality (Lemma 5.4.1), we have $c_1(R^2 h_* \Omega_{\mathcal{X}/S}^2, h_{L^2}) = -c_1(R^2 h_* \Omega_{\mathcal{X}/S}^2, h_{L^2})$. Therefore

$$c_1(R^2 h_* \Omega_{\mathcal{X}/S}^2, h_{L^2}) = 0. \quad (7.1.5)$$

By (7.1.2), (7.1.3), (7.1.5) and by the Serre duality (Lemma 5.4.1), we have

$$\begin{aligned} \omega_{H^4} &= c_1(R^3 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) + 2c_1(R^2 h_* \Omega_{\mathcal{X}/S}^2, h_{L^2}) + 3c_1(R^1 h_* \Omega_{\mathcal{X}/S}^3, h_{L^2}) + 4c_1(R^0 h_* \Omega_{\mathcal{X}/S}^4, h_{L^2}) \\ &= -2c_1(R^3 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) - 4c_1(R^4 h_* \Omega_{\mathcal{X}/S}^0, h_{L^2}), \\ \omega_{H^5} &= 2c_1(R^3 h_* \Omega_{\mathcal{X}/S}^2, h_{L^2}) + 3c_1(R^2 h_* \Omega_{\mathcal{X}/S}^3, h_{L^2}) \\ &= -2c_1(R^1 h_* \Omega_{\mathcal{X}/S}^2, h_{L^2}) - 3c_1(R^2 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}), \\ \omega_{H^6} &= 3c_1(R^3 h_* \Omega_{\mathcal{X}/S}^3, h_{L^2}) = -3c_1(R^1 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}), \\ \omega_{H^7} &= 0, \quad \omega_{H^8} = 4c_1(R^4 h_* \Omega_{\mathcal{X}/S}^4, h_{L^2}) = -4c_1(h_* \mathcal{O}_{\mathcal{X}}, h_{L^2}). \end{aligned} \quad (7.1.6)$$

By (7.1.4) and (7.1.6), we obtain the desired result. $\square$

Let X be a fiber of $f : \mathcal{X} \rightarrow S$. The blowup of X along the fixed locus X^ι is denoted by $\tau : \tilde{X} \rightarrow X$. The involution of $\tilde{X}$ induced from ι is denoted by $\tilde{\iota}$. Then the quotient $Z := \tilde{X}/\tilde{\iota}$ is a fiber of $h : \mathcal{Z} \rightarrow S$. Let E be the exceptional divisor and let $j : E \hookrightarrow \tilde{X}$ be the inclusion. Let $H^q(X, \Omega_X^p)_\pm$ and $H^q(\tilde{X}, \Omega_{\tilde{X}}^p)_\pm$ be the (± 1) -eigenspaces of $H^q(X, \Omega_X^p)$ and $H^q(\tilde{X}, \Omega_{\tilde{X}}^p)$ with respect to the ι and $\tilde{\iota}$ -action, respectively. By [80, Theorem 7.31], we have

$$\tau^* \oplus j_*(\tau|_E)^* : H^q(X, \Omega_X^p) \oplus H^{q-1}(X^\iota, \Omega_{X^\iota}^{p-1}) \cong H^q(\tilde{X}, \Omega_{\tilde{X}}^p), \quad (7.1.7)$$

where $j_* : H^{q-1}(E, \Omega_E^{p-1}) \rightarrow H^q(\tilde{X}, \Omega_{\tilde{X}}^p)$ is the Gysin map. Since $\tau : \tilde{X} \rightarrow X$ is a μ_2 -equivariant map with respect to ι and $\tilde{\iota}$ and since μ_2 acts trivially on X^ι and E , the isomorphism (7.1.7) induces the isomorphism of eigenspaces

$$H^q(X, \Omega_X^p)_\pm \oplus H^{q-1}(X^\iota, \Omega_{X^\iota}^{p-1}) \cong H^q(\tilde{X}, \Omega_{\tilde{X}}^p)_\pm. \quad (7.1.8)$$

Note that the pullback of the projection $\tilde{X} \rightarrow Z$ induces an isomorphism

$$H^q(\tilde{X}, \Omega_{\tilde{X}}^p)_+ \cong H^q(Z, \Omega_Z^p). \quad (7.1.9)$$

By (7.1.8) and (7.1.9), we have the canonical isomorphism

$$H^q(X, \Omega_X^p)_+ \oplus H^{q-1}(X^\iota, \Omega_{X^\iota}^{p-1}) \cong H^q(Z, \Omega_Z^p). \quad (7.1.10)$$

(See [24, proof of Theorem 4.1].)

Let $\left(R^q f_* \Omega_{\mathcal{X}/S}^p\right)_\pm$ be the (± 1) -eigenbundles of $R^q f_* \Omega_{\mathcal{X}/S}^p$. By (7.1.10), we have an isomorphism of holomorphic vector bundles

$$\phi : \left(R^q f_* \Omega_{\mathcal{X}/S}^p\right)_+ \oplus R^{q-1} f_* \Omega_{\mathcal{X}/S}^{p-1} \cong R^q h_* \Omega_{\mathcal{Z}/S}^p.$$

Similarly, we have an isomorphism of holomorphic vector bundles

$$\psi : \left(R^p f_* \Omega_{\mathcal{X}/S}^q \right)_+ \oplus R^{p-1} f_* \Omega_{\mathcal{X}^\iota/S}^{q-1} \cong R^p h_* \Omega_{\mathcal{X}/S}^q.$$

In the category of C^∞ complex vector bundles on S , we can identify $R^q f_* \Omega_{\mathcal{X}/S}^p$ with the C^∞ complex vector bundle whose fiber is the space of harmonic (p, q) -forms. Similarly, $R^{q-1} f_* \Omega_{\mathcal{X}^\iota/S}^{p-1}$ and $R^q h_* \Omega_{\mathcal{X}/S}^p$ can be identified with C^∞ complex vector bundles whose fiber is the space of harmonic forms. Let $c : \left(R^q f_* \Omega_{\mathcal{X}/S}^p \right)_+ \rightarrow \left(R^p f_* \Omega_{\mathcal{X}/S}^q \right)_+$, $c : R^{q-1} f_* \Omega_{\mathcal{X}^\iota/S}^{p-1} \rightarrow R^{p-1} f_* \Omega_{\mathcal{X}^\iota/S}^{q-1}$ and $c : R^q h_* \Omega_{\mathcal{X}/S}^p \rightarrow R^p h_* \Omega_{\mathcal{X}/S}^q$ be the homomorphisms of C^∞ complex vector bundles induced from the complex conjugation of harmonic forms. Then the following diagram of C^∞ complex vector bundles commutes:

$$\begin{array}{ccc} \left(R^q f_* \Omega_{\mathcal{X}/S}^p \right)_+ \oplus R^{q-1} f_* \Omega_{\mathcal{X}^\iota/S}^{p-1} & \xrightarrow{\phi} & R^q h_* \Omega_{\mathcal{X}/S}^p \\ c \downarrow & & \downarrow c \\ \left(R^p f_* \Omega_{\mathcal{X}/S}^q \right)_+ \oplus R^{p-1} f_* \Omega_{\mathcal{X}^\iota/S}^{q-1} & \xrightarrow{\psi} & R^p h_* \Omega_{\mathcal{X}/S}^q. \end{array} \quad (7.1.11)$$

Lemma 7.1.4. *The following identity of $(1, 1)$ -forms on S holds:*

$$\begin{aligned} & c_1(R^q h_* \Omega_{\mathcal{X}/S}^p, h_{L^2}) - c_1(R^p h_* \Omega_{\mathcal{X}/S}^q, h_{L^2}) \\ &= c_1 \left(\left(R^q f_* \Omega_{\mathcal{X}/S}^p \right)_+, h_{L^2} \right) - c_1 \left(\left(R^p f_* \Omega_{\mathcal{X}/S}^q \right)_+, h_{L^2} \right) \\ & \quad + c_1(R^{q-1} f_* \Omega_{\mathcal{X}^\iota/S}^{p-1}, h_{L^2}) - c_1(R^{p-1} f_* \Omega_{\mathcal{X}^\iota/S}^{q-1}, h_{L^2}). \end{aligned}$$

Proof. Let $\alpha_1, \dots, \alpha_r$ be a local holomorphic frame of $\left(R^q f_* \Omega_{\mathcal{X}/S}^p \right)_+ \oplus R^{q-1} f_* \Omega_{\mathcal{X}^\iota/S}^{p-1}$ and let $\beta_1, \dots, \beta_r$ be a local holomorphic frame of $\left(R^p f_* \Omega_{\mathcal{X}/S}^q \right)_+ \oplus R^{p-1} f_* \Omega_{\mathcal{X}^\iota/S}^{q-1}$. We define smooth functions $\tilde{c}_1(\phi)$ and $\tilde{c}_1(\psi)$ by

$$\tilde{c}_1(\phi) = \log \frac{\det(\langle \phi(\alpha_i), \phi(\alpha_j) \rangle_{L^2})}{\det(\langle \alpha_i, \alpha_j \rangle_{L^2})} \quad \text{and} \quad \tilde{c}_1(\psi) = \log \frac{\det(\langle \phi(\beta_i), \phi(\beta_j) \rangle_{L^2})}{\det(\langle \beta_i, \beta_j \rangle_{L^2})}. \quad (7.1.12)$$

Then $\tilde{c}_1(\phi)$ and $\tilde{c}_1(\psi)$ are independent of the choices of holomorphic frames $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_r$ and are globally defined on S . Note that $\tilde{c}_1(\phi)$ and $\tilde{c}_1(\psi)$ are the Bott-Chern secondary forms such that

$$\begin{aligned} & c_1(R^q h_* \Omega_{\mathcal{X}/S}^p, h_{L^2}) - c_1 \left(\left(R^q f_* \Omega_{\mathcal{X}/S}^p \right)_+, h_{L^2} \right) - c_1(R^{q-1} f_* \Omega_{\mathcal{X}^\iota/S}^{p-1}, h_{L^2}) = -dd^c \tilde{c}_1(\phi), \\ & c_1(R^p h_* \Omega_{\mathcal{X}/S}^q, h_{L^2}) - c_1 \left(\left(R^p f_* \Omega_{\mathcal{X}/S}^q \right)_+, h_{L^2} \right) - c_1(R^{p-1} f_* \Omega_{\mathcal{X}^\iota/S}^{q-1}, h_{L^2}) = -dd^c \tilde{c}_1(\psi). \end{aligned} \quad (7.1.13)$$

Note that $c(\alpha_1), \dots, c(\alpha_r)$ are local C^∞ sections of $\left(R^p f_* \Omega_{\mathcal{X}/S}^q\right)_+ \oplus R^{p-1} f_* \Omega_{\mathcal{X}'/S}^{q-1}$ and are not holomorphic sections. Similarly, $c(\phi(\alpha_1)), \dots, c(\phi(\alpha_r))$ are local C^∞ sections of $R^p h_* \Omega_{\mathcal{X}/S}^q$ and are not holomorphic sections. Let $C = (c_{ij})$ be a $r \times r$ -matrix defined by

$$c(\alpha_i) = \sum_{j=1}^r c_{ij} \beta_j. \quad (7.1.14)$$

Then c_{ij} are C^∞ functions. By the above commutative diagram (7.1.11), we have

$$c(\phi(\alpha_i)) = \psi(c(\alpha_i)) = \sum_{j=1}^r c_{ij} \psi(\beta_j). \quad (7.1.15)$$

By (7.1.14), we have

$${}^t(\langle \alpha_i, \alpha_j \rangle_{L^2}) = (\overline{\langle \alpha_i, \alpha_j \rangle_{L^2}}) = (\langle c(\alpha_i), c(\alpha_j) \rangle_{L^2}) = C \cdot (\langle \beta_i, \beta_j \rangle_{L^2}) \cdot {}^t \overline{C}.$$

Therefore

$$\det(\langle \alpha_i, \alpha_j \rangle_{L^2}) = |\det C|^2 \cdot \det(\langle \beta_i, \beta_j \rangle_{L^2}). \quad (7.1.16)$$

Similarly, by (7.1.15), we have

$$\det(\langle \phi(\alpha_i), \phi(\alpha_j) \rangle_{L^2}) = |\det C|^2 \cdot \det(\langle \psi(\beta_i), \psi(\beta_j) \rangle_{L^2}). \quad (7.1.17)$$

By (7.1.12), (7.1.16) and (7.1.17), we have

$$\begin{aligned} \tilde{c}_1(\phi) &= \log \frac{\det(\langle \phi(\alpha_i), \phi(\alpha_j) \rangle_{L^2})}{\det(\langle \alpha_i, \alpha_j \rangle_{L^2})} \\ &= \log \frac{\det(\langle \phi(\beta_i), \phi(\beta_j) \rangle_{L^2})}{\det(\langle \beta_i, \beta_j \rangle_{L^2})} \\ &= \tilde{c}_1(\psi). \end{aligned} \quad (7.1.18)$$

By (7.1.13) and (7.1.18), we have

$$\begin{aligned} &c_1(R^q h_* \Omega_{\mathcal{X}/S}^p, h_{L^2}) - c_1 \left(\left(R^q f_* \Omega_{\mathcal{X}/S}^p \right)_+, h_{L^2} \right) - c_1(R^{q-1} f_* \Omega_{\mathcal{X}'/S}^{p-1}, h_{L^2}) \\ &= -dd^c \tilde{c}_1(\phi) \\ &= -dd^c \tilde{c}_1(\psi) \\ &= c_1(R^p h_* \Omega_{\mathcal{X}/S}^q, h_{L^2}) - c_1 \left(\left(R^p f_* \Omega_{\mathcal{X}/S}^q \right)_+, h_{L^2} \right) - c_1(R^{p-1} f_* \Omega_{\mathcal{X}'/S}^{q-1}, h_{L^2}), \end{aligned}$$

which completes the proof. $\square$

Lemma 7.1.5. *The following identities hold:*

$$\begin{aligned} c_1(R^1 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) &= c_1(R^2 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}), \\ c_1(R^2 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) &= \frac{1}{2} \{c_1(R^1 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) - c_1(f_* \Omega_{\mathcal{X}^\iota/S}^1, h_{L^2})\} + \frac{1}{2} c_1(R^3 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}). \end{aligned}$$

Proof. By (7.1.3), the first identity holds. Since $R^3 h_* \mathbb{C} \otimes \mathcal{O}_S$ is endowed with the Hodge filtration

$$R^3 h_* \mathbb{C} \otimes \mathcal{O}_S = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \mathcal{F}^2 \supset \mathcal{F}^3 \supset \mathcal{F}^4 = 0$$

with

$$\mathcal{F}^p / \mathcal{F}^{p+1} = R^{3-p} h_* \Omega_{\mathcal{X}/S}^p,$$

we deduce from (7.1.3) that

$$\mathcal{F}^1 = \mathcal{F}^0 = R^3 h_* \mathbb{C} \otimes \mathcal{O}_S, \quad \mathcal{F}^3 = \mathcal{F}^4 = 0. \quad (7.1.19)$$

By the short exact sequence

$$0 \rightarrow \mathcal{F}^{p+1} \rightarrow \mathcal{F}^p \rightarrow R^{3-p} h_* \Omega_{\mathcal{X}/S}^p \rightarrow 0,$$

we have

$$\begin{aligned} c_1(\mathcal{F}^1, h_{L^2}) - c_1(\mathcal{F}^2, h_{L^2}) &= c_1(R^2 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}), \\ c_1(\mathcal{F}^2, h_{L^2}) - c_1(\mathcal{F}^3, h_{L^2}) &= c_1(R^1 h_* \Omega_{\mathcal{X}/S}^2, h_{L^2}). \end{aligned} \quad (7.1.20)$$

By (7.1.19) and (7.1.20), we have

$$c_1(R^3 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) = c_1(R^2 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) + c_1(R^1 h_* \Omega_{\mathcal{X}/S}^2, h_{L^2}). \quad (7.1.21)$$

Since the fibers of $f : \mathcal{X} \rightarrow S$ are manifolds of $K3^{[2]}$ -type, we have

$$R^q f_* \Omega_{\mathcal{X}/S}^p = 0 \quad (7.1.22)$$

for p, q with $p + q = 1, 3, 5$ or 7 . Then we get

$$\begin{aligned} &c_1(R^2 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) \\ &= \frac{1}{2} \{c_1(R^2 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) - c_1(R^1 h_* \Omega_{\mathcal{X}/S}^2, h_{L^2})\} + \frac{1}{2} c_1(R^3 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) \\ &= \frac{1}{2} \{c_1(R^2 f_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) - c_1(R^1 f_* \Omega_{\mathcal{X}/S}^2, h_{L^2})\} \\ &\quad + \frac{1}{2} \{c_1(R^1 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) - c_1(f_* \Omega_{\mathcal{X}^\iota/S}^1, h_{L^2})\} + \frac{1}{2} c_1(R^3 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) \\ &= \frac{1}{2} \{c_1(R^1 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) - c_1(f_* \Omega_{\mathcal{X}^\iota/S}^1, h_{L^2})\} + \frac{1}{2} c_1(R^3 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}), \end{aligned}$$

where the first equality follows from (7.1.21), the second equality follows from Lemma 7.1.4, and the last equality follows from (7.1.22). This completes the proof. $\square$

Let $h_{X,0}$ be an ι -invariant Ricci-flat Kähler metric on X with Kähler form $\omega_{X,0}$. It is locally written as

$$\omega_{X,0} = \frac{i}{2} \sum_{j,k} h_{X,0} \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k} \right) dz^j \wedge d\bar{z}^k.$$

The Riemannian metric associated with $h_{X,0}$ is denoted by g . The hermitian metric on $\wedge^{p,q} T^*X$ attached to the Ricci-flat Kähler metric is also denoted by $h_{X,0}$.

Let I be the complex structure of X . Since (X, g) is hyperkähler, there are complex structures J and K of X such that (X, I, g) , (X, J, g) and (X, K, g) are manifolds of $K3^{[2]}$ -type and $IJ = -JI = K$. The Kähler forms with respect to J and K are given by

$$\omega_J = g(-, J(-)), \quad \text{and} \quad \omega_K = g(-, K(-)),$$

respectively. Set $\sigma_I = \omega_J + i\omega_K$. This is a holomorphic 2-form on X (cf. [36, §23]). Since $H^0(X, \Omega_X^2) = \mathbb{C}\eta$, there exists a complex number $\lambda \in \mathbb{C}$ such that $\eta = \frac{\lambda}{2}\sigma_I$. Note that the L^2 -norm of η^2 is given by

$$\|\eta^2\|_{L^2}^2 = \int_X h_{X,0}(\eta^2, \eta^2) \frac{\omega_{X,0}^4}{4!} = \int_X \eta^2 \wedge \bar{\eta}^2,$$

and the volume of $(X, \omega_{X,0})$ is defined by

$$\text{Vol}(X, \omega_{X,0}) = \int_X \frac{\omega_{X,0}^4}{4!}.$$

Since $\omega_{X,0}$ is Ricci-flat, it follows from [36, Corollary 23.9] that

$$|\lambda|^2 = \frac{1}{2} \left(\frac{\eta^2 \wedge \bar{\eta}^2}{\omega_{X,0}^4/4!} \right)^{\frac{1}{2}} = \frac{1}{2} \left(\frac{\|\eta^2\|_{L^2}^2}{\text{Vol}(X, \omega_{X,0})} \right)^{\frac{1}{2}}. \quad (7.1.23)$$

Similarly, there exists a complex number $\mu \in \mathbb{C}$ such that $\theta = \frac{\mu}{2}\bar{\sigma}_I$ in $H^2(X, \mathcal{O}_X)$. We identify the cohomology class θ with its harmonic representative. The L^2 -norm of θ^2 is given by

$$\|\theta^2\|_{L^2}^2 = \int_X h_{X,0}(\theta^2, \theta^2) \frac{\omega_{X,0}^4}{4!} = \int_X \theta^2 \wedge \bar{\theta}^2,$$

and we have

$$|\mu|^2 = \frac{1}{2} \left(\frac{\|\theta^2\|_{L^2}^2}{\text{Vol}(X, \omega_{X,0})} \right)^{\frac{1}{2}}. \quad (7.1.24)$$

Lemma 7.1.6. *For any $\alpha \in A^{1,1}(X)$, the following identity holds:*

$$h_{X,0}(\eta \wedge \alpha, \eta \wedge \alpha) = |\lambda|^2 h_{X,0}(\alpha, \alpha).$$

Proof. Fix $p \in X$. Since (X, g) is hyperkähler, the real tangent space $T_{p,\mathbb{R}}X$ at p is equipped with the structure of a quaternionic hermitian vector space (see [36, §23.2]). Therefore, there exist two tangent vectors $e, f \in T_{p,\mathbb{R}}X$ such that

$$e, Ie, Je, Ke, f, If, Jf, Kf$$

form an orthonormal basis of $(T_{p,\mathbb{R}}X, g)$. Set

$$v_1 = \frac{e - iIe}{2}, \quad v_2 = \frac{-Ke - iJe}{2}, \quad v_3 = \frac{f - iIf}{2}, \quad v_4 = \frac{-Kf - iJf}{2}.$$

Then v_1, v_2, v_3, v_4 are of type $(1, 0)$ with respect to the complex structure I and form a $\mathbb{C}$ -basis of the holomorphic tangent space T_pX at p . Moreover, we have

$$h_{X,0}(v_i, v_j) = \frac{1}{2}\delta_{ij}.$$

Let v^1, v^2, v^3, v^4 be the dual basis. Then we have

$$h_{X,0}(v^i, v^j) = 2\delta_{ij}, \quad \text{and} \quad \sigma_I = iv^1 \wedge v^2 + iv^3 \wedge v^4.$$

Let $\alpha = \frac{1}{2} \sum_{i,j} \alpha_{ij} v^i \wedge \bar{v}^j \in \wedge^{1,1} T_p^* X$. Then we have

$$h_{X,0}(\sigma_I \wedge \alpha, \sigma_I \wedge \alpha) = 4 \sum_{i,j} |\alpha_{ij}|^2 = 4h_{X,0}(\alpha, \alpha),$$

which completes the proof. □

Lemma 7.1.7. *The following identities hold:*

$$\begin{aligned} & c_1 \left((R^1 f_* \Omega_{\mathcal{X}/S}^1)_-, h_{L^2} \right) - c_1 \left((R^3 f_* \Omega_{\mathcal{X}/S}^1)_+, h_{L^2} \right) \\ &= -\frac{21-t}{4} c_1(R^4 f_* \mathcal{O}_{\mathcal{X}}, h_{L^2}) - \frac{21-t}{4} dd^c \log \text{Vol}(\mathcal{X}/S, \omega), \\ & c_1 \left((R^1 f_* \Omega_{\mathcal{X}/S}^1)_-, h_{L^2} \right) - c_1 \left((R^1 f_* \Omega_{\mathcal{X}/S}^3)_+, h_{L^2} \right) \\ &= -\frac{21-t}{4} c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}) - \frac{21-t}{4} dd^c \log \text{Vol}(\mathcal{X}/S, \omega). \end{aligned}$$

Proof. The first identity is proved in Lemma 5.3.5. We prove the second identity in the same manner as in Lemma 5.3.5.

Let X be a fiber of $f : \mathcal{X} \rightarrow S$. Since X is a manifold of $K3^{[2]}$ -type, it follows from [36, Proposition 24.1] and [37, Main Theorem] that the cup product induces a μ_2 -equivariant isomorphism

$$\text{Sym}^2 H^2(X, \mathbb{C}) \cong H^4(X, \mathbb{C}).$$

Since the involution $\iota : X \rightarrow X$ is antisymplectic, we have the canonical isomorphism

$$H^1(X, \Omega_X^1)_- \otimes H^0(X, \Omega_X^2) \cong H^1(X, \Omega_X^3)_+ \quad \alpha \otimes \eta \rightarrow \alpha \wedge \eta.$$

Therefore, we have an isomorphism of holomorphic vector bundles

$$(R^1 f_* \Omega_{\mathcal{X}/S}^1)_- \otimes f_* \Omega_{\mathcal{X}/S}^2 \cong (R^1 f_* \Omega_{\mathcal{X}/S}^3)_+. \quad (7.1.25)$$

Set $N = \frac{2l-t}{2}$. Let $s_1, \dots, s_N$ be a local holomorphic frame of $(R^1 f_* \Omega_{\mathcal{X}/S}^1)_-$. Then $s_1 \wedge \dots \wedge s_N$ is a nowhere vanishing local holomorphic section of $\det (R^1 f_* \Omega_{\mathcal{X}/S}^1)_-$. Let η be a nowhere vanishing local holomorphic section of $f_* \Omega_{\mathcal{X}/S}^2$. By (7.1.25), $s_1 \wedge \eta, \dots, s_N \wedge \eta$ are local holomorphic frame of $(R^1 f_* \Omega_{\mathcal{X}/S}^3)_+$ and $(s_1 \wedge \eta) \wedge \dots \wedge (s_N \wedge \eta)$ is a nowhere vanishing local holomorphic section of $\det (R^1 f_* \Omega_{\mathcal{X}/S}^3)_+$.

It suffices to show that

$$\begin{aligned} & -\log \|(s_1 \wedge \eta) \wedge \dots \wedge (s_N \wedge \eta)\|_{L^2}^2 \\ &= -\frac{N}{2} \log \|\eta\|_{L^2}^2 - \log \|s_1 \wedge \dots \wedge s_N\|_{L^2}^2 + \log \left(2^N \text{Vol}(\mathcal{X}/S, \omega_{\mathcal{X}/S})^{\frac{N}{2}} \right). \end{aligned} \quad (7.1.26)$$

We may assume that S consists of one point, and we set $X = \mathcal{X}$.

By Lemma 5.4.2, the L^2 -metrics on $H^1(X, \Omega_X^p)$ ($p = 1, 3$) and $H^0(X, \Omega_X^2)$ depend only on the choice of the Kähler class $[\omega_X]$ and are independent of the Kähler form ω_X itself. Therefore we may assume that $\omega_X = \omega_{X,0}$ is Ricci-flat.

By Lemma 7.1.6, we have

$$h_{X,0}(\eta \wedge \alpha, \eta \wedge \alpha) = |\lambda|^2 h_{X,0}(\alpha, \alpha)$$

for each $\alpha \in A^{1,1}(X)$. By integrating both sides, we have

$$\|\eta \wedge \alpha\|_{L^2}^2 = |\lambda|^2 \|\alpha\|_{L^2}^2.$$

Therefore we have an isometry

$$(\det H^1(X, \Omega_X^1)_-, |\lambda|^{2N} \det h_{L^2}) \cong (\det H^1(X, \Omega_X^3)_+, \det h_{L^2}).$$

By the formula (7.1.23), we obtain the formula (7.1.26). $\square$

Lemma 7.1.8. *The following identity holds:*

$$c_1(R^4 h_* \mathcal{O}_{\mathcal{X}}, h_{L^2}) = -c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}).$$

Proof. Recall that $\tau : \widetilde{\mathcal{X}} \rightarrow \mathcal{X}$ is the blowup of $\mathcal{X}$ along $\mathcal{X}^\iota$. Let $\pi : \widetilde{\mathcal{X}} \rightarrow \mathcal{Z} = \widetilde{\mathcal{X}}/\tilde{\iota}$ be the natural projection. Set $\tilde{f} = f \circ \tau : \widetilde{\mathcal{X}} \rightarrow S$. For $s \in S$, the fiber of f , $\tilde{f}$ and h are denoted by X_s , $\tilde{X}_s$ and Z_s , respectively. Since $h^{0,4}(X_s) = h^{0,4}(\tilde{X}_s) = h^{0,4}(Z_s) = 1$, $R^4 f_* \mathcal{O}_{\mathcal{X}}$, $R^4 \tilde{f}_* \mathcal{O}_{\widetilde{\mathcal{X}}}$ and $R^4 h_* \mathcal{O}_{\mathcal{Z}}$ are invertible sheaves.

Let $\alpha \in H^4(X_s, \mathcal{O}_{X_s})$. Since ι_s is an antisymplectic involution on X_s , α is ι_s -invariant and $\tau^* \alpha \in H^4(\tilde{X}_s, \mathcal{O}_{\tilde{X}_s})$ is $\tilde{\iota}_s$ -invariant. Therefore, there exists a unique element $\beta \in H^4(Z_s, \mathcal{O}_{Z_s})$ such that

$$\tau^* \alpha = \pi^* \beta. \quad (7.1.27)$$

The correspondence from α to β induces a natural isomorphism of holomorphic line bundles on S

$$R^4 f_* \mathcal{O}_{\mathcal{X}} \cong R^4 h_* \mathcal{O}_{\mathcal{Z}}. \quad (7.1.28)$$

Since $\pi : \widetilde{\mathcal{X}} \rightarrow \mathcal{Z}$ is a double cover, we have

$$\|\beta\|_{L^2}^2 = \int_{Z_s} \beta \wedge \bar{\beta} = \frac{1}{2} \int_{\tilde{X}_s} \pi^* \beta \wedge \overline{\pi^* \beta} = \frac{1}{2} \int_{X_s} \alpha \wedge \bar{\alpha} = \frac{1}{2} \|\alpha\|_{L^2}^2, \quad (7.1.29)$$

where the third equality holds by (7.1.27). By (7.1.28), (7.1.29) and the Serre duality (Lemma 5.4.1), we have

$$c_1(R^4 h_* \mathcal{O}_{\mathcal{Z}}, h_{L^2}) = c_1(R^4 f_* \mathcal{O}_{\mathcal{X}}, h_{L^2}) = -c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}),$$

which completes the proof. $\square$

Lemma 7.1.9. *The following identity holds:*

$$c_1(R^3 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) = -\frac{21-t}{4} c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}) - c_1(f_* \Omega_{\mathcal{X}^\iota/S}^2, h_{L^2}).$$

Proof. By the Serre duality (Lemma 5.4.1) and Lemma 7.1.4, we have

$$\begin{aligned} & c_1(R^3 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) \\ &= \frac{1}{2} \{ c_1(R^3 h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) - c_1(R^1 h_* \Omega_{\mathcal{X}/S}^3, h_{L^2}) \} \\ &= \frac{1}{2} \left\{ c_1 \left((R^3 f_* \Omega_{\mathcal{X}/S}^1)_+, h_{L^2} \right) - c_1 \left((R^1 f_* \Omega_{\mathcal{X}/S}^3)_+, h_{L^2} \right) \right\} \\ & \quad + \frac{1}{2} \{ c_1(R^2 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) - c_1(f_* \Omega_{\mathcal{X}^\iota/S}^2, h_{L^2}) \}. \end{aligned} \quad (7.1.30)$$

By Lemma 7.1.7 and the Serre duality (Lemma 5.4.1), we have

$$c_1 \left((R^3 f_* \Omega_{\mathcal{X}/S}^1)_+, h_{L^2} \right) - c_1 \left((R^1 f_* \Omega_{\mathcal{X}/S}^3)_+, h_{L^2} \right) = -\frac{21-t}{2} c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}). \quad (7.1.31)$$

By the Serre duality (Lemma 5.4.1), we have

$$c_1(R^2 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) = -c_1(f_* \Omega_{\mathcal{X}^\iota/S}^2, h_{L^2}). \quad (7.1.32)$$

Substituting (7.1.31) and (7.1.32) into (7.1.30), we obtain the desired formula. $\square$

Lemma 7.1.10. *The following identity holds:*

$$2c_1(R^2 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) - c_1(R^3 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) = -4c_1(h_* \mathcal{O}_{\mathcal{Z}}, h_{L^2}).$$

Proof. By [25, proof of Proposition 5.9], we have

$$\sum_{k=0}^8 (-1)^k k c_1(R^k h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) = 0. \quad (7.1.33)$$

By (7.1.3), we have

$$c_1(R^1 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) = c_1(R^7 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) = 0. \quad (7.1.34)$$

By the Poincaré duality, we have

$$\begin{aligned} c_1(R^4 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) &= 0, \\ c_1(R^5 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) &= -c_1(R^3 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}), \\ c_1(R^6 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) &= -c_1(R^2 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}), \\ c_1(R^8 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) &= -c_1(h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) = -c_1(h_* \mathcal{O}_{\mathcal{X}}, h_{L^2}). \end{aligned} \quad (7.1.35)$$

Substituting (7.1.34), (7.1.35) into (7.1.33), we obtain the desired formula. $\square$

Recall that $\omega_{H^*}(\mathcal{X}^\iota/S)$ is defined by

$$\omega_{H^*}(\mathcal{X}^\iota/S) = c_1(f_* \Omega_{\mathcal{X}^\iota/S}^1, h_{L^2}) - c_1(R^1 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) - 2c_1(f_* K_{\mathcal{X}^\iota/S}, h_{L^2}). \quad (7.1.36)$$

Proposition 7.1.11. *The following identity holds:*

$$\sum_{k=0}^8 (-1)^k \omega_{H^k} = \frac{29-t}{2} c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}) - \omega_{H^*}(\mathcal{X}^\iota/S).$$

Proof. By Lemmas 7.1.5 and 7.1.9, we have

$$\begin{aligned} & 2 \sum_{q=1}^3 (-1)^q c_1(R^q h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) \\ &= \frac{21-t}{2} c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}) \\ & \quad - \{c_1(f_* \Omega_{\mathcal{X}^\iota/S}^1, h_{L^2}) - c_1(R^1 f_* \mathcal{O}_{\mathcal{X}^\iota}, h_{L^2}) - 2c_1(f_* K_{\mathcal{X}^\iota/S}, h_{L^2})\} \\ & \quad - \{2c_1(R^2 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2}) - c_1(R^3 h_* \mathbb{C} \otimes \mathcal{O}_S, h_{L^2})\}. \end{aligned} \quad (7.1.37)$$

Substituting (7.1.36) and Lemma 7.1.10 into (7.1.37), we have

$$2 \sum_{q=1}^3 (-1)^q c_1(R^q h_* \Omega_{\mathcal{X}/S}^1, h_{L^2}) = \frac{21-t}{2} c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}) - \omega_{H^*}(\mathcal{X}^\iota/S) + 4c_1(h_* \mathcal{O}_{\mathcal{X}}, h_{L^2}). \quad (7.1.38)$$

By (7.1.38) and by Lemmas 7.1.3 and 7.1.8, we have

$$\sum_{k=0}^8 (-1)^k \omega_{H^k} = \frac{29-t}{2} c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}) - \omega_{H^*}(\mathcal{X}^\iota/S),$$

which completes the proof. $\square$

For a compact Kähler manifold M , we set

$$h^{p,q}(M) = \dim H^q(M, \Omega_M^p), \quad b_k(M) = \dim H^k(M, \mathbb{C}), \quad \chi(M) = \sum_{k \geq 0} (-1)^k b_k(M).$$

Let X and Z be the fibers of $f : \mathcal{X} \rightarrow S$ and $h : \mathcal{Z} \rightarrow S$, respectively. Let $H^q(X, \Omega_X^p)_\pm$ be the (± 1) -eigenspaces of $H^q(X, \Omega_X^p)$ with respect to the μ_2 -action induced from $\iota : X \rightarrow X$. Set

$$h^{p,q}(X)_\pm = \dim H^q(X, \Omega_X^p)_\pm, \quad b_k(X)_\pm = \sum_{p+q=k} h^{p,q}(X)_\pm.$$

Recall that $t = \text{Tr}(\iota^*|_{H^{1,1}(X)})$. Therefore

$$h^{1,1}(X)_+ - h^{1,1}(X)_- = t, \quad h^{1,1}(X)_+ + h^{1,1}(X)_- = h^{1,1}(X) = 21,$$

and

$$h^{1,1}(X)_+ = \frac{21+t}{2}, \quad h^{1,1}(X)_- = \frac{21-t}{2}. \quad (7.1.39)$$

By [24, proof of Theorem 4.1], we have

$$H^q(X, \Omega_X^p)_+ \oplus H^{q-1}(X^\iota, \Omega_{X^\iota}^{p-1}) \cong H^q(Z, \Omega_Z^p).$$

Since

$$b_k(Z) = \sum_{p+q=k} h^{p,q}(Z) = \sum_{p+q=k} (h^{p,q}(X)_+ + h^{p-1,q-1}(X^\iota)) = b_k(X)_+ + b_{k-2}(X^\iota),$$

we have

$$\chi(Z) = \sum_{k=0}^8 (-1)^k b_k(X)_+ + \chi(X^\iota). \quad (7.1.40)$$

Proposition 7.1.12. *The following identity holds:*

$$\chi(Z) = \frac{3t^2 + 717}{4}.$$

Proof. By [6, Theorem 2], we have

$$\chi(X^\iota) = \frac{t^2 + 23}{2}. \quad (7.1.41)$$

By [37, Main Theorem], we have $b_k(X) = 0$ for $k = 1, 3, 5, 7$, and

$$b_k(X)_+ = 0 \quad (k = 1, 3, 5, 7). \quad (7.1.42)$$

Recall that ι is antisymplectic. By (7.1.39), we have

$$\begin{aligned} b_0(X)_+ &= b_8(X)_+ = 1, \\ b_2(X)_+ &= h^{1,1}(X)_+ = \frac{21+t}{2} = b_6(X)_+. \end{aligned} \quad (7.1.43)$$

On the other hand,

$$b_2(X)_- = h^{2,0}(X) + h^{1,1}(X)_- + h^{0,2}(X) = \frac{25-t}{2}. \quad (7.1.44)$$

By [36, Proposition 24.1], the cup product induces a μ_2 -equivariant isomorphism

$$\mathrm{Sym}^2 H^2(X, \mathbb{C}) \cong H^4(X, \mathbb{C}).$$

Therefore

$$H^4(X, \mathbb{C})_+ \cong \mathrm{Sym}^2 H^2(X, \mathbb{C})_+ \oplus \mathrm{Sym}^2 H^2(X, \mathbb{C})_-,$$

and, by (7.1.43), (7.1.44), we have

$$\begin{aligned} b_4(X)_+ &= \binom{b_2(X)_+}{2} + b_2(X)_+ + \binom{b_2(X)_-}{2} + b_2(X)_-, \\ &= \frac{t^2 - 4t + 579}{4}. \end{aligned} \quad (7.1.45)$$

By (7.1.42), (7.1.43) and (7.1.45), we have

$$\sum_{k=0}^8 (-1)^k b_k(X)_+ = \frac{t^2 + 671}{4}. \quad (7.1.46)$$

Substituting (7.1.41), (7.1.46) into (7.1.40), we have

$$\chi(Z) = \frac{3t^2 + 717}{4}.$$

This completes the proof. $\square$

Proof of Theorem 7.1.2. By Lemma 7.1.8 and by Propositions 7.1.11 and 7.1.12, we have

$$\begin{aligned} &\sum_{k=0}^8 (-1)^k \omega_{H^k} - \frac{\chi}{12} \omega_{WP} \\ &= \frac{29-t}{2} c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}) - \omega_{H^*(\mathcal{X}^\iota/S)} - \frac{t^2 + 239}{16} c_1(h_* K_{\mathcal{X}/S}, h_{L^2}) \\ &= -\frac{(t+1)(t+7)}{16} c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}) - \omega_{H^*(\mathcal{X}^\iota/S)}. \end{aligned} \quad (7.1.47)$$

By (7.1.47) and by Theorem 5.3.11, we have

$$dd^c \log \tau_{M,\mathcal{K},\mathcal{X}/S} = -\frac{(t+1)(t+7)}{16} c_1(f_* \Omega_{\mathcal{X}/S}^4, h_{L^2}) - \omega_{H^*(\mathcal{X}^\iota/S)} = \sum_{k=0}^8 (-1)^k \omega_{H^k} - \frac{\chi}{12} \omega_{WP}.$$

This is the desired result. $\square$

7.1.3 The BCOV invariant of the Calabi-Yau fourfolds Z

Let M_0 be a primitive hyperbolic 2-elementary sublattice of L_{K3} . In this subsection, we assume that

- (1) $\text{rk}(M_0) \leq 17$.
- (2) $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible.

Here $M_0^\perp$ denotes the orthogonal complement of M_0 in L_{K3} . As before, we define a sublattice $\widetilde{M}_0$ of $L_2 = L_{K3} \oplus \mathbb{Z}e$ by

$$\widetilde{M}_0 = M_0 \oplus \mathbb{Z}e.$$

Then $\widetilde{M}_0$ is an admissible sublattice of L_2 . Let $\mathcal{K} \subset \text{KT}(\widetilde{M}_0)$ be a natural chamber.

Let (Y, σ) be a 2-elementary K3 surface of type M_0 . Then $(Y^{[2]}, \sigma^{[2]})$ is a manifold of $K3^{[2]}$ -type with involution of type $(\widetilde{M}_0, \mathcal{K})$. We set $(X_Y, \iota_Y) = (Y^{[2]}, \sigma^{[2]})$. The blowup of X_Y along the fixed locus $X_Y^{\iota_Y}$ is denoted by $\widetilde{X}_Y$, and the involution of $\widetilde{X}_Y$ induced from ι_Y is denoted by $\tilde{\iota}_Y$. The Calabi-Yau 4-fold of Camere-Garbagnati-Mongardi is denoted by

$$Z_Y = \widetilde{X}_Y / \tilde{\iota}_Y.$$

Recall that $\mathcal{M}_{M_0}^\circ = \mathcal{M}_{M_0} \setminus \bar{\mathcal{D}}_{M_0^\perp}$ is a coarse moduli space of 2-elementary K3 surfaces of type M_0 . We define a function u on $\mathcal{M}_{M_0}^\circ$ by

$$u(Y, \sigma) = \tau_{BCOV}(Z_Y) \cdot \tau_{\widetilde{M}_0, \mathcal{K}}(X_Y, \iota_Y)^{-1} \quad ((Y, \sigma) \in \mathcal{M}_{M_0}^\circ).$$

Lemma 7.1.13. *The function $\log u$ on $\mathcal{M}_{M_0}^\circ$ is pluriharmonic.*

Proof. Let $\Pi_{M_0^\perp} : \Omega_{M_0^\perp} \rightarrow \mathcal{M}_{M_0}$ be the natural projection. It suffices to show that $\Pi_{M_0^\perp}^* u$ is a smooth function on $\Omega_{M_0^\perp} \setminus \mathcal{D}_{M_0^\perp}$ and $\log \Pi_{M_0^\perp}^* u$ is a pluriharmonic function on $\Omega_{M_0^\perp} \setminus \mathcal{D}_{M_0^\perp}$.

Let $x \in \Omega_{M_0^\perp} \setminus \mathcal{D}_{M_0^\perp}$ and set $p = \Pi_{M_0^\perp}(x)$. Let $(Y, \sigma) \in \bar{\pi}_{M_0}^{-1}(p)$. By the local Torelli theorem, we may assume that the Kuranishi space $S := \text{Def}(Y, \sigma)$ is an open neighborhood of x in $\Omega_{M_0^\perp}$. Its Kuranishi family is denoted by $g : (\mathcal{Y}, \sigma) \rightarrow S$. Let $f : (\mathcal{X}, \iota) \rightarrow S$ be the family of $K3^{[2]}$ -type manifolds with involution of type $(\widetilde{M}_0, \mathcal{K})$ such that

$$(X_s, \iota_s) = (X_{Y_s}, \iota_{Y_s})$$

for each $s \in S$. Let $h : \mathcal{Z} \rightarrow S$ be the family of Calabi-Yau 4-folds constructed in the same manner as in §7.1.1.

Then $(\Pi_{M_0^\perp}^* u)|_S = \tau_{BCOV, \mathcal{Z}/S} \cdot \tau_{\widetilde{M}_0, \mathcal{K}, \mathcal{Z}/S}^{-1}$. Since both $\tau_{BCOV, \mathcal{Z}/S}$ and $\tau_{\widetilde{M}_0, \mathcal{K}, \mathcal{Z}/S}$ are smooth functions on S , so is $(\Pi_{M_0^\perp}^* u)|_S$. By (7.1.1) and Theorem 7.1.2, $\log(\Pi_{M_0^\perp}^* u)|_S$ ($= \log \tau_{BCOV, \mathcal{Z}/S} - \log \tau_{\widetilde{M}_0, \mathcal{K}, \mathcal{Z}/S}$) is a pluriharmonic function on S . $\square$

Let C be an irreducible projective curve on $\mathcal{M}_{M_0}^*$ such that C satisfies Assumption 6.2.5. Namely, C is not contained in $\bar{\mathcal{Q}}_{M^\perp} \cup (\mathcal{M}_{M_0}^* \setminus \mathcal{M}_{M_0})$. Let $p \in C \cap \bar{\mathcal{Q}}_{M_0^\perp}$ and assume that p is a smooth point of C . By Proposition 6.2.6, there exists a smooth projective curve B , a morphism $\varphi : B \rightarrow \mathcal{M}_{M_0}^*$, a smooth projective variety $\mathcal{X}$ of complex dimension 5, an involution $\iota : \mathcal{X} \rightarrow \mathcal{X}$, and a surjective morphism $f : \mathcal{X} \rightarrow B$ such that

- (1) $\varphi(B) = C$.
- (2) $f \circ \iota = f$.
- (3) There exists a Zariski open subset $B^\circ \neq \emptyset$ of B such that $\varphi(B^\circ) \subset \mathcal{M}_{M_0}^\circ$ and for each $b \in B^\circ$, the fiber $(f^{-1}(b), \iota|_{f^{-1}(b)})$ is a manifold of $K3^{[2]}$ -type with antisymplectic involution of type $(\widetilde{M}_0, \mathcal{K})$ with period $\varphi(b)$.
- (4) $f : \mathcal{X} \rightarrow B$ is a normal crossing degeneration. Namely, its singular fibers are simple normal crossing divisor on $\mathcal{X}$.

Let $\Sigma_f = \{s \in \mathcal{X}; df(x) = 0\}$ be the critical locus of f and let $\Delta_f = f(\Sigma_f)$ be the discriminant locus of f . The fixed locus of ι is denoted by $\mathcal{X}^\iota$. Write

$$\mathcal{X}^\iota = \mathcal{X}_H^\iota \sqcup \mathcal{X}_V^\iota,$$

where any connected component of $\mathcal{X}_H^\iota$ is flat over B and $\mathcal{X}_V^\iota \subset f^{-1}(\Delta_f)$.

The blowup of $\mathcal{X}$ along $\mathcal{X}_H^\iota$ is denoted by $\widetilde{\mathcal{X}}$. Let $\tilde{\iota}$ be the involution of $\widetilde{\mathcal{X}}$ induced from ι . Set $\mathcal{Z} = \widetilde{\mathcal{X}}/\tilde{\iota}$ and let $h : \mathcal{Z} \rightarrow B$ be the morphism induced from $\widetilde{\mathcal{X}} \rightarrow \mathcal{X} \xrightarrow{f} B$. By [24, Theorem 3.6] and (3), $h|_{h^{-1}(B^\circ)} : h^{-1}(B^\circ) \rightarrow B^\circ$ is a family of Calabi-Yau 4-folds.

Let $0 \in B \setminus B^\circ$ and let (Δ', t) be a local coordinate on B centered at 0 such that $\Delta' \setminus \{0\} \subset B^\circ$. We define functions $\tau_{BCOV, \mathcal{Z}/B}$ and $\tau_{\widetilde{M}_0, \mathcal{K}, \mathcal{Z}/B}$ on B° by

$$\tau_{BCOV, \mathcal{Z}/B}(b) = \tau_{BCOV}(Z_b), \quad \tau_{\widetilde{M}_0, \mathcal{K}, \mathcal{Z}/B}(b) = \tau_{\widetilde{M}_0, \mathcal{K}}(X_b, \iota_b) \quad (b \in B^\circ),$$

Then $\varphi^*u = \tau_{BCOV, \mathcal{Z}/B} \cdot \tau_{\widetilde{M}_0, \mathcal{K}, \mathcal{Z}/B}^{-1}$. By Theorem 4.2.5 and Theorem 6.2.4, there exists $a_0 \in \mathbb{Q}$ such that

$$\log \varphi^*u|_{\Delta'}(t) = a_0 \log |t|^2 + O(\log \log |t|^{-1}) \quad (t \rightarrow 0). \quad (7.1.48)$$

Let (Δ, s) be a local coordinate on C centered at p such that $\Delta^* \subset \mathcal{M}_{M_0}^\circ$. Since $\varphi : B \rightarrow C$ is a finite map with $\varphi(0) = p$, there exists $\nu \in \mathbb{Z}_{>0}$ such that $\varphi^*s = t^\nu \varepsilon(t)$, where $\varepsilon(t) \in \mathbb{C}\{t\}$ is a unit. Replacing t with $t\varepsilon(t)^{\frac{1}{\nu}}$, we may assume $\varepsilon(t) = 1$. Namely, $\varphi^*s = t^\nu$. Then, by (7.1.48), we have

$$\log u|_{\Delta}(s) = a \log |s|^2 + O(\log \log |s|^{-1}) \quad (s \rightarrow 0), \quad (7.1.49)$$

where $a = a_0/\nu$.

Since $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible, it follows from (7.1.49) and Lemma 6.4.9 the equation of currents on $\mathcal{M}_{M_0}$

$$dd^c \log u = a \delta_{\bar{\mathcal{D}}_{M_0^\perp}}, \quad (7.1.50)$$

where $\delta_{\bar{\mathcal{D}}_{M_0^\perp}}$ is the Dirac delta current with support $\bar{\mathcal{D}}_{M_0^\perp}$.

Theorem 7.1.14. *If $r(M_0) \leq 17$ and $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible, then there exists a constant C_{M_0} depending only on M_0 such that*

$$\tau_{BCOV}(Z_Y) = C_{M_0} \tau_{\widetilde{M_0}, \mathcal{K}}(X_Y, \iota_Y)$$

for any 2-elementary K3 surface (Y, σ) of type M_0 .

Proof. We show that $\log u$ is a constant function on $\mathcal{M}_{M_0}^\circ$. Since $r(M_0) \leq 17$, we have $\text{codim } \mathcal{M}_{M_0}^* \setminus \mathcal{M}_{M_0} \geq 2$. Therefore, we can choose an irreducible projective curve C on $\mathcal{M}_{M_0}^*$ that satisfies the following conditions:

- (1) $C \subset \mathcal{M}_{M_0}$ (2) $C \cap \bar{\mathcal{D}}_{M_0^\perp} \neq \emptyset$ (3) $\text{Sing } C \cap \bar{\mathcal{D}}_{M_0^\perp} = \emptyset$
- (4) C intersects $\bar{\mathcal{D}}_{M_0^\perp}$ transversally.

Let $\varphi : \tilde{C} \rightarrow C$ be the normalization. Note that $\varphi^* \partial \log u$ is a meromorphic 1-form on $\tilde{C}$ and that its residue at $p \in \varphi^{-1}(C \cap \bar{\mathcal{D}}_{M_0^\perp})$ is equal to a by (7.1.50) and (4). By the residue theorem, we have

$$\sum_{p \in \varphi^{-1}(C \cap \bar{\mathcal{D}}_{M_0^\perp})} a = 0.$$

By (2), we obtain $a = 0$, and $\log u$ extends to a pluriharmonic function on $\mathcal{M}_{M_0}$. Since $r(M_0) \leq 17$, we have $\text{codim } \mathcal{M}_{M_0}^* \setminus \mathcal{M}_{M_0} \geq 2$. Therefore $\log u$ extends to a pluriharmonic function on $\mathcal{M}_{M_0}^*$ by [34, Satz 4]. Since $\mathcal{M}_{M_0}^*$ is compact, $\log u$ is constant by the maximal principle. This completes the proof. $\square$

By Fu-Zhang [30, Theorem A], the BCOV invariant of Calabi-Yau manifolds is a birational invariant. After the result of Fu-Zhang [30, Theorem A] and Theorem 7.1.14, we expect that the invariant $\tau_{M,\mathcal{K}}$ of $K3^{[2]}$ -type manifolds with involution of type $(M, \mathcal{K})$ is also a birational invariant.

Conjecture 7.1.15. Let M be an admissible sublattice of L_2 and let $\mathcal{K}_1, \mathcal{K}_2 \in \text{KT}(M)$ be Kähler-type chambers. Let (X_1, ι_1) and (X_2, ι_2) be manifolds of $K3^{[2]}$ -type with involution of type $(M, \mathcal{K}_1)$ and $(M, \mathcal{K}_2)$, respectively. If there exists a μ_2 -equivariant birational map between (X_1, ι_1) and (X_2, ι_2) , then $\tau_{M, \mathcal{K}_1}(X_1, \iota_1) = \tau_{M, \mathcal{K}_2}(X_2, \iota_2)$.

7.2 Some examples

By Theorems 6.4.12, 7.1.14 and 4.1.10, the BCOV invariant of the Calabi-Yau 4-folds of Camere-Garbagnati-Mongardi can be expressed as the Petersson norm of an explicit automorphic form. In this section, we prove that the BCOV invariant of the Calabi-Yau 4-fold Z of Camere-Garbagnati-Mongardi is expressed as the Petersson norm of a Borcherds lift if every small deformation of Z remains to be a Calabi-Yau 4-fold of Camere-Garbagnati-Mongardi.

7.2.1 The BCOV invariant of the universal covering of the Hilbert scheme of 2-points on an Enriques surface

Example 7.2.1. Consider an Enriques surface S . Its universal covering $\tilde{S}$ is a K3 surface and the covering involution $\sigma : \tilde{S} \rightarrow \tilde{S}$ is antisymplectic. By [63, Theorem 4.2.2], the σ -invariant sublattice $H^2(\tilde{S}, \mathbb{Z})^\sigma$ of $H^2(\tilde{S}, \mathbb{Z})$ is isomorphic to $M_0 := U(2) \oplus E_8(2)$ (see Section 2.1.1 for the notation). Therefore $(\tilde{S}, \sigma)$ is a 2-elementary K3 surface of type M_0 . By [66, Theorem 3.1] and [24, 6.1], $Z_{\tilde{S}} = \widetilde{S^{[2]}}$, where $\widetilde{S^{[2]}}$ is the universal covering of the Hilbert scheme of 2-points on the Enriques surface S .

By [24, Theorem 5.1 and A], we have

$$\dim H^1(Z_{\tilde{S}}, T_{Z_{\tilde{S}}}) = \dim H^3(Z_{\tilde{S}}, \Omega_{Z_{\tilde{S}}}^1) = 10.$$

Therefore the dimension of the Kuranishi space of $Z_{\tilde{S}}$ is equal to that of S and every small deformation of $Z_{\tilde{S}}$ is induced from a small deformation of S .

Let us recall the construction of the Borcherds Φ -function [20]. Set $\Lambda = U \oplus E_8(2)$. Then $M_0^\perp = U(2) \oplus \Lambda$, where the orthogonal complement is considered in L_{K3} . An element of $M_0^\perp$ is denoted by (m, n, λ) , where $m, n \in \mathbb{Z}$ and $\lambda \in \Lambda$.

Set

$$\mathcal{C}_\Lambda = \{s \in \Lambda_{\mathbb{R}}; x^2 > 0\}.$$

The tube domain $\Lambda_{\mathbb{R}} + i\mathcal{C}_\Lambda$ can be identified with the period domain $\Omega_{M_0^\perp}$ via the isomorphism

$$\Lambda_{\mathbb{R}} + i\mathcal{C}_\Lambda \rightarrow \Omega_{M_0^\perp} \quad y \mapsto \left[\left(\frac{1}{2}, -\frac{y^2}{2}, y \right) \right].$$

Set $\rho = (0, 1, 0), \rho' = (1, 0, 0) \in \Lambda$. The connected component of $\mathcal{C}_\Lambda$ whose closure contains ρ is denoted by $\mathcal{C}_\Lambda^+$.

Let $\mathbb{H}$ be the upper half plane. Set $q = e^{2\pi i \tau}$ for $\tau \in \mathbb{H}$. The Dedekind η -function is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n>0} (1 - q^n).$$

Let $\{c(n)\}_{n \geq -1}$ be the series defined by the generating function

$$\sum_{n \geq -1} c(n) q^n = \frac{\eta(2\tau)^8}{\eta(\tau)^8 \eta(4\tau)^8}.$$

Set

$$\Pi^+ = \mathbb{Z}_{>0}\rho \cup \{r \in \Lambda; (\rho, r) > 0\}.$$

We define the infinite product $\Phi(y)$ by

$$\Phi(y) = e^{2\pi i(\rho, y)} \prod_{r \in \Pi^+} (1 - e^{2\pi i(r, y)})^{(-1)^{(r, \rho - \rho')} c\left(\frac{r^2}{2}\right)}.$$

By [20, Theorem 3.2] and [21, Theorem 13.3 and Example 13.7], $\Phi(y)$ converges absolutely for $y \in \Lambda_{\mathbb{R}} + i\mathcal{C}_{\Lambda}^+$ with $(\text{Im } y)^2 \gg 0$ and extends to an automorphic form on $\Omega_{M_0^\perp}^+(\cong \Lambda_{\mathbb{R}} + i\mathcal{C}_{\Lambda}^+)$ for $O^+(M_0^\perp)$ of weight 4 with zero divisor $\mathcal{D}_{M_0^\perp} \cap \Omega_{M_0^\perp}^+$.

Fix a vector l_{Λ} with $l_{\Lambda}^2 \geq 0$. Set

$$B_{\Lambda}([z]) = \frac{(z, \bar{z})}{|(z, l_{\Lambda})|^2}.$$

For an automorphic form f on $\Omega_{M_0^\perp}^+$ for $O^+(M_0^\perp)$ of weight w , recall that the Petersson norm $\|f\|$ is defined by

$$\|f([z])\|^2 = B_{\Lambda}([z])^w |f([z])|^2.$$

Then $\|f\|$ is a C^∞ function on $\mathcal{M}_{M_0}^\circ = \Omega_{M_0^\perp}^+/O^+(M_0^\perp)$.

Theorem 7.2.2. *There exists a constant C_{M_0} depending only on $M_0 = U(2) \oplus E_8(2)$ such that*

$$\tau_{BCOV}(\widetilde{S^{[2]}}) = C_{M_0} \|\Phi([S])\|$$

for any Enriques surface S , where $\|\Phi([S])\|$ is the Petersson norm of the Borchers Φ -function evaluated at the period of S .

Proof. Note that $\mathcal{M}_{M_0}^\circ$ can be regarded as a coarse moduli space of Enriques surfaces. By [61, (1.10)], $\bar{\mathcal{D}}_{M_0^\perp}$ is irreducible. Moreover, we have $\text{rk}(M_0) = 10 \leq 17$. By Theorems 7.1.14 and 6.4.12, there exist constants C_1, C_2 depending only on $M_0 = U(2) \oplus E_8(2)$ such that

$$\tau_{BCOV}(\widetilde{S^{[2]}}) = C_1 \tau_{\widetilde{M_0}, \mathcal{K}}(X_{\widetilde{S}}, \iota_{\widetilde{S}}), \quad \tau_{\widetilde{M_0}, \mathcal{K}}(X_{\widetilde{S}}, \iota_{\widetilde{S}}) = C_2 \tau_{M_0}(\widetilde{S}, \sigma)^{-2} \quad (7.2.1)$$

for any Enriques surface S . By [82, Theorem 8.3], there exists a constant C_3 depending only on $M_0 = U(2) \oplus E_8(2)$ such that

$$\tau_{M_0}(\widetilde{S}, \sigma) = C_3 \|\Phi([S])\|^{-\frac{1}{2}} \quad (7.2.2)$$

for any Enriques surface S . Combining (7.2.1) and (7.2.2), we obtain the desired result. $\square$

7.2.2 The case where Y^σ is a union of rational curves

Example 7.2.3. Let $k = 1, \dots, 6$ and set

$$\Lambda_k = I_2 \oplus I_{10-k}(-1),$$

where I_N is a positive definite odd unimodular lattice of rank N . Namely, Λ_k is an odd unimodular lattice of signature $(2, 10 - k)$. Let

$$M_0 = \Lambda_k(2)^\perp.$$

Here the orthogonal complement is considered in L_{K3} . We define

$$(r, l) := (\text{rk}(M_0), \dim_{\mathbb{Z}_2}((M_0)^\vee/M_0)) = (10 + k, 12 - k)$$

and

$$g := 11 - \frac{r + l}{2} = 0.$$

By Proposition 4.1.4, the fixed locus Y^σ of a 2-elementary K3 surface (Y, σ) of type $M_0 = \Lambda_k(2)^\perp$ is the union of k -rational curves. In this case, $\bar{\mathcal{D}}_{M_0^\perp}$ is also irreducible by [87, 11.3] and [28, FIGURE 1].

By [24, Theorem 5.1 and A], we have

$$\dim H^1(Z_Y, T_{Z_Y}) = \dim H^3(Z_Y, \Omega_{Z_Y}^1) = 10 - k.$$

On the other hand, we have

$$\dim \Omega_{M_0^\perp} = 10 - k.$$

Therefore the dimension of the Kuranishi space of Z_Y is equal to that of (Y, σ) . Hence every small deformation of Z_Y is induced from a deformation of the 2-elementary K3 surface (Y, σ) of type M_0 .

Let us recall the automorphic form constructed in [84, Theorem 4.2] and [35, Theorem 5.4]. For $k = 1, \dots, 6$, we set

$$L_k = I_1 \oplus -I_{9-k}.$$

Then $\Lambda_k = U(-1) \oplus L_k$. An element of $M_0^\perp = \Lambda_k(2)$ is denoted by (m, n, l) , where $m, n \in \mathbb{Z}$ and $l \in L_k$. Set

$$\mathcal{C}_{L_k} = \{x \in L_{k, \mathbb{R}}; x^2 > 0\}.$$

Note that Ω_{Λ_k} is canonically isomorphic to $\Omega_{M_0^\perp}$. The tube domain $L_{k, \mathbb{R}} + i\mathcal{C}_{L_k}$ can be identified with Ω_{Λ_k} via the isomorphism

$$L_{k, \mathbb{R}} + i\mathcal{C}_{L_k} \rightarrow \Omega_{\Lambda_k} \quad y \mapsto \left[\left(\frac{y^2}{2}, 1, y \right) \right]. \quad (7.2.3)$$

Let $\{h, d_1, \dots, d_{9-k}\}$ be a basis of $L_k(2)$ such that

$$h^2 = 1, \quad (h, d_i) = 0, \quad (d_i, d_j) = -\delta_{ij}.$$

Set

$$\varrho_k = \frac{1}{2}(3h - d_1 - \dots - d_{9-k}) \in L_k(2)^\vee = L_k\left(\frac{1}{2}\right)$$

and

$$\Pi_k = \{d \in L_k(2); d^2 = -2, (\varrho_k, d) = 1\}.$$

Let $\mathcal{W}_k$ be the Weyl chamber of $\mathcal{C}_{L_k(2)} = \{x \in L_{k,\mathbb{R}}; x^2 > 0\}$ containing ϱ_k . By [84, Proposition 4.1], we have

$$\mathcal{W}_k = \{v \in L_k(2) \otimes \mathbb{R}; v^2 > 0, (v, d) > 0 \text{ for all } d \in \Pi_k\}.$$

Let $\mathcal{C}_{L_k}^+$ be the connected component of $\mathcal{C}_{L_k}$ containing $\mathcal{W}_k$. The connected component of Ω_{Λ_k} corresponding to $L_{k,\mathbb{R}} + i\mathcal{C}_{L_k}^+$ via (7.2.3) is denoted by $\Omega_{\Lambda_k}^+$.

We define the series $\{c_k^{(0)}(l)\}_{l \in \mathbb{Z}}$ and $\{c_k^{(1)}(l)\}_{l \in \frac{k}{4} + \mathbb{Z}}$ by the generating functions

$$\sum_{l \in \mathbb{Z}} c_k^{(0)}(l) q^l = \frac{\eta(2\tau)^8 \theta_{A_1^+}(\tau)^k}{\eta(\tau)^8 \eta(4\tau)^8}, \quad \sum_{l \in \frac{k}{4} + \mathbb{Z}} c_k^{(1)}(l) q^l = -8 \frac{\eta(4\tau)^8 \theta_{A_1^+ + \frac{1}{2}}(\tau)^k}{\eta(2\tau)^{16}},$$

where

$$\theta_{A_1^+ + \frac{\varepsilon}{2}}(\tau) = \sum_{n \in \mathbb{Z}} q^{(n + \frac{\varepsilon}{2})^2}.$$

We define the infinite product $\Phi_k(y)$ by

$$\Phi_k(y) = e^{2\pi i(\rho_k, y)} \prod_{\delta \in \{0,1\}} \prod_{\lambda \in \Pi_k^{+(\delta)}} (1 - e^{2\pi i(\lambda, y)})^{c_k^{(\delta)}\left(\frac{r^2}{2}\right)},$$

where

$$\Pi_k^{+(\delta)} = \{\lambda \in \delta \varrho_k + L_k(2); \lambda^2 \geq 2(\delta - 1), (\lambda, v) > 0 \text{ for all } v \in \mathcal{W}_k\}.$$

By [21, Theorem 13.3], $\Phi_k(y)$ converges absolutely for $y \in \Lambda_{\mathbb{R}} + i\mathcal{W}_k$ with $(\text{Im } y)^2 \gg 0$ and extends to an automorphic form on $\Omega_{M_0^\perp}^+ (= \Omega_{\Lambda_k}^+ \cong \Lambda_k \otimes \mathbb{R} + i\mathcal{C}_{\Lambda_k}^+)$ for $O^+(M_0^\perp)$ of weight $k + 4$ with zero divisor $\mathcal{D}_{M_0^\perp} \cap \Omega_{M_0^\perp}^+$. In fact, Φ_k is identified with a Borcherds product by [87, Example 8.7].

Theorem 7.2.4. *There exists a constant C_k depending only on $M_0 = \Lambda_k(2)^\perp$ such that*

$$\tau_{BCOV}(Z_Y) = C_k \|\Phi_k([Y])\|^{k+1}$$

for any 2-elementary K3 surface (Y, σ) of type M_0 , where $\|\Phi_k([Y])\|$ is the Petersson norm of Φ_k evaluated at the period of (Y, σ) .

Proof. By Theorem 7.1.14 and 6.4.12, there exist constants $C_{k,1}$, $C_{k,2}$ depending only on $M_0 = \Lambda_k(2)^\perp$ such that

$$\tau_{BCOV}(Z_Y) = C_{k,1}\tau_{\widetilde{M}_0, \mathcal{K}}(X_Y, \iota_Y), \quad \tau_{\widetilde{M}_0, \mathcal{K}}(X_Y, \iota_Y) = C_{k,2}\tau_{M_0}(Y, \sigma)^{-2(k+1)} \quad (7.2.4)$$

for any 2-elementary K3 surface (Y, σ) of type M_0 . By [84, Theorem 4.2.(1)] and [87, Theorem 0.1], there exists a constant $C_{k,3}$ depending only on $M_0 = \Lambda_k(2)^\perp$ such that

$$\tau_{M_0}(Y, \sigma) = C_{k,3}\|\Phi_k([Y])\|^{-\frac{1}{2}} \quad (7.2.5)$$

for any 2-elementary K3 surface (Y, σ) of type M_0 . Combining (7.2.4) and (7.2.5), we obtain the desired result. $\square$

According to the genus one mirror symmetry conjecture ([26, Conjectures 1 and 2]), for a Calabi-Yau manifold, the BCOV invariant of its mirror family is conjecturally equivalent to a certain generating series of the genus one Gromov-Witten invariants of the Calabi-Yau manifold. For Calabi-Yau hypersurfaces in the projective spaces, this conjecture is verified by Zinger [89] [90], Fang-Lu-Yoshikawa [27] in dimension 3, and by Zinger [89] [90], Eriksson-Freixas i Montplet-Mourougane [26] in arbitrary dimension. For Enriques surfaces, an analogue of the genus one mirror symmetry conjecture holds true by Oberdieck [65] and Yoshikawa [82].

At least to the author, it is unclear what is the mirror of the Calabi-Yau 4-folds in Examples 7.2.1 and 7.2.3. According to Camere-Garbagnati-Mongardi [24, Corollary 5.8], even if $(\check{Y}, \check{\sigma})$ is a lattice theoretic mirror of a 2-elementary K3 surface (Y, σ) of type M_0 , the Hodge diamond of $Z_{\check{Y}}$ is not mirror to the one of Z_Y . Namely, $h^{1,1}(Z_Y) \neq h^{3,1}(Z_{\check{Y}})$.

Let Z be the Calabi-Yau 4-fold in Examples 7.2.1 and 7.2.3. By Theorems 7.2.2 and 7.2.4, we expect that if there exists a mirror Calabi-Yau 4-fold of Z , then its genus one Gromov-Witten invariants are encoded by a power of Φ or Φ_k in a certain way.

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