

Some aspects of multidimensional continued fraction algorithms

By

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Abstract

In this paper, we give a resume of our recent works mainly related to algorithms of multidimensional continued fraction expansions, which are expected to have periodicity properties such as Lagrange's theorem.

§ 1. Introduction

Many kinds of algorithms of continued fraction expansions of dimension $s(\geq 2)$ have been studied starting with K.G.J.Jacobi(1804-1851), for example, see [14]. For $s = 1$, we know Lagrange's theorem related to periodic¹ continued fractions and real quadratic irrationals. But, even for real cubic irrationalities, there appeared no suitable algorithms (of dimension 2). In this section, we roughly explain why classical algorithms (the Jacobi-Perron algorithm, etc.) do not work well related to the periodicity. On the other hand, in a series of papers, we gave some good candidates of algorithms that seem to work well related to the periodicity, cf. [15–19]. Our objective in this paper is to introduce our new algorithms announced in [15–19] together with some new results in [7, 20].

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¹Here and throughout our paper, we mean purely periodic or eventually periodic by the word "periodic".

Let K be a real algebraic number field of degree $s + 1$, and $\bar{\alpha} = (1, \alpha^{(1)}, \dots, \alpha^{(s)})$ be a $\mathbb{Q}$ -basis of K . Suppose

$$(1.1) \quad [\overline{a_0}; \overline{a_1}, \overline{a_2}, \dots] \quad (\overline{a_0} \in \mathbb{Z}^s, \overline{a_n} \in \mathbb{Z}_{\geq 0}^s, n \geq 1)$$

be a continued fraction of dimension s of $\bar{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(s)})$ obtained by a certain algorithm ALGOR. We define $\overline{\alpha_n} = (\alpha_n^{(1)}, \dots, \alpha_n^{(s)}) \in K^s (n \geq 0)$ by

$$(1.2) \quad \bar{\alpha} = [\overline{a_0}; \overline{a_1}, \overline{a_2}, \dots, \overline{a_{n-1}}, \overline{a_n} + \overline{\alpha_n}].$$

Confer with [11] for the meaning of the continued fraction on the right-hand side of (1.2). Notice that we do not assume the convergence of the continued fraction (1.1) for the definition of $\overline{\alpha_n}$, and $\overline{\alpha_n} = \overline{\alpha_n}(\bar{\alpha}; \text{ALGOR})$ depends on $\bar{\alpha}$ and the algorithm ALGOR. We introduce a function $rdh = rdh(\bar{\alpha}; \text{ALGOR})$

$$rdh : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Q}$$

which measures a quality of an algorithm ALGOR in a sense related to the periodicity of the continued fraction (1.1) of $\bar{\alpha}$ obtained by the algorithm, cf. ([15–19]). We need some definitions:

$$dh(r) := \max\{\lfloor \log_{10} |p| \rfloor + 1, \lfloor \log_{10} |q| \rfloor + 1\}$$

for $r = p/q \in \mathbb{Q}$ ($p, q \in \mathbb{Z}$ are coprime). The function dh can be extended to $\mathbb{Q}[x]$ by setting

$$dh(\phi) := \max_{0 \leq i \leq m} \{dh(c_i)\}$$

for $\phi = \sum_{i=0}^m c_i x^i \in \mathbb{Q}[x]$. We put

$$\begin{aligned} dh(\overline{\alpha_n}) &= dh(n; \bar{\alpha}; \text{ALGOR}) \\ &:= \max_{1 \leq i \leq s} \{dh(\phi_{\alpha_n^{(i)}})\}, \\ rdh(n) &:= dh(\overline{\alpha_n})/dh(\bar{\alpha}) \quad (n \geq 0), \end{aligned}$$

where $\phi_{\alpha} \in \mathbb{Q}[x]$ denotes the monic minimal polynomial of $\alpha \in K$. Notice that the function $rdh : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Q}$ depends on $\bar{\alpha}$ and an algorithm ALGOR. It is clear that the set of polynomials $p(x) \in \mathbb{Q}[x]$ such that

$$\deg p(x) \leq d, \quad dh(p(x)) \leq M$$

becomes a finite set for any given numbers d, M , so that we have the following

Claim: The continued fraction (1.1) is periodic if and only if rdh is bounded.

We conjectured that the function rdh is not only unbounded but also the following explosion phenomenon takes place:

Explosion Phenomenon: $\lim_{n \rightarrow \infty} rdh(n; \bar{\alpha}; \text{ALGOR}) = \infty$ holds even for some $\bar{\alpha} \in K^2$ with $K = \mathbb{Q}(\sqrt[3]{d})$ for some classical algorithms (the Jacobi-Perron algorithm, the modified Jacobi-Perron algorithm (that is 2-dimensional Brun's algorithm), etc.)

In fact, by numerical experiments by PC, the graph of the function $y = rdh(n)$ ($0 \leq n \leq 50000$) looks like a line $y = cx$ ($c > 0$) and $rdh(50000)$ attains a few thousands in some cases. On the other hand we gave some new algorithms for which we can expect

$$rdh(n; \bar{\alpha}; \text{ALGOR}) \leq c(K).$$

Moreover, it seems very likely that the constant $c(K)$ depends only on the degree $[K, \mathbb{Q}]$ for some algorithms. For instance, we do not have found an example such that $rdh(n; \bar{\alpha})$ exceeds ten for our algorithms when K is a real cubic field. We also refer in Section 2 some other experiments related to the periodicity of some classical multidimensional continued fraction algorithms.

In Sections 3-5 in this paper, we survey our recent works mainly related to periodic continued fractions. In Section 3, we describe two algorithms AJPA ($\deg K \leq 4$) and AJPA(2) ($\deg K \leq 5$). We give an algorithm of dimension 2 which generalizes the so called slow continued fraction algorithm in Section 4. The algorithm given in this section could play important roles related to 2-dimensional Sturmian words (or stepped surfaces), etc.

In Section 5, we refer to continued fractions for quadratic elements over the rational function field $k(t)$ in the formal Laurent series $k((t^{-1}))$, where k denotes a field.

§ 2. Numerical experiments for some classical algorithms

The Jacobi-Perron algorithm is defined as follows. For $\mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n$ ($1, x_1, \dots, x_n$ are linearly independent over $\mathbb{Q}$), we define a transformation $\bar{T}$ by

$$\bar{T}(x_1, \dots, x_n) = (u_1, \dots, u_n),$$

where

$$u_i := \begin{cases} \frac{1}{x_1} - \left\lfloor \frac{1}{x_1} \right\rfloor, & \text{if } i = n, \\ \frac{x_{i+1}}{x_1} - \left\lfloor \frac{x_{i+1}}{x_1} \right\rfloor, & \text{if } i \neq n. \end{cases}$$

The transformation $\bar{T}$ gives rise to an algorithm, which is the so called Jacobi-Perron algorithm. In [4] Elsner and Hasse gave numerical experiments for 36 pairs of cubic irrationalities on the Jacobi-Perron algorithm by means of a computer. Among them, they found periodicity for 14 pairs and did not find any periodicity for the other 22 pairs. From the numerical experiments in [19] we can not expect the periodicity for $(\sqrt{m} + \sqrt{n} - \lfloor \sqrt{m} + \sqrt{n} \rfloor, \sqrt{m} - \sqrt{n} - \lfloor \sqrt{m} - \sqrt{n} \rfloor, \sqrt{mn} - \lfloor \sqrt{mn} \rfloor)$ by Jacobi-Perron Algorithm for all m, n with $2 \leq n < m \leq 11$ and $m, n, m/n \notin \mathbb{Q}^{\times 2}$ except $m = 10$ and $n = 8$. In [16] we have a similar result about the modified Jacobi-Perron Algorithm by Podsypanin [12]. One can find the explosion phenomenon in [16], [19].

§ 3. AJPA

In [16] we introduce a new multidimensional continued fraction algorithm called algebraic Jacobi-Perron algorithm(AJPA) as follows motivated by the algorithms given in [15]. Let K be a real number field over $\mathbb{Q}$ with $\deg_{\mathbb{Q}}(K) = d$. We mean by X_K the set defined by

$$X_K := \{(\alpha_1, \dots, \alpha_{d-1}) \in (K \cap [0, 1])^{d-1} \mid \text{there exists a integer } i \text{ with } 1 \leq i \leq d-1 \text{ such that } K = \mathbb{Q}(\alpha_i) \text{ and } 1, \alpha_1, \dots, \alpha_{d-1} \text{ are linearly independent over } \mathbb{Q}\}.$$

The function $\nu(\theta)$ is defined for $\theta \in K$ by

$$(3.1) \quad \nu(\theta) := \begin{cases} \frac{\theta}{|N_{K/\mathbb{Q}}(\theta)|^{\frac{1}{d-1}}} & \text{if } K = \mathbb{Q}(\theta), \\ -1 & \text{if } K \neq \mathbb{Q}(\theta), \end{cases}$$

where $N_{K/\mathbb{Q}}(\theta)$ is the norm of θ over $\mathbb{Q}$.

For $\alpha = (\alpha_1, \dots, \alpha_{d-1}) \in X_K$, we define $\rho(\alpha)$ by

$$\rho(\alpha) = \max\{\nu(\alpha_i) \mid 1 \leq i \leq d-1\}.$$

We denote by $\omega(\alpha)$ the number $i \in \{1, \dots, d-1\}$ uniquely determined by

$$\alpha_i = \max\{\alpha_j \mid \rho(\alpha) = \nu(\alpha_j)\}.$$

We remark that $\sharp\{\alpha_j \mid \rho(\alpha) = \nu(\alpha_j)\} = 1$. We define a transformation $T_{(K)}$ on X_K as follows:

For $\alpha = (\alpha_1, \dots, \alpha_{d-1}) \in X_K$, $T_{(K)}(\alpha) = (\beta_1, \dots, \beta_{d-1})$ with

$$\beta_i := \begin{cases} \frac{1}{\alpha_{\omega(\alpha)}} - \left\lfloor \frac{1}{\alpha_{\omega(\alpha)}} \right\rfloor & \text{if } i = \omega(\alpha), \\ \frac{\alpha_i}{\alpha_{\omega(\alpha)}} - \left\lfloor \frac{\alpha_i}{\alpha_{\omega(\alpha)}} \right\rfloor & \text{if } i \neq \omega(\alpha) \end{cases} \quad (i = 1, \dots, d-1).$$

The transformation $T_{(K)}$ gives rise to an algorithm of continued fraction of dimension $d - 1$, which will be also referred to AJPA. In [16] we gave some computer experiments by which we can expect that the expansion obtained by our algorithm for $\underline{\alpha} = (\alpha_1, \dots, \alpha_s) \in K^s$ (with some natural conditions on $\underline{\alpha}$) becomes periodic for any real number field K as far as $s + 1 = \deg_{\mathbb{Q}}(K) \leq 4$. But, it seems very likely that the algorithm will not work well if $\deg_{\mathbb{Q}}(K) = 5$. For each real valued function $\nu^*(\theta)$ on K instead of the function $\nu(\theta)$, we can define an algorithm in the similar manner as above. For an algebraic number θ , we mean by $\phi_{\theta} \in \mathbb{Q}[x]$ the monic minimal polynomial of θ . In [18] we define $\nu'(\theta)$ for $\theta \in K$ by

$$(3.2) \quad \nu'(\theta) := \begin{cases} \frac{\theta}{|D(\theta)|^{\frac{1}{d-1}}} & \text{if } K = \mathbb{Q}(\theta), \\ -1 & \text{if } K \neq \mathbb{Q}(\theta), \end{cases}$$

where $D(\theta) = \left[\frac{d}{dx} \phi_{\theta}(x) \right]_{x=\theta}$ is the differential coefficient of $\phi_{\theta}(x)$ at $x = \theta$. We introduced the algorithm AJPA2, which is associated with ν' . From the numerical experiments in [18] we can expect that the resulting expansion of $\underline{\alpha} \in X_K$ by AJPA2 always becomes periodic for any real number field K with $\deg_{\mathbb{Q}}(K) = s + 1 \leq 5$. But, it seems very likely that the algorithm will not work well if $\deg_{\mathbb{Q}}(K) = 6$.

§ 4. Slow continued fraction algorithm

In this section we report some of the results announced in [7]. We define a transformation T on the interval $[0, 1]$ as follows:

$$T(x) := \begin{cases} \frac{x}{1-x} & \text{if } x \in I_0, \\ \frac{2x-1}{x} & \text{if } x \in I_1, \end{cases}$$

where $I_0 = [0, \frac{1}{2}]$ and $I_1 = (\frac{1}{2}, 1]$. $\epsilon : [0, 1] \rightarrow \{0, 1\}$ is defined by $x \in I_{\epsilon(x)}$. The algorithm $(T, \epsilon, [0, 1])$ are considered by many authors (for example see [10]). $(T, \epsilon, [0, 1])$ is related to the continued fraction algorithm. The transformation F on $[0, 1]$ defines $F(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$. Let $x = [0, k_1, k_2, \dots]$ be the regular continued fraction expansion of x . Then, we see that $T^{k_1+k_2}(x) = F^2(x)$. Therefore, $(T, \epsilon, [0, 1])$ is a kind of slow continued fraction algorithm. $(T, \epsilon, [0, 1])$ is also related to Farey partitions. Let L be a real quadratic number field. We can also define the algorithm $(T, \epsilon, [0, 1] \cap L)$. Then, every element in $[0, 1] \cap L$ is eventually periodic. In [7] we extend the algorithm $(T, \epsilon, [0, 1] \cap L)$ to certain multidimensional algorithms. Let K be a real cubic field. Let

$\Delta_K = \{(x, y) \in K^2 \mid 1, x, y \text{ are linearly independent over } \mathbf{Q}, 0 < x, y \text{ and } x + y < 1\}$.
We put

$$Ind = \{(i, j) \mid i, j \in \{0, 1, 2\}, i \neq j\}.$$

We denote by Δ and $\Delta(i, j)$ for $(i, j) \in Ind$ the regions

$$\begin{aligned}\Delta &:= \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, x + y \leq 1\}, \\ \Delta(1, 2) &:= \{(x, y) \in \Delta \mid x \geq y\}, \\ \Delta(2, 1) &:= \{(x, y) \in \Delta \mid x \leq y\}, \\ \Delta(0, 1) &:= \{(x, y) \in \Delta \mid 2x + y - 1 \leq 0\}, \\ \Delta(1, 0) &:= \{(x, y) \in \Delta \mid 2x + y - 1 \geq 0\}, \\ \Delta(0, 2) &:= \{(x, y) \in \Delta \mid x + 2y - 1 \leq 0\}, \\ \Delta(2, 0) &:= \{(x, y) \in \Delta \mid x + 2y - 1 \geq 0\}.\end{aligned}$$

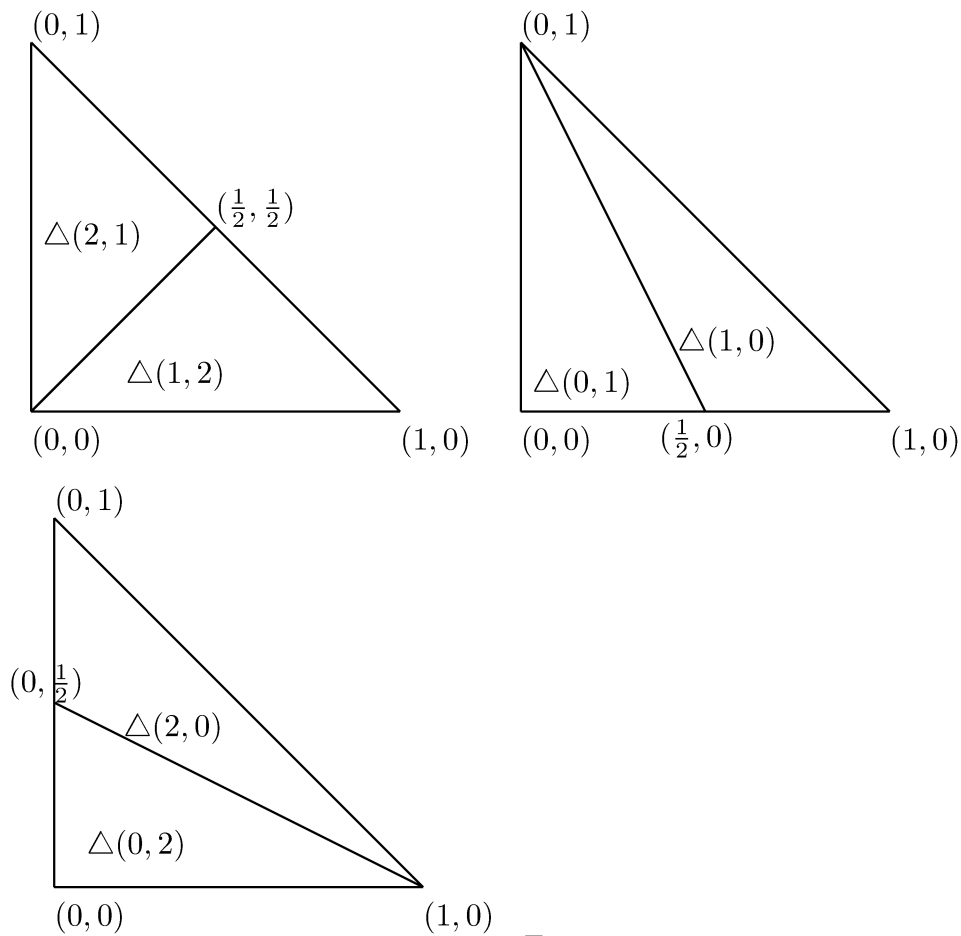


Figure 1.

For each $(i, j) \in \text{Ind}$, let us introduce the maps $T_{ij} : \Delta(i, j) \rightarrow \Delta$ as follows:

$$\begin{aligned} T_{12}(x, y) &:= \left(\frac{x-y}{1-y}, \frac{y}{1-y} \right), \\ T_{21}(x, y) &:= \left(\frac{x}{1-x}, \frac{y-x}{1-x} \right), \\ T_{10}(x, y) &:= \left(\frac{2x+y-1}{x+y}, \frac{y}{x+y} \right), \\ T_{01}(x, y) &:= \left(\frac{x}{1-x}, \frac{y}{1-x} \right), \\ T_{20}(x, y) &:= \left(\frac{x}{x+y}, \frac{x+2y-1}{x+y} \right), \\ T_{02}(x, y) &:= \left(\frac{x}{1-y}, \frac{y}{1-y} \right). \end{aligned}$$

We define the value $v(\alpha, \beta, i, j)$ for $r \in \mathbb{R}^+$, $(\alpha, \beta) \in \Delta_K$ and $i, j \in \{0, 1, 2\}$ with $i \neq j$ as follows:

$$v(\alpha, \beta, i, j) := \begin{cases} \frac{|\alpha^{5/2}\beta^{5/2}|}{|N(\alpha)N(\beta)|}, & \text{if } \{i, j\} = \{1, 2\}, \\ \frac{|\alpha^{5/2}(1-\alpha-\beta)^{5/2}|}{|N(\alpha)N(1-\alpha-\beta)|}, & \text{if } \{i, j\} = \{0, 1\}, \\ \frac{|\beta^{5/2}(1-\alpha-\beta)^{5/2}|}{|N(\beta)N(1-\alpha-\beta)|}, & \text{if } \{i, j\} = \{0, 2\}. \end{cases}$$

We remark that the element $(i_0, j_0) \in \text{Ind}$ is uniquely determined by an equality $v(\alpha, \beta, i_0, j_0) = \max\{v(\alpha, \beta, i, j)\}$. We define $\epsilon_K(\alpha, \beta)$ for $(\alpha, \beta) \in \Delta_K$ by

$$\epsilon_K(\alpha, \beta) := \begin{cases} (1, 2), & \text{if } \{i_0, j_0\} = \{1, 2\} \text{ and } (\alpha, \beta) \in \Delta(1, 2), \\ (2, 1), & \text{if } \{i_0, j_0\} = \{1, 2\} \text{ and } (\alpha, \beta) \in \Delta(2, 1), \\ (1, 0), & \text{if } \{i_0, j_0\} = \{0, 1\} \text{ and } (\alpha, \beta) \in \Delta(1, 0), \\ (0, 1), & \text{if } \{i_0, j_0\} = \{0, 1\} \text{ and } (\alpha, \beta) \in \Delta(0, 1), \\ (2, 0), & \text{if } \{i_0, j_0\} = \{0, 2\} \text{ and } (\alpha, \beta) \in \Delta(2, 0), \\ (0, 2), & \text{if } \{i_0, j_0\} = \{0, 2\} \text{ and } (\alpha, \beta) \in \Delta(0, 2). \end{cases}$$

Notice that $\epsilon_K(\alpha, \beta)$ is well-defined since $1, \alpha, \beta$ is linearly independent over $\mathbb{Q}$. We define a transformation T_K on Δ_K by

$$T_K(\alpha, \beta) := T_{i_0 j_0}(\alpha, \beta), \text{ if } \epsilon_K(\alpha, \beta) = (i_0, j_0).$$

Thus, we have seen that an algorithm $(\Delta_K, T_K, \epsilon_K)$ can be defined.

Conjecture 4.1 ([7]). *Every element in Δ_K is eventually periodic by the algorithm $(T_K, \epsilon_K, \Delta_K)$ for every real cubic field K .*

The algorithm $(T, \epsilon, [0, 1])$ has connection with substitutions on Sturmian sequences. We define the substitution $\phi_i (i = 0, 1)$ by

$$\phi_0 : \begin{cases} 0 \rightarrow 0 \\ 1 \rightarrow 01, \end{cases} \quad \phi_1 : \begin{cases} 0 \rightarrow 01 \\ 1 \rightarrow 1. \end{cases}$$

For $x \in [0, 1]$, $S(x)$ is defined to be an infinite word $s_1 s_2 \dots s_n \dots$, where $s_n = \lfloor nx \rfloor - \lfloor (n-1)x \rfloor$. The word $S(x)$ is referred to as a homogeneous Sturmian sequence.

The following theorem is given by many authors (for example see [10]).

Theorem 4.2. *For $x \in [0, 1]$, $\phi_{\epsilon(x)}(S(T(x))) = S(x)$ holds.*

Theorem 4.2 is naturally extended to Theorem 4.3 using the algorithm $(T_2, \epsilon_2, \Delta)$ and stepped surfaces introduced in [8] and [1].

For $\bar{x} \in \mathbb{Z}^3$, $i = 0, 1, 2$, we mean by $(\bar{x}, i^*)$ a unit square defined by

$$(\bar{x}, i^*) := \{\bar{x} + t\bar{e}_j + u\bar{e}_k \mid t, u \in [0, 1], \{i, j, k\} = \{0, 1, 2\}\}$$

where $\bar{e}_i (i = 0, 1, 2)$ is the canonical basis of $\mathbb{R}^3$.

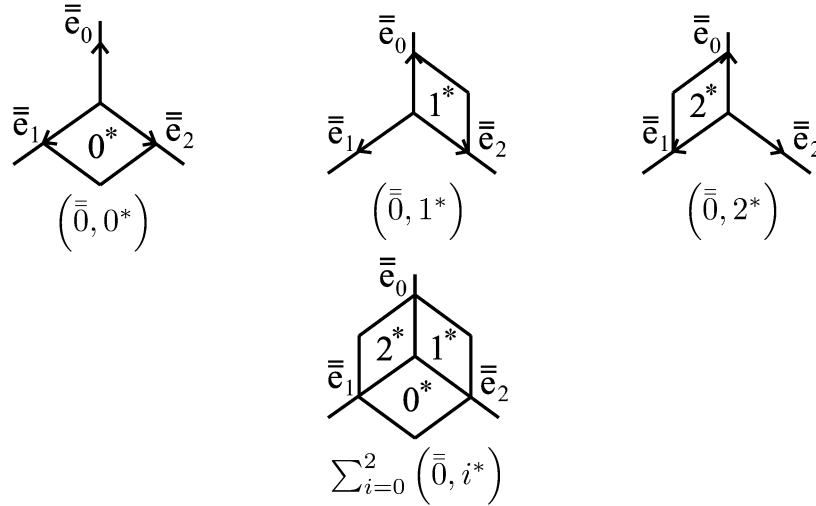


Figure 2. $(\bar{0}, i^*)$, $i = 0, 1, 2$ and $\sum_{i=0}^2 (\bar{0}, i^*)$.

We denote by $\mathcal{G}$ the $\mathbb{Z}$ -free module generated by all the squares:

$$\mathcal{G} := \left\{ \sum m_{\bar{x}, i} (\bar{x}, i^*) \mid \bar{x} \in \mathbb{Z}^3, i \in \{0, 1, 2\}, m_{\bar{x}, i} \in \mathbb{Z} \right\}.$$

Let $\bar{\alpha} = {}^t(\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}) \in \mathbb{R}_{>0}^3$ and $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}$ be linearly independent over $\mathbb{Q}$. Notice that without loss of generality, we may assume $\alpha^{(0)} + \alpha^{(1)} + \alpha^{(2)} = 1$.

We put

$$\mathcal{P}_{\bar{\alpha}} := \{\bar{x} \in \mathbb{R}^3 \mid \langle \bar{\alpha}, \bar{x} \rangle = 0\}$$

where $\langle \cdot, \cdot \rangle$ means the inner product. The so-called '*stepped surface*' $\mathcal{S}_{\bar{\alpha}}$ is defined by

$$\mathcal{S}_{\bar{\alpha}} := \bigcup_{j=0}^2 \{(\bar{x}, i^*) \mid \langle \bar{\alpha}, \bar{x} \rangle \geq 0, \langle \bar{\alpha}, (\bar{x} - \bar{e}_i) \rangle < 0\}$$

Let σ be a substitution over $\{0, 1, 2\}$ and M_{σ} be its incidence matrix of σ . We suppose that σ is unimodular, i.e., $\det M_{\sigma} = \pm 1$.

The dual substitution Θ_{σ} of σ , which is an endomorphism on $\mathcal{G}$ introduced in [1], can be defined by

$$(4.1) \quad \begin{aligned} \Theta_{\sigma}(\bar{x}, i^*) &:= M_{\sigma}^{-1}(\bar{x}) + \sum_{j=0}^2 \sum_{S: \sigma(j)=PiS} (M_{\sigma}^{-1}(f(S)), j^*), \\ \Theta_{\sigma}\left(\sum (\bar{x}, i^*)\right) &:= \sum (\Theta_{\sigma}(\bar{x}, i^*)) \end{aligned}$$

for $i = 0, 1, 2$, where $f(w) := {}^t(|w|_0, |w|_1, |w|_2)$ ($|w|_i$ is the number of occurrences of a symbol i appearing in a finite word $w \in \{0, 1, 2\}^*$), and P (resp. S) means that the prefix (resp. suffix) of i of $\sigma(j) = PiS$ respectively.

We consider six substitutions σ_{ϵ} for $\epsilon \in Ind$ defined by

$$\begin{aligned} \sigma_{(1,0)} &: \begin{cases} 0 \rightarrow 0 \\ 1 \rightarrow 01 \\ 2 \rightarrow 2 \end{cases}, & \sigma_{(2,1)} &: \begin{cases} 0 \rightarrow 0 \\ 1 \rightarrow 1 \\ 2 \rightarrow 12 \end{cases}, & \sigma_{(0,2)} &: \begin{cases} 0 \rightarrow 20 \\ 1 \rightarrow 1 \\ 2 \rightarrow 2 \end{cases}, \\ \sigma_{(2,0)} &: \begin{cases} 0 \rightarrow 0 \\ 1 \rightarrow 1 \\ 2 \rightarrow 02 \end{cases}, & \sigma_{(0,1)} &: \begin{cases} 0 \rightarrow 10 \\ 1 \rightarrow 1 \\ 2 \rightarrow 2 \end{cases}, & \sigma_{(1,2)} &: \begin{cases} 0 \rightarrow 0 \\ 1 \rightarrow 21 \\ 2 \rightarrow 2 \end{cases}. \end{aligned}$$

Then, in view of (4.1), we have the dual substitutions Θ_{ϵ} on $\mathcal{G}$ of σ_{ϵ} , which are

given by

$$\begin{aligned}
\Theta_{(1,0)} : \left\{ \begin{array}{l} \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) \mapsto \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) + (\bar{e}_1 - \bar{e}_0, 1^*) \end{array} \right. & , \quad \Theta_{(2,0)} : \left\{ \begin{array}{l} \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) \mapsto \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) + (\bar{e}_2 - \bar{e}_0, 2^*) \end{array} \right. \\
\Theta_{(2,1)} : \left\{ \begin{array}{l} \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) \mapsto \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) + (\bar{e}_2 - \bar{e}_1, 2^*) \end{array} \right. & , \quad \Theta_{(0,1)} : \left\{ \begin{array}{l} \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) \mapsto \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) + (\bar{e}_0 - \bar{e}_1, 0^*) \end{array} \right. \\
\Theta_{(0,2)} : \left\{ \begin{array}{l} \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) \mapsto \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) + (\bar{e}_0 - \bar{e}_2, 0^*) \end{array} \right. & , \quad \Theta_{(1,2)} : \left\{ \begin{array}{l} \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) \mapsto \left(\begin{array}{c} \bar{0}, 0^* \\ \bar{0}, 1^* \\ \bar{0}, 2^* \end{array} \right) + (\bar{e}_1 - \bar{e}_2, 1^*) \end{array} \right.
\end{aligned}$$

Theorem 4.3 ([7]). *Let $(x, y) \in \Delta_K$. Let $(u, v) = T_K(x, y)$. Then, $\Theta_{\epsilon_K(x, y)}$ gives a bijection from $\mathcal{S}_{(u, v, 1-u-v)}$ to $\mathcal{S}_{(x, y, 1-x-y)}$ for every real cubic field K .*

We remark that multidimensional continued fraction algorithms and substitutions on stepped surfaces are considered in [6], [8], [9], [3] and [5].

§ 5. Formal power series

Let k be a field. Throughout the paper, t denotes an indeterminate. For $0 \neq \alpha = \sum_{i=-\infty}^m b_i t^i \in k((t^{-1}))$ with $b_m \neq 0$ we define $|\alpha|$ and $\lfloor \alpha \rfloor$ by

$$\begin{aligned}
|\alpha| &:= e^m \quad (|0| := 0), \\
\lfloor \alpha \rfloor &:= \sum_{i=0}^m b_i t^i \quad (\lfloor \alpha \rfloor := 0 \text{ for } m < 0).
\end{aligned}$$

One can consider an algorithm of simple continued fraction expansion in $k((t^{-1}))$ as well as in $\mathbb{R}$ (see, for example [2], [13]). For $\alpha \in k((t^{-1}))$ we set

$$\begin{aligned}
\alpha_0 &= \alpha, \quad a_0 = \lfloor \alpha_0 \rfloor, \\
\alpha_{i+1} &= \frac{1}{\alpha_i - a_i}, \quad a_{i+1} = \lfloor \alpha_{i+1} \rfloor.
\end{aligned}$$

Then, we get

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}} = [a_0; a_1, a_2, \dots] \quad (\alpha \in k((t^{-1})) \setminus k(t), a_n \in k[t]).$$

If there exist $m, n \in \mathbb{Z}_{\geq 0}$ with $m \neq n$ and $\epsilon \in k$ such that $\alpha_m = \epsilon \alpha_n$, we call $\alpha \in k((t^{-1}))$ *pseudoperiodic*. W. M. Schmidt gave the following criterion whether $\alpha \in k((t^{-1}))$ is pseudoperiodic or not:

Theorem 5.1 (Schmidt [13]). *Suppose $\text{char } k \neq 2$. Suppose α is quadratic over $k(t)$ and satisfies*

$$A\alpha^2 + B\alpha + C = 0$$

where A, B, C are relatively prime. The discriminant of this equation is $D = B^2 - 4AC$. Then α has a pseudoperiodic continued fraction if and only if the relation

$$Y^2 - DZ^2 \in k^\times$$

has a nontrivial solution, i.e., a solution $Y, Z \in k[t]$ with $Z \neq 0$.

As a consequence of the theorem, Schmidt showed that for a field k of characteristic 0 there are quadratic elements in $k((t^{-1}))$ whose continued fraction is not pseudoperiodic. The same is true when $\text{char } k = p (\neq 0)$ and k is transcendental over $\mathbb{F}_p$. In [17] we investigated what continued fraction expansions quadratic elements in $k((t^{-1}))$ have. We introduced there a new class $\mathcal{C}$ of continued fractions which contains the pseudoperiodic continued fractions and we gave a conjecture which says that if an element α of $k((t^{-1}))$ is quadratic over $\mathbb{Q}(t)$ then α belongs to the new class $\mathcal{C}$. We gave some examples which support our conjecture. The following Theorem 5.2 is one of these examples. We remark that the following element $\sigma \in \mathbb{Q}((t^{-1}))$ in the theorem is quadratic over $\mathbb{Q}(t)$ and its continued fraction expansion $\in \mathcal{C}$ is not pseudoperiodic.

Theorem 5.2 ([17]). *Let $\sigma = \sigma(t)$ be the element in $\mathbb{Q}((t^{-1}))$ satisfying*

$$(5.1) \quad \sigma^2 - (t^2 + 1) = 0$$

such that the coefficient of t in σ is 1. Let $t^3\sigma = [A_0; A_1, A_2, \dots]$ be the simple continued fraction expansion of $t^3\sigma$. Let $\tau_n = [0; A_n, A_{n+1}, \dots]$ ($n \in \mathbb{Z}_{\geq 0}$).

1. *Then,*

$$A_0 = t^4 + \frac{1}{2}t^2 - \frac{1}{8},$$

for $n \geq 1$

$$(5.2) \quad A_n = \frac{\gamma_{n+1}((4n^2 + 4n)t^2 + 2n^2 + 2n + 1)}{8(n^4 + 2n^3 + n^2)},$$

where $\{\gamma_j\}_{j=1}^\infty$ is defined by as follows: $\gamma_1 = -1$ and for $j \geq 1$

$$(5.3) \quad \gamma_j \gamma_{j+1} = -64j^4.$$

2.

$$(5.4) \quad \tau_n = \frac{\gamma_n(8n^2t^3\sigma - 8n^2t^4 - 4n^2t^2 + 1)}{8(4n^4 - 4n^3)t^2 - 8n^2}.$$

There are many multidimensional continued fraction algorithms on formal power series (see [21]). But it seems to be difficult to get Lagrange type theorems related to continued fractions obtained by any known algorithms for elements in the field $\mathbb{Q}((t^{-1}))$. In [20] we consider AJPA in $k((t^{-1}))$ in the similar manner to number fields on $\mathbb{Q}$. In [20] we give an example (α, β) , where $\alpha, \beta \in \mathbb{Q}((t^{-1}))$ are elements of a cubic field on $\mathbb{Q}(t)$ and the expansion of (α, β) by AJPA is not eventually periodic, but we can describe the expansion such as Theorem 5.2. In [20] we give some conjectures related to Lagrange type theorem by AJPA.

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