

グラフの辺取りゲーム

(ニムとケイレスの一般化)

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1. Introduction.

We first define a new game played on graphs. Let G be a finite graph without loops or multiple edges, and $\mathcal{K}$ a set of graphs. This game is played by two players on the graph G . Each player in turn removes a set of edges which induces a graph isomorphic to a graph in $\mathcal{K}$. The winner is a player who removes edges such that the resulting graph contains no graph of $\mathcal{K}$, that is, the player who cannot move loses. We call this game an *edge-removing game of normal $\mathcal{K}$ type*. If we change the rule to one where the player who removes the last edges loses, then the game is called an *edge-removing game of misere $\mathcal{K}$ type*. In this paper, we shall discuss only games of normal type.

We call the complete bipartite graph $K_{1,n} = K(1,n)$ the *star* of order $n+1$, and denote by P_n the path of order n . If $\mathcal{K}$ is the set of all stars, then we call this game the *edge-removing game of normal star type*, or simply *ER-game of star type*. If we play ER-game of star type on a graph consisting of some stars, then this game is nothing but the game of Nim. Similarly ER-game of star type played on a graph consisting of some paths is equivalent to the game called Kayles. So ER-game of star type is a generalization of these

two games. In this paper we give some results on ER-game of star type played on double stars, forks and trees.

2. ER-game of star type played on doubles stars

In order to solve ER-game of normal $\mathcal{H}$ type played on a graph G , it suffices to determine the Sprague-Grundy number $g(G)$ of G , which is often called the Grundy number [1,2,3]. The Grundy number is defined inductively as follows: If a graph G_1 contains no graph of $\mathcal{H}$, then $g(G_1) = 0$. Let $H_1, H_2, \dots, H_m$ be the set of all graphs which can be obtained from a graph G by one move. Then

$$g(G) = \min\{\{0,1,2,3,\dots\} - \{g(H_i) \mid 1 \leq i \leq m\}\}$$

By this definition, we can easily show that $g(G) \leq |E(G)|$ by induction on $|E(G)|$. It is well-known that if a graph G consists of the components $D_1, \dots, D_r$, then

$$\begin{aligned} g(G) &= \text{the nim-sum of } g(D_1), g(D_2), \dots, g(D_r). \\ &= g(D_1) \dot{+} g(D_2) \dot{+} \dots \dot{+} g(D_r). \end{aligned}$$

Namely if

$$g(D_k) = \sum_{i \geq 0} x_k(i) 2^i, \quad x_k(i) \in \{0,1\}$$

then

$$g(G) = \sum_{i \geq 0} y(i) 2^i, \quad y(i) \equiv \sum_{k=1}^r x_k(i) \pmod{2} \quad \text{and} \quad y(i) \in \{0,1\}.$$

Moreover, it is easy to see that the player going second can win if and only if $g(G) = 0$.

We denote by $(\dots)$ an order set, that is, $(x_1, x_2, \dots, x_k) = (y_1, y_2, \dots, y_k)$ means that $x_i = y_i$ for all $i, 1 \leq i \leq k$.

Theorem A. [1,2,3] The Grundy numbers of stars $K_{1,n}$ and paths P_n of ER-game of star type are given by the following statements.

- (i) $g(K_{1,n}) = n$.
- (ii) $g(P_{n+12}) = g(P_n)$ if $n \geq 72$, and
 $(g(P_k) \mid 72 \leq k < 84) = (7, 4, 1, 2, 8, 1, 4, 7, 2, 1, 8, 4)$.

For convenience, we denote by $K_{1,0}$ a graph with one vertex and no edge. The double star $DS(n,m)$ is a graph obtained from two stars $K_{1,n}$ and $K_{1,m}$ by joining their two centers by a new edge. Then the order of $DS(n,m)$ is $n+m+2$ and its size is $n+m+1$. We now give a conjecture on the Grundy numbers of double stars.

Conjecture B. Suppose that ER-game of star type is played on a double star $DS(n,m)$. Then

- (i) For every positive odd integer n , there exists an integer $M = M(n)$ for which $g(DS(n,m)) = n+m+1$ if $m \geq M$.
- (ii) For every positive even integer n , there exists integers $p = p(n)$ and $M = M(n)$ for which $g(DS(n,m+p)) = g(DS(n,m)) + p$ if $m \geq M$.

We shall show that the conjecture is true if $n = 2^k - 1$, $n = 2^k$ or $1 \leq n \leq 10$. Moreover, by making use of computer, we observe that if $n < 50$ and $m \leq 5000$ then the conjecture holds and that $M(n) < 800$ except $n = 33$ ($M(n) = 1953$), $n = 34$ ($M(n) = 2141$) and $n = 48$ ($M(n) = 2157$), furthermore, we may give a conjecture on $p(n)$ that $p(n) = 2^{k+1}$ if $2^k \leq n < 2^{k+1}$ except $n = 24$ ($p(n) = 64$).

Theorem 1. Suppose that ER-game of star type is played on a double star. Then

- (i) For every integers $k \geq 1$ and $m \geq 0$, we have

$$g(DS(2^k - 1, m)) = 2^k + m.$$

(ii) For every integers $k \geq 1$ and $h \geq 0$, we have

$$g(DS(2^k, h \cdot 2^{k+1} + s)) = h \cdot 2^{k+1} + 2^k + s + 1,$$

where $-1 \leq s \leq 2^k - 1$ and

$$g(DS(2^k, h \cdot 2^{k+1} + 2^k + s)) = h \cdot 2^{k+1} + s + 1,$$

where $0 \leq s \leq 2^k - 2$. In particular,

$$g(DS(2^k, m + 2^{k+1})) = g(DS(2^k, m)) + 2^{k+1} \text{ for all } m \geq 0.$$

Proof We first prove Statement (i). For convenience, let $n = 2^k - 1$. We shall prove that $g(DS(n, m)) = n + m + 1$ by induction on m . Since a double star $DS(n, 0)$ is a star $K_{1, n+1}$, $g(DS(n, 0)) = n + 1$ by Theorem A.

Suppose that $1 \leq m \leq n$. For every integer j , $0 \leq j \leq n$, let $r = m + j$. Then $0 \leq r \leq n$ and $r + m = j$. We can remove a star from the double star $DS(n, m)$ such that the resulting graph is $K_{1, r} \cup K_{1, m}$, whose Grundy number is $r + m = j$. By the induction hypothesis, we have that $g(DS(n, r)) = n + r + 1$ for every $0 \leq r < m$. Therefore $g(DS(n, m)) \geq n + m + 1$. Since $g(DS(n, m)) \leq |E(DS(n, m))| = n + m + 1$, we can conclude that $g(DS(n, m)) = n + m + 1$.

Next assume that $n < m$. For every integer j , $0 \leq j \leq n$, let $r = n + j$. Then $0 \leq r \leq n$ and $n + r = j$. We can remove a star from the double star $DS(n, m)$ such that the resulting graph is $K_{1, n} \cup K_{1, r}$, whose Grundy number is $n + r = j$. By the same argument as above, we can also show that $g(DS(n, m)) = n + m + 1$.

For convenience, we denote the star $K_{1, l}$ by $K(1, l)$. In order to prove Statement (ii) we need to show that the following equation holds.

$$g(DS(h 2^{k+1} - 1, m)) = h 2^{k+1} + m$$

for every integers $0 \leq m \leq 2^k$ and $h \geq 1$. We prove the above equation by induction on m . Let $0 \leq j < 2^k$ and $0 \leq t \leq h - 1$. If $m < 2^k$, then $r = j + 1 < 2^k$, and we can remove stars from $DS(h 2^{k+1} - 1, m)$ such that the resulting graphs are $K(1, t 2^{k+1} + r) \cup K(1, m)$ and $K(1, t 2^{k+1} + 2^k + r) \cup K(1, m)$, whose Grundy numbers are $t 2^{k+1} + j$ and $t 2^{k+1} + 2^k + j$, respectively. If $m = 2^k$ then we can remove stars from $DS(h 2^{k+1} - 1, m)$ such that the resulting graphs are $K(1, t 2^{k+1} + j) \cup K(1, 2^k)$ and $K(1, t 2^{k+1} + 2^k + j) \cup K(1, 2^k)$, whose Grundy numbers are $t 2^{k+1} + 2^k + j$ and $t 2^{k+1} + j$, respectively. By the induction hypothesis, we have that $g(DS(h 2^{k+1} - 1, y)) = h 2^{k+1} + y$ for every $0 \leq y < m$. Thus $g(DS(h 2^{k+1} - 1, m)) \geq h 2^{k+1} + m$, and so $g(DS(h 2^{k+1} - 1, m)) = h 2^{k+1} + m$.

We now prove Statement (ii) by induction on $h 2^{k+1} + s$ or $h 2^{k+1} + 2^k + s$. By Theorem A and the above statement, $g(DS(2^k, 0)) = 2^k + 1$ and $g(DS(2^k, h 2^{k+1} - 1)) = h 2^{k+1} + 2^k$. Consider a double star $DS(2^k, h 2^{k+1} - 1)$, $0 \leq h$, $0 \leq s \leq 2^k - 1$. For every integers $0 \leq t \leq r - 1$, and $0 \leq x < 2^k$, we have that

$$g(K(1, 2^k)) + g(K(1, t 2^{k+1} + 2^k + x)) = t 2^{k+1} + x$$

and

$$g(K(1, 2^k)) + g(K(1, t 2^{k+1} + x)) = t 2^{k+1} + 2^k + x.$$

Moreover,

$$g(K(1, x')) + g(K(1, h 2^{k+1})) = h 2^{k+1} + x' \text{ for } 0 \leq x' \leq 2^k,$$

and for every integer y , $0 \leq y < s$, it follows from the induction hypothesis that

$$g(DS(2^k, h 2^{k+1} + y)) = h 2^{k+1} + 2^k + y + 1.$$

Therefore $g(DS(2^k, h 2^{k+1} + s)) \geq h 2^{k+1} + 2^k + s + 1$, and thus $g(DS(2^k, h 2^{k+1} + s)) = h 2^{k+1} + 2^k + s + 1$.

We next consider a double star $DS(2^k, h 2^{k+1} + 2^k + s)$, $0 \leq s \leq 2^k - 2$.

By the same argument as above, we can easily show that

$$\begin{aligned} & \{g(K(1, 2^k)) \dot{+} g(K(1, y)) \mid 0 \leq y \leq h 2^{k+1} + 2^k + s\} \\ &= \{0, 1, 2, \dots, h 2^{k+1} + s\} \cup \{h 2^{k+1} + 2^k, \dots, (h+1)2^{k+1} - 1\}. \end{aligned}$$

Thus $g(DS(2^k, h 2^{k+1} + 2^k + s)) \geq h 2^{k+1} + s + 1$. It is obvious that for every $0 < t < 2^k$, $DS(t, h 2^{k+1} + 2^k + s)$ contains $K(1, t) \cup K(1, h 2^{k+1} + r)$, $r = t + (s + 1)$, whose Grundy number is $h 2^{k+1} + s + 1$. Therefore $g(DS(t, h 2^{k+1} + 2^k + s)) \neq h 2^{k+1} + s + 1$. Consequently we can conclude that $g(DS(2^k, h 2^{k+1} + 2^k + s)) = h 2^{k+1} + s + 1$.

Theorem 2. The Grundy numbers of double stars $DS(n, m)$, $n \leq 10$, of ER-game of star type are given by the following statements.

(i) If $n = 0$, $n = 1$, $n = 3$, $n = 5$ and $m \geq 15$, $n = 7$ or $n = 9$ and $m \geq 95$ then

$$g(DS(n, m)) = n + m + 1.$$

(ii) Let $p = 4, 8$ or 16 according as $n = 2$, $n = 4, 6$, or $n = 8, 10$. Suppose that $m \geq 15$ if $n = 6$, and $m \geq 110$ if $n = 10$. Then

$$g(DS(n, m + p)) = g(DS(n, m)) + p.$$

Note that Theorem 2 holds for $n = 0, 1, 2, 3, 4, 7, 8$ by Theorem A and Theorem 1. We shall prove the following proposition instead of remaining Statement (ii) of Theorem 2.

Proposition 3. Consider $g(DS(n, m))$ with $n = 2, 4, 6, 8$ or 10 . Let t and s be integers such that $0 \leq t$, and $0 \leq s < 4$, $0 \leq s < 8$, or $0 \leq s < 16$ according as $n = 2$, $n = 4, 6$ or $n = 8, 10$. Then the following statements hold.

(i) $g(DS(2,4t+s)) = 4t+3, 4t+4, 4t+1, 4t+6$ if $s = 0, 1, 2, 3$, respectively.

(ii) $g(DS(4,8t+s)) = 8t+5, 8t+6, 8t+7, 8t+8, 8t+1, 8t+2, 8t+3, 8t+12$ if $s = 0, 1, 2, \dots, 7$, respectively.

(iii) $g(DS(6,15+8t+s)) = 8t+22, 8t+23, 8t+24, 8t+21, 8t+26, 8t+25, 8t+28, 8t+27$ if $s = 0, 1, 2, \dots, 7$, respectively.

(iv) $g(DS(8,16t+s)) = 8t+s+9, 8t+s-7$ or $8t+24$ if $0 \leq s \leq 7, 8 \leq s \leq 14$ or $s = 15$, respectively.

(v) $g(DS(10,110+8t+s)) = 8t+117, 8t+122, 8t+123, 8t+124, 8t+121, 8t+126, 8t+127, 8t+128, 8t+125, 8t+130, 8t+119, 8t+132, 8t+129$ if $s = 0, 1, \dots, 15$, respectively.

2. Grundy Numbers of Forks and trees

A fork $F(n,m)$ is defined to be a graph which is obtained from a star $K_{1,n}$ and a path P_m by joining the center of the star to one of the end vertices of the path by a new edge. Then the order of $F(n,m)$ is $n+m+1$ and its size is $n+m$. Note that $F(0,m) = P_{m+1}$, $F(1,m) = P_{m+2}$, $F(n,0) = K_{1,n}$ and $F(n,1) = K_{1,n+1}$ and these Grundy numbers are given by Theorem A.

Theorem 2. The Grundy numbers of forks $F(n,m)$ with $n \leq 10$ or $m \leq 10$ of ER-game of star type are given by the following statements.

(i) If $n = 2$ and $m \geq 152$, $n = 3$ and $m \geq 141$, $n = 4$ and $m \geq 142$, $n = 5$ and $m \geq 286$, $n = 6$ and $m \geq 286$, $n = 7$ and $m \geq 215$, $n = 8$ and $m \geq 112$, $n = 9$ and $m \geq 141$, or $n = 10$ and $m \geq 190$ then

$$g(F(n,m+12)) = g(F(n,m)).$$

(ii) If $m = 2, m = 3$ and $n \geq 2$, $m = 4$ and $n \geq 5$, $m = 6$ and $n \geq 15$,

or $m = 10$ and $n \geq 30$ then

$$g(F(n, m)) = n + m.$$

(iii) Let $p = 4, 16$ or 8 according as $m = 5, m = 7, 8$ or $m = 9$. Suppose that $n \geq 8, 9, 10$ or 15 if $m = 5, 7, 8,$ or $9,$ respectively. Then

$$g(F(n+p, m)) = g(F(n, m)) + p.$$

Conjecture C. (i) For every positive integer n , there exists an integer $M = M(n)$ for which $g(F(n, m+12)) = g(F(n, m))$ if $m \geq M$.

(ii) For every positive even integer m , there exists integers $p = p(m)$ and $M = M(m)$ for which $g(F(n+p, m)) = g(F(n, m)) + p$ if $n \geq M$.

We finally give some remarks on ER-game of star type on trees and propose a related problem. The Grundy number of every tree with order less than 10 is non-zero, and there exist 16 trees of order 10 and seven trees of order 11 whose Grundy numbers are equal to 0. These trees are given below. Let T_i denote a tree of order 10 or 11 whose Grundy number is 0, and let $V(T_i) = \{1, 2, \dots, 9, a\}$ or $\{1, 2, \dots, 9, a, b\}$. If the order of T_i is 10, then T_i contains a set of edge $F = \{12, 23, 34, 45, 56\}$, and so we denote only $F_i = E(T) - F$.

$F_1 = \{67, 78, 89, 2a\}, F_2 = \{67, 78, 89, 3a\}, F_3 = \{67, 78, 79, 4a\}, F_4 = \{67, 58, 89, 2a\},$
 $F_5 = \{67, 78, 79, 6a\}, F_6 = \{67, 78, 69, 4a\}, F_7 = \{67, 58, 89, 3a\}, F_8 = \{37, 78, 89, 1a\},$
 $F_9 = \{37, 78, 89, 2a\}, F_{10} = \{67, 48, 29, 3a\}, F_{11} = \{37, 78, 29, 7a\}, F_{12} = \{47, 78, 29, 3a\},$
 $F_{13} = \{47, 78, 79, 5a\}, F_{14} = \{67, 78, 79, 7a\}, F_{15} = \{67, 68, 69, 4a\}$ and $F_{16} = \{47, 78, 29, 2a\}.$

If the order of T_i is 11, then the edge set E_i of T_i are given as follows:

$$\begin{aligned}
E_1 &= \{12,23,34,45,56,67,58,89,4a,ab\}, \quad E_2 = \{12,23,34,45,56,37,78,29,4a,4b\}, \\
E_3 &= \{12,23,34,45,56,57,78,59,4a,4b\}, \quad E_4 = \{12,23,34,45,46,67,48,89,3a,3b\}, \\
E_5 &= \{12,23,34,45,56,57,58,59,4a,4b\}, \quad E_6 = \{12,13,14,15,16,67,68,89,7a,7b\} \text{ and} \\
E_7 &= \{12,23,35,56,34,37,78,29,2a,2b\}.
\end{aligned}$$

Problem Characterize trees whose Grundy numbers are equal to 0.

ER-game of path type (i.e. $\mathcal{K}$ is the set of all paths) will be deal with other paper. Is it possible to solve ER-games of the following $\mathcal{K}$ type on certain class of graphs: $\mathcal{K}$ is the set of all cycles, $\mathcal{K}$ is the set of all trees, $\mathcal{K}$ is the set of all matchings, $\mathcal{K}$ is the set of all forests, and so on.

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