

Convergence of stochastic processes on varying metric spaces

Kohei Suzuki

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Contents

1	Introduction	5
1.1	Overview	5
1.2	Convergence of stochastic processes	6
1.3	Riemannian Curvature Dimension condition	7
1.4	Convergence of Brownian motions on RCD spaces converging in the mGH sense	8
1.5	Convergence of stochastic processes on metric spaces conver- gene in the Lipschitz distance	9
1.6	Notation	11
2	Convergence of continuous stochastic processes	12
2.1	Weak convergence of probability measures	12
2.2	Convergence of continuous stochastic processes	15
2.3	Probability measures on $C([0, \infty); X)$	16
2.4	Tightness of the laws of continuous stochastic processes	17
3	Riemannian Curvature Dimension conditions	20
3.1	L^2 -Wasserstein Space	20
3.2	Cheeger's L^2 -energy functional	21
3.3	$\text{RCD}^*(K, N)$ condition	23
4	Convergence of Brownian motions on $\text{RCD}^*(K, N)$ spaces	29
4.1	Brownian motions on $\text{RCD}^*(K, N)$ spaces	29
4.2	Sturm's $\mathbb{D}$ -distance and measured Gromov–Hausdorff conver- gence	31
4.2.1	Sturm's $\mathbb{D}$ -distance	32
4.2.2	mGH-convergence	33
4.2.3	Relation between two convergences	33
4.3	Mosco convergence of Cheeger energies	35

4.4	Equivalence of convergence of Brownian motions and mGH-convergence of the underlying spaces	37
4.5	Proof of Theorem 4.4.1 and Corollary 4.4.2	38
4.5.1	Sketch of proof of Theorem 4.4.1	38
4.5.2	Proof of Theorem 4.4.1	39
4.5.3	Proof of Corollary 4.4.2	50
5	Convergence of continuous stochastic processes on compact metric spaces converging in the Lipschitz distance	51
5.1	Lipschitz distance	51
5.2	Lipschitz–Prokhorov distance	55
5.3	Relative compactness	62
5.4	Examples	72
5.4.1	Brownian motions on Riemannian manifolds	72
5.4.2	Uniformly elliptic diffusions on Riemannian manifolds	75

Chapter 1

Introduction

1.1 Overview

Convergence of stochastic processes is one of basic and important themes in probability theory, which is defined as the weak convergence of probability measures on path spaces. The most typical and important limit theorem is the so-called *Donsker's invariance principle* Donsker [29], which states that random walks on $\mathbb{Z}^d$ converge to the Brownian motion on $\mathbb{R}^d$ in some proper scaling limit. There are a long history and a numbers of results of convergence of stochastic processes. For example in recent studies, in Stroock–Varadhan [78], Stroock–Zheng [79] and Burdzy–Chen [21], they studied approximations of diffusion processes on $\mathbb{R}^d$ by discrete Markov chains on $\mathbb{Z}^d$. In Bass–Kumagai–Uemura [11] and Chen–Kim–Kumagai [22], they investigated approximations of jump processes on proper metric spaces by Markov chains on discrete graphs, and on ultra-metric spaces in Suzuki [84]. In Stroock–Varadhan [78], and Trevisan [87, 88], they studied stability with respect to coefficients of martingale problems. In Ogura [69], he considered convergence of Brownian motions on Riemannian manifolds converging in the sense of Kasue–Kumura's spectral convergence [51]. In Albeverio and Kusuoka [1], they considered diffusion processes associated with SDEs on thin tubes in $\mathbb{R}^d$ shrinking to one-dimensional spider graphs. In Kumagai–Sturm [60], they constructed diffusions by Γ -limit of non-local Dirichlet forms. In Croydon [27, 28], he studied convergence of random walks on trees converging in the sense of the measured Gromov–Hausdorff. In Athreya–Lohr–Winter [8], they studied convergence of diffusions on trees converging in the sense of the measured Gromov–Hausdorff. In Kotani–Sunada [58], and Ishiwata–Kawabi–Kotani [45] (see references therein for

complete references), they studied invariance principles of random walks on crystal lattices. There are many studies about scaling limits of random processes on random environments (see, e.g., Kumagai [59] and references therein). See, e.g., Billingsley [14] for more historically comprehensive articles.

By recent development of geometric analysis, one can deal with differential calculus on *non-smooth spaces* which appear, e.g., in limit spaces of smooth Riemannian manifolds. These spaces are no more manifolds, but have some metric measure structures and one can consider differential geometric objects such as gradients, Laplacians, heat equations, curvatures and so on (see, e.g., Villani [90] and Heinonen–Koskela–Shanmugalingam–Tyson [39] for comprehensive descriptions). In these studies, it is important to know stability of these analytical objects on varying metric spaces, such as spectra of Laplacian (Fukaya [33], Cheeger–Colding [23, 24, 25], Kuwae–Shioya [62, 63], Shioya [77], Gigli–Mondino–Savaré [35]), heat kernels (Kasue–Kumura [51, 52]), and various analytical objects (Honda [40, 41, 42, 43]). From probabilistic view points, most of such analytical objects have rich connections to Markov processes (see, e.g., Bakry–Gentil–Ledoux [10]), and it is natural to consider stability of Markov processes (or, stochastic processes in general) under varying metric spaces.

In this thesis, we deal with convergence of stochastic processes on varying metric spaces in the sense of measured Gromov–Hausdorff and in the sense of Lipschitz. We have two objectives in this thesis. The first, which will be discussed in Chapter 4, is to show equivalence between the measured Gromov–Hausdorff convergence of metric measure spaces and the weak convergence of the laws of Brownian motions under the Riemannian curvature dimension condition (Theorem 4.4.1). The second, which will be discussed in Chapter 5, is to formulate a notion of convergence of stochastic processes on varying metric spaces in the Lipschitz sense, and study topological properties of this new topology. As preparation for Chapter 4 and Chapter 5, we study basics of convergence of stochastic processes in Chapter 2 and the Riemannian curvature dimension condition in Chapter 3.

1.2 Convergence of stochastic processes

Given a continuous stochastic process S on a complete separable metric space X with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the process S becomes a measurable map from Ω to the space of continuous paths $C([0, \infty); X)$. Thus S induces a probability measure $\mathbb{S}$ on the space of continuous paths $C([0, \infty); X)$

as the push-forwarded measure $\mathbb{S}_n = S_{\#}\mathbb{P}$. A sequence of stochastic processes S_n *converges to* S if $\mathbb{S}_n$ converges to $\mathbb{S}$ weakly as probability measures on $C([0, \infty); X)$. In other words, $\mathbb{S}_n(f) \rightarrow \mathbb{S}(f)$ for any bounded continuous function f on $C([0, \infty); X)$, where $\mathbb{S}_n(f)$ means the integral of f with respect to $\mathbb{S}_n$. We sometimes call it *the weak convergence* of S_n to S .

To show the weak convergence of $\{\mathbb{S}_n\}_{n \in \mathbb{N}}$, usually we have two steps. The first is to show relative compactness of $\{\mathbb{S}_n\}_{n \in \mathbb{N}}$, or in other words, existence of a converging subsequence. The second is uniqueness of limit points. The relative compactness is equivalent to tightness, that is, for any $\varepsilon > 0$, there exists a compact set $K \subset C([0, \infty); X)$ so that $\inf_n \mathbb{S}_n(K) \geq 1 - \varepsilon$. If all S_n 's are Markov processes, one of sufficient conditions for tightness is a priori heat kernel estimate. On the other hand, uniqueness of limit points can be obtained by the weak convergence of finite-dimensional distributions (abbreviated as CFD). If all S_n 's are Markov processes associated with symmetric Dirichlet forms $\mathcal{E}_n$, the CFD follows from Mosco convergence of $\mathcal{E}_n$ (Mosco [67, 68], Kuwae–Shioya [62, 63], Gigli–Mondino–Savaré [35]).

1.3 Riemannian Curvature Dimension condition

To define any types of curvatures of spaces, we usually need twice differentiable structures because we use information of Hessian of metrics. Recent geometric analysis, however, enables us to investigate curvatures on more general spaces, which do not have necessarily usual differentiable structures. In particular, various generalized notions that Ricci curvatures are bounded from below were introduced by several authors such as Sturm [82, 83], Lott–Villani [65], Bacher–Sturm [9], Ambrosio–Gigli–Savaré [4], Erbar–Kuwada–Sturm [30] and Ambrosio–Mondino–Savaré [7], or Ollivier [70], Bonciocat–Sturm [19] and Bonciocat [18]. A basic idea for these generalization of Ricci curvature bound is to use convex analysis on the space of probability measures on the underlying spaces. They characterize “Ricci $\geq K$ ” by “Convexity of entropy on the space of probability measures”, and the latter notion do not need to assume the twice differentiable structure on spaces.

The $\text{RCD}^*(K, N)$ (*Riemannian curvature dimension*) condition was first introduced by Erbar–Kuwada–Sturm [30] and Ambrosio–Mondino–Savaré [7], which is the class of metric measure spaces satisfying *the reduced curvature dimension condition* $\text{CD}^*(K, N)$ (Bacher–Sturm [19]) with the Cheeger energies being *quadratic*. Roughly speaking, $\text{RCD}^*(K, N)$ condition is a generalization of Ricci curvature is bounded from below by K and dimension is bounded from above by N to non-smooth metric measure spaces. When

we consider complete Riemannian manifolds, the $\text{RCD}^*(K, N)$ condition is equivalent to the condition that Ricci curvature is bounded from below by K and dimension is bounded from above by N (Sturm [82, 83]). We will recall the precise definition of the $\text{RCD}^*(K, N)$ in Section 3.3.

Typical examples for $\text{RCD}^*(K, N)$ spaces are measured Gromov–Hausdorff limit spaces of N -dimensional complete Riemannian manifolds with Ricci $\geq K$, and N -dimensional Alexandrov spaces with curvature $\geq K/(N-1)$ (see Kuwae–Machigashira–Shioya [61], Petrunin [72] and Zhang–Zhu [91]). Note that the Sierpiński gasket $(K, \rho_{\mathcal{H}}, \mu)$ equipped with the harmonic geodesic metric $\rho_{\mathcal{H}}$ and the Kusuoka measure μ does not satisfy the $\text{CD}^*(K, N)$ condition (see Kajino [48]).

As in the case of Riemannian manifolds whose Ricci curvatures are bounded from below, $\text{RCD}^*(K, N)$ spaces also have many useful functional inequalities or estimate of geometric quantities such as Poincaré inequality, Sobolev inequalities, Bishop–Gromov inequalities and Bonnet–Myers diameter estimates. We recall these inequalities in Section 3.3. In $\text{RCD}^*(K, N)$ spaces, we have a canonical energy form, called *Cheeger energy*. This energy form becomes strongly local regular symmetric Dirichlet forms and induces a unique diffusion process. We call it *Brownian motions*.

1.4 Convergence of Brownian motions on RCD spaces converging in the mGH sense

Brownian motions are canonical Markov processes in the sense that they are characterized only by metric measure structures of the underlying spaces. The one-dimensional Brownian motion $\mathbb{B}$ is, for example, a unique Markov process associated with the following energy form:

$$\mathbb{B} \rightsquigarrow \frac{1}{2} \int_{\mathbb{R}} |\nabla f|^2 dx, \quad (1.4.1)$$

where ∇ means the usual differentiation and dx means the Lebesgue measure. On a complete Riemannian manifold (M, g) , the Brownian motion is defined by unique Markov process associated with the following energy form:

$$\mathbb{B} \rightsquigarrow \frac{1}{2} \int_M |\nabla_g f|^2 dm_g, \quad (1.4.2)$$

where ∇_g denotes the gradient operator and m_g denotes the Riemannian volume measure associated with the metric g . A more general framework

is the class of metric measure spaces $\mathcal{X} = (X, d, m)$ satisfying the RCD condition. The Brownian motion on $\mathcal{X}$ is defined by a unique Markov process associated with the following Cheeger energy:

$$(\textbf{Brownian motion}) \quad \mathbb{B} \rightsquigarrow \frac{1}{2} \int_X |\nabla f|_w^2 dm \quad (\textbf{metric measure structure}), \quad (1.4.3)$$

where $|\nabla f|_w$ means the minimal weak upper gradient of f (Ambrosio–Gigli–Savaré [4]). We refer the reader to Fukushima–Oshima–Takeda [34] as a standard reference of Markov processes and energy forms.

The correspondence (1.4.3) means that the Brownian motions are characterized only by information of metric measure structures of the underlying spaces. Therefore some properties of the Brownian motions and of the underlying metric measure spaces must be related to each other.

In Chapter 4, we focus on relations between convergence of the underlying spaces and convergence of the Brownian motions. The following is the main questions in this chapter:

- (Q) Does convergence of the laws of Brownian motions follow from some geometrical convergence of the underlying spaces (or, vice versa)?

Since there is no standard notion of convergence of stochastic processes on varying spaces, we should clarify in what sense the convergence of the Brownian motions in the question (Q) is. See the statement (ii) of Theorem 4.4.1 for the precise meaning.

As one of the main results of this theses, which will be stated in Theorem 4.4.1 in Chapter 4, we show that the weak convergence of the laws of Brownian motions in some common space is *equivalent* to the measured Gromov–Hausdorff convergence of the underlying metric measure spaces under the $\text{RCD}^*(K, N)$ condition with uniform diameter bounds.

1.5 Convergence of stochastic processes on metric spaces convergene in the Lipschitz distance

One of main themes in this these is to formulate a convergence of continuous stochastic processes on compact metric spaces converging *in the Lipschitz distance*.

For compact metric spaces X and Y , the *Lipschitz distance* $d_L(X, Y)$ is defined to be the infimum of $\varepsilon \geq 0$ such that an ε -isometry $f : X \rightarrow Y$ exists.

Here a bi-Lipschitz homeomorphism $f : X \rightarrow Y$ is called an ε -isometry if

$$|\log \text{dil}(f)| + |\log \text{dil}(f^{-1})| \leq \varepsilon,$$

where $\text{dil}(f)$ denotes the smallest Lipschitz constant of f , called *the dilation of f* :

$$\text{dil}(f) = \sup_{\substack{x, y \in X \\ x \neq y}} \frac{d(f(x), f(y))}{d(x, y)}.$$

Let $\mathcal{M}$ be the set of isometry classes of compact metric spaces. It is well-known that $(\mathcal{M}, d_L)$ is a complete metric space. See, e.g., Burago–Burago–Ivanov [20] and Gromov [37] for details of the Lipschitz distance. We say that X_n converges to X in the sense of Lipschitz if $d_L(X_n, X) \rightarrow 0$ as $n \rightarrow \infty$.

The main object of this chapter is to introduce a new distance among pairs (X, P) where X is a compact metric space and P is the law of a continuous stochastic process on X . Let $\mathcal{PM}$ be the set of all pairs (X, P) modulo by an isomorphism relation (defined in Section 5.2). We will define (in Section 5.2) a new distance on $\mathcal{PM}$, which we will call *the Lipschitz–Prokhorov distance d_{LP}* , as a kind of mixture of the Lipschitz distance and the Prokhorov distance. The Lipschitz distance is a distance on the set of isometry classes of metric spaces, which was first introduced by Gromov (see e.g., [37]).

We summarize our results as follows:

- (A) $(\mathcal{PM}, d_{LP})$ is a complete metric space (Theorem 5.2.5 and Theorem 5.2.7);
- (B) Relative compactness in $(\mathcal{PM}, d_{LP})$ follows from bounds for sectional curvatures, diameters and volumes of Riemannian manifolds and uniform heat kernel estimates of Markov processes (Theorem 5.3.4);
- (C) Sequences in a relatively compact set are convergent if the corresponding Dirichlet forms of Markov processes are Mosco-convergent in the sense of Kuwae–Shioya [62] (Theorem 5.3.9);
- (D) Examples for
 - Brownian motions on Riemannian manifolds (Section 5.4.1);
 - uniformly elliptic diffusions on Riemannian manifolds (Section 5.4.2).

We use heat kernel estimates by Kasue–Kumura [52] to show tightness and use Mosco convergence to show the weak convergence of the finite-dimensional distributions of the Brownian motions.

1.6 Notation

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Let $\mathbb{R}$ be the set of real numbers.

Let (X, d) be a complete separable metric space. We denote by $\mathcal{B}(X)$ the family of all Borel sets in (X, d) . We write $B_r(x) = \{y \in X : d(x, y) < r\}$ for a open ball centered at $x \in X$ with radius $r > 0$.

Let $C(X)$ denote the set of real-valued continuous functions on X . Let $C_b(X)$, $C_0(X)$ and $C_{bs}(X)$ denote the subsets of $C(X)$ consisting of bounded functions, functions with compact support and bounded functions with bounded support, respectively. We denote by $\mathcal{B}_b(X)$ the set of real-valued bounded Borel-measurable functions on X . Let $\mathcal{P}(X)$ denote the set of Borel probability measures on X .

We say that $\gamma : [a, b] \rightarrow X$ is a *curve on X* if $\gamma : [a, b] \rightarrow X$ is continuous. A curve $\gamma : [a, b] \rightarrow X$ is said to be *connecting x and y* if $\gamma_a = x$ and $\gamma_b = y$. A curve $\gamma : [a, b] \rightarrow X$ is said to be a *minimal geodesic* if

$$d(\gamma_t, \gamma_s) = \frac{|s - t|}{|b - a|} d(\gamma_a, \gamma_b) \quad a \leq t \leq s \leq b.$$

In particular, if $\frac{d(\gamma_a, \gamma_b)}{|b - a|}$ can be replaced to 1, we say that γ is *unit-speed*.

Let (Y, d_Y) be a complete separable metric space. For a Borel measurable map $f : X \rightarrow Y$, let $f_{\#}m$ denote the push-forward measure on Y :

$$f_{\#}m(B) = m(f^{-1}(B)) \quad \text{for any Borel set } B \in \mathcal{B}(Y).$$

For $f \in L^1(X, m)$, we sometimes write $m(f) := \int_X f \, dm$.

Chapter 2

Convergence of continuous stochastic processes

In this chapter, we study convergence of continuous stochastic processes on a fixed underlying space. We refer the reader to Billingsley [14] for a standard reference. In the case of càdlàg processes, we refer the reader to Jacod–Shriyaev [46] and Ethier–Kurtz [32].

2.1 Weak convergence of probability measures

Let (X, d) be a complete separable metric space. Recall that $C_b(X)$ denotes the set of bounded continuous functions on X and $\mathcal{P}(X)$ denotes the set of Borel probability measures on X .

A sequence of probability measures $P_n \in \mathcal{P}(X)$ *converges weakly* to $P \in \mathcal{P}(X)$ if

$$\int_X f dP_n \rightarrow \int_X f dP \quad \forall f \in C_b(X).$$

We sometimes write $P(f) := \int_X f dP$ shortly. In this notation, P_n converges weakly to P if $P_n(f) \rightarrow P(f)$ for any $f \in C_b(X)$. We note that the weak convergence is a topological notion. This means that the weak convergence does not depend on specific metrics d on X , but depends only on the topology generated by d . The weak convergence gives rise to a topology in $\mathcal{P}(X)$ and this topology is metrizable. There are several ways to equip metrics (see, e.g., Bogachev [17, Theorem 8.3.2]). We recall the so-called *Lévy–Prokhorov metric* (or, just *Prokhorov metric*):

Definition 2.1.1 (Lévy–Prokhorov metric) For $\mu_1, \mu_2 \in \mathcal{P}(X)$, the Lévy–Prokhorov metric $d_P(\mu_1, \mu_2)$ is defined as follows:

$$d_P(\mu_1, \mu_2) := \inf\{\varepsilon > 0 : \mu_1(A) \leq \mu_2(A^\varepsilon) + \varepsilon, \mu_2(A) \leq \mu_1(A^\varepsilon) + \varepsilon\},$$

where $A^\varepsilon := \{x \in X : d(x, A) := \inf_{y \in A} d(x, y) < \varepsilon\}$.

The Prokhorov metric d_P induces the equivalent topology of the weak convergence. In other words, $d_P(\mu_n, \mu) \rightarrow 0$ if and only if $\mu_n \rightarrow \mu$ weakly. Note that, without the condition of separability of (X, d) , the topology induced by d_P is stronger than the weak convergence (see [32, Chapter 3, Theorem 3.1]). It is known that $(\mathcal{P}(X), d_P)$ becomes a complete and separable metric space whenever (X, d) is complete and separable (see, e.g., [32, Chapter 3, Theorem 1.7]).

The notion of the weak convergence can be realized as the almost-sure convergence of random variables on a certain probability space.

Proposition 2.1.2 (Skorokhod representation theorem) Let $P_n, P \in \mathcal{P}(X)$ for $n \in \mathbb{N}$. Assume that P_n converges weakly to P . Then there exists a probability space $(\Omega, \mathcal{F}, Q)$ on which are defined X -valued measurable maps R_n and R with each distributions $R_{n\#}Q = P_n$ and $R_{\#}Q = P$ so that

$$\lim_{n \rightarrow \infty} R_n = R \quad Q\text{-a.s.}$$

Here recall that $\#$ means the push-forward. See [32, Chapter 3, Theorem 1.8] for a proof.

To show the weak convergence, the following equivalent statements are sometimes useful, called *Portmanteau theorem*:

Proposition 2.1.3 (Portmanteau theorem) Let $P_n, P \in \mathcal{P}(X)$. Then the following are equivalent:

(i) P_n converges weakly to P , that is, for any $f \in C_b(X)$,

$$P_n(f) \rightarrow P(f).$$

(ii) For any bounded uniformly continuous real-valued function f on X ,

$$P_n(f) \rightarrow P(f).$$

(iii) For any closed set $A \subset X$,

$$\limsup_{n \rightarrow \infty} P_n(A) \leq P(A).$$

(iv) For any open set $B \subset X$,

$$\liminf_{n \rightarrow \infty} P_n(B) \geq P(B).$$

(v) For any subset $C \subset X$ satisfying $P(\partial C) = 0$,

$$\lim_{n \rightarrow \infty} P_n(C) = P(C),$$

where ∂C denotes the boundary of C , i.e., $\partial C := \overline{C} \setminus C$.

One of the important question is which kind of subsets in $\mathcal{P}(X)$ are compact, or relatively compact (sometimes called precompact) with respect to the topology of the weak convergence. Here we mean that a subset $S \subset \mathcal{P}(X)$ is relatively compact (or, precompact) if, for every sequence $\{P_n\}_{n \in \mathbb{N}} \subset S$, there exists a converging subsequence $\{P_{n'}\}_{n' \in \mathbb{N}} \subset \{P_n\}_{n \in \mathbb{N}}$ and $P' \in \mathcal{P}(X)$ (P' is not necessarily required in S) so that

$$P_{n'} \xrightarrow{weak} P'.$$

We define a notion of *tightness* for a subset $S \subset \mathcal{P}(X)$.

Definition 2.1.4 A subset $S \subset \mathcal{P}(X)$ is *tight* if, for any $\varepsilon > 0$, there exists a compact set $K \subset X$ so that

$$\inf_{P \in S} P(K) \geq 1 - \varepsilon.$$

The following *Prokhorov theorem* states that tightness implies relative compactness. If, moreover, (X, d) is complete and separable, then a subset $S \subset \mathcal{P}(X)$ is relatively compact if and only if S is *tight*.

Proposition 2.1.5 (Prokhorov theorem) *Let (X, d) be a metric space. The following statements hold.*

- (i) *A subset $S \subset \mathcal{P}(X)$ is relatively compact if S is tight.*
- (ii) *If (X, d) is complete and separable, S is relatively compact if and only if S is tight.*

See [32, Chapter 3, Theorem 2.2, Corollary 2.3] for proofs.

2.2 Convergence of continuous stochastic processes

In this section, we recall convergence of continuous stochastic process. The convergence of continuous stochastic processes is defined as the weak convergence of probability measures on the space of continuous paths $C([0, \infty); X) := \{w : [0, \infty) \rightarrow X, \text{continuous}\}$.

Let (X, d) be a complete and separable metric space. We equip $C([0, \infty); X)$ with the following metric:

$$\rho(w, v) := \int_0^\infty e^{-T} \left(\sup_{t \in [0, T]} (d(w(t), v(t)) \wedge 1) \right) dT,$$

where $a \wedge b := \max\{a, b\}$. We sometimes write $\hat{d} = d \wedge 1$ for short. The metric ρ induces the topology of the uniform convergence on compacts. It is known that $(C([0, \infty); X), \rho)$ is a complete and separable metric space.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family of maps $\{S_t\}_{t \geq 0}$ is said to be *stochastic process on X* if $S_t : \Omega \rightarrow X$ is $\mathcal{F}/\mathcal{B}(X)$ -measurable for any $t \geq 0$. Recall that $\mathcal{B}(X)$ denotes the family of Borel sets in X , and $\mathcal{F}/\mathcal{B}(X)$ -measurable means that $S_t^{-1}(A) \in \mathcal{F}$ for any $A \in \mathcal{B}(X)$. A stochastic process $\{S_t\}_{t \geq 0}$ on X is said to be *continuous* if there exists a measurable set $\Omega_0 \in \mathcal{F}$ so that $P(\Omega_0) = 1$ and

$$S_t(\omega) \text{ is continuous in } t \text{ for any } \omega \in \Omega_0.$$

Given a continuous stochastic process S , without loss of generality, we may assume that S is continuous for any $\omega \in \Omega$ by replacing $(\Omega, \mathcal{F}, P)$ to $(\Omega_0, \mathcal{F}|_{\Omega_0}, P|_{\Omega_0})$ where $\mathcal{F}|_{\Omega_0} := \sigma\{A \cap \Omega_0 : A \in \mathcal{F}\}$ and $P|_{\Omega_0}$ is the restricted measure of P on $\mathcal{F}|_{\Omega_0}$. We note that the definition of continuous stochastic processes does not assume that S is a *random variable* from Ω to $C([0, \infty); X)$, i.e., $S^{-1}(C) \in \mathcal{F}$ for any $C \in \mathcal{B}(C([0, \infty); X))$. However, under the assumption of the separability of (X, d) , if $\{S_t\}_{t \geq 0}$ is continuous, then S becomes automatically a $C([0, \infty); X)$ -valued random variable.

Proposition 2.2.1 *If $\{S_t\}_{t \geq 0}$ is a continuous stochastic process, then S is a $C([0, \infty); X)$ -valued random variable.*

Proof. By the separability of (X, d) , we have that $C([0, \infty); X)$ is also separable under the topology of the uniform convergence on compacts. Thus it suffices to show that, for each closed ball $\overline{B}_r(v) := \{w \in C([0, \infty); X) : \rho(v, w) \leq r\} \subset C([0, \infty); X)$, it holds that $S^{-1}(\overline{B}_r(v)) \in \mathcal{F}$. Since

$$S^{-1}(\overline{B}_r(v)) = \{\omega \in \Omega : S(\omega) \in \overline{B}_r(v)\} = \bigcap_{q \in \mathbb{Q}_+} \{\omega : \hat{d}(S_q(\omega), v(q)) \leq r\},$$

($\mathbb{Q}_+$ means the set of non-negative rationals and $\hat{d} := d \wedge 1$) and S_q is measurable for each q by the definition of stochastic processes, we have the desired result. $\square$

By Proposition 2.2.1, we can consider the push-forward measure of P by S . We denote the push-forward measure of P with respect to S by $S_{\#}P$, i.e., $S_{\#}P(A) := P(S^{-1}(A))$ for $A \in \mathcal{B}(C([0, \infty); X))$. The probability measure $S_{\#}P$ on $C([0, \infty); X)$ is called *the law of $\{S_t\}_{t \geq 0}$* .

Definition 2.2.1 (Convergence of continuous stochastic processes)

Let $\{S_t^n\}_{t \geq 0}$ and $\{S_t\}_{t \geq 0}$ be continuous stochastic processes on X with probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$ and $(\Omega, \mathcal{F}, P)$ respectively. We say that $\{S_t^n\}_{t \geq 0}$ *converges to $\{S_t\}_{t \geq 0}$* if

$$\text{the laws } S_{\#}^n P^n \text{ converges weakly to } S_{\#}P$$

in $\mathcal{P}(C([0, \infty); X))$. We sometimes say that $\{S_t^n\}_{t \geq 0}$ *converges in law to $\{S_t\}_{t \geq 0}$* .

2.3 Probability measures on $C([0, \infty); X)$

As we have seen in the previous section, the convergence of continuous stochastic processes is defined as the weak convergence of their laws in $C([0, \infty); X)$. Thus, to study convergence of stochastic processes, we need to know topological properties about the weak convergence of probability measures on $C([0, \infty); X)$.

It is sometimes useful to consider subclass of Borel σ -fields instead of the whole Borel σ -fields. A *determining class* $\mathcal{D} \subset \mathcal{B}(C([0, \infty); X))$ means that, for any two probability measures P, Q on $C([0, \infty); X)$, if $P(A) = Q(A)$ for any $A \in \mathcal{D}$, then $P = Q$. A determining set $\mathcal{D}$ says that it suffices for coincidence of P and Q to see values only on $\mathcal{D}$.

We recall the class of cylinder sets. For $k \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \dots < t_k < \infty$, let $\pi_{t_1, t_2, \dots, t_k} : C([0, \infty); X) \rightarrow X^k$ be defined by

$$\pi_{t_1, t_2, \dots, t_k}(w) = (w(t_1), w(t_2), \dots, w(t_k)).$$

Let $\mathcal{C}$ be *class of cylinder sets*, i.e.,

$$\mathcal{C} := \{\pi_{t_1, t_2, \dots, t_k}^{-1}(A) : A \in \mathcal{B}(X^k), k \in \mathbb{N}, 0 \leq t_1 < t_2 < \dots < t_k < \infty\}.$$

The following Proposition states that the class of cylinder sets in $C([0, \infty); X)$ is a determining set.

Proposition 2.3.1 *Let P and Q be Borel probability measures on $C([0, \infty); X)$. If $P(A) = Q(A)$ for any $A \in \mathcal{C}$, then $P = Q$.*

See [14, Example 1.2, 1.3] for a proof.

For a probability measure P on $C([0, \infty); X)$, the class of push-forwarded probability measures $\pi_{t_1, t_2, \dots, t_k \#} P$ on X^k is called *finite-dimensional distributions* of P . To show the weak convergence of P_n to P in $\mathcal{P}(C([0, \infty); X))$, it is not enough to show the weak convergence of finite-dimensional distributions $\pi_{t_1, t_2, \dots, t_k \#} P_n \rightarrow \pi_{t_1, t_2, \dots, t_k \#} P$ (see, e.g., [14, Example 1.3, 2.5]). However, if we know that $\{P_n\}_{n \in \mathbb{N}}$ is relatively compact, it suffices for the uniqueness of limit points of $\{P_n\}_{n \in \mathbb{N}}$ to show the weak convergence of finite-dimensional distributions.

Proposition 2.3.2 *Let P_n and P be probability measures on $C([0, \infty); X)$. Assume the following:*

- (i) $\{P_n\}_{n \in \mathbb{N}}$ is tight
- (ii) each finite-dimensional distributions of P_n converges weakly to those of P , i.e., for any $k \in \mathbb{N}$ and $t_1, t_2, \dots, t_k \in [0, \infty)$,

$$\pi_{t_1, t_2, \dots, t_k \#} P_n \xrightarrow{\text{weak}} \pi_{t_1, t_2, \dots, t_k \#} P \quad \text{in } \mathcal{P}(X^k).$$

Then P_n converges weakly to P .

Proof. By the equivalence of tightness and relative compactness from Proposition 2.1.5, there exists a convergence subsequence (we also write $\{P_n\}_{n \in \mathbb{N}}$) to some P' . By the assumption (ii), $P' = P$ on the class of cylinder sets $\mathcal{C}$. By Proposition 2.3.1, the class of cylinder sets $\mathcal{C}$ is a determining class. Thus we conclude $P' = P$. We finish the proof. $\square$

2.4 Tightness of the laws of continuous stochastic processes

In Proposition 2.3.2 in the previous section, we see that tightness and the weak convergence of finite-dimensional distributions imply the weak convergence. The main objective in this section is to characterize tightness of the laws of continuous stochastic processes and give some sufficient conditions.

For $v \in C([0, \infty); X)$, $T > 0$ and $0 < \delta < T$, the modulus of continuity $m(v, \delta, T)$ is defined by

$$m(v, \delta, T) := \sup\{d(v(s), v(t)) : t, s \in [0, T], |t - s| \leq \delta\}.$$

Proposition 2.4.1 *A sequence $\{P_n\}_{n \in \mathbb{N}} \subset \mathcal{P}(C([0, \infty); X))$ is tight if and only if the following two statements hold*

- (i) *For any $\varepsilon > 0$, there exists a positive real number a and some $x_0 \in X$ so that*

$$\sup_{n \in \mathbb{N}} P_n(v : d(v(0), x_0) > a) \leq \varepsilon.$$

- (ii) *For any $\eta > 0$ and $\varepsilon > 0$ and $T > 0$, there exists a positive real number $0 < \delta < T$ and $n_0 \in \mathbb{N}$ so that*

$$\sup_{n \geq n_0} P_n(v : m(v, \delta, T) \geq \varepsilon) \leq \eta.$$

See [14, Theorem 8.2] for a proof. A key for the proof is to apply the Ascoli–Arzèà theorem.

We note that, by Proposition 2.4.1, when we consider the laws $\{\mathbb{S}_n\}_{n \in \mathbb{N}}$ of continuous stochastic processes $\{\{S_t^n\}_{t \geq 0} : n \in \mathbb{N}\}$ with probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$, the sequence of the laws $\{\mathbb{S}_n\}_{n \in \mathbb{N}}$ is tight if and only if

- (i) for any $\varepsilon > 0$, there exists a positive real number a and some $x_0 \in X$ so that

$$\sup_{n \in \mathbb{N}} P_n(\omega : d(S_0^n(\omega), x_0) > a) \leq \varepsilon;$$

- (ii) for any $\eta > 0$ and $\varepsilon > 0$ and $T > 0$, there exists a positive real number $0 < \delta < T$ and $n_0 \in \mathbb{N}$ so that

$$\sup_{n \geq n_0} P_n(\omega : m(S_0(\omega), \delta, T) \geq \varepsilon) \leq \eta.$$

A useful criteria for tightness is to estimate moments of processes.

Proposition 2.4.2 *The laws $\{\mathbb{S}_n\}_{n \in \mathbb{N}} \subset \mathcal{P}(C([0, \infty); X))$ is tight if*

- (i) *the initial distribution is tight, i.e.,*

$$\text{the laws of } \{S_0^n\}_{n \in \mathbb{N}} \text{ is tight in } \mathcal{P}(X);$$

- (ii) *for each $T > 0$, there exist constants $C > 0$, $\gamma > 0$, $\alpha > 1$ so that, for any $t \in [0, T]$*

$$\sup_{n \in \mathbb{N}} E_n(\hat{d}(S_{t+h}^n, S_t^n)^\gamma) \leq Ch^\alpha \quad (0 < \forall h \leq 1).$$

Here we mean that E_n is expectation, i.e., $E_n(f) := \int_{\Omega_n} f(\omega) dP_n(\omega)$. Recall $\hat{d} := d \wedge 1$.

See [14, Theorem 12.3] and [32, Chapter 3 Corollary 8.10].

Remark 2.4.3 In application, we often consider stochastic processes to be Markov processes. In this case, if we know a priori heat kernel estimate, we can check the moment estimate (ii) of Proposition 2.4.2. In Chapter 4, we use Gaussian heat kernel estimate to show tightness of Brownian motions on $\text{RCD}^*(K, N)$ spaces. See Lemma 4.5.1 for precise arguments. In Chapter 5, we use a kind of general heat kernel estimate to show tightness. See Theorem 5.3.4. Moreover, if we know Mosco convergence of Dirichlet forms, or equivalently, L^2 -convergence of semigroups, we can show the weak convergence of finite dimensional distributions of the corresponding Markov processes. See Lemma 4.5.2 in Chapter 4, and Theorem 5.3.9 in Chapter 5.

Chapter 3

Riemannian Curvature Dimension conditions

In Erbar–Kuwada–Sturm [30] and Ambrosio–Mondino–Savaré [7], they introduced Riemannian Curvature Dimension conditions $(\text{RCD}^*(K, N))$ for metric measure spaces. These conditions generalize the notion of Ricci curvatures bounded below and dimension bounded above by using convex analysis of Entropy on Wasserstein spaces. In this section, we recall the definition of $\text{RCD}^*(K, N)$ spaces and state properties satisfied by $\text{RCD}^*(K, N)$ spaces.

In this chapter, we say that (X, d, m) is a *metric measure space* if

- (i) (X, d) is a complete separable metric space;
- (ii) m is a non-zero Borel measure on X which is locally finite in the sense that $m(B_r(x)) < \infty$ for all $x \in X$ and sufficiently small $r > 0$.

Let $\text{supp}[m] = \{x \in X : m(B_r(x)) > 0, \forall r > 0\}$ denote the support of m .

3.1 L^2 -Wasserstein Space

Let (X_i, d_i) ($i = 1, 2$) be complete separable metric spaces. For $\mu_i \in \mathcal{P}(X_i)$, a probability measure $q \in \mathcal{P}(X_1 \times X_2)$ is called a *coupling* of μ_1 and μ_2 if

$$\pi_{1\#}q = \mu_1 \quad \text{and} \quad \pi_{2\#}q = \mu_2,$$

where π_i ($i = 1, 2$) is the projection $\pi_i : X_1 \times X_2 \rightarrow X_i$ as $(x_1, x_2) \mapsto x_i$. We denote by $\Pi(\mu, \nu)$ the set of all coupling of μ and ν .

Let (X, d) be a complete separable metric space. Let $\mathcal{P}_2(X)$ be the subset of $\mathcal{P}(X)$ consisting of all Borel probability measures μ on X with finite second moment:

$$\int_X d^2(x, \bar{x}) d\mu(x) < \infty \quad \text{for some (and thus any) } \bar{x} \in X.$$

We endow $\mathcal{P}_2(X)$ with the quadratic transportation distance W_2 , called *L^2 -Wasserstein distance*, defined as follows:

$$W_2(\mu, \nu) = \left(\inf_{q \in \Pi(\mu, \nu)} \int_{X \times X} d^2(x, y) dq(x, y) \right)^{1/2}. \quad (3.1.1)$$

A coupling $q \in \Pi(\mu, \nu)$ is called *an optimal coupling* if q attains the infimum in the equality (3.1.1). It is known that, for any μ, ν , there always exists an optimal coupling q of μ and ν (e.g., [90, §4]).

It is known that $(\mathcal{P}_2(X), W_2)$ is a complete separable metric space (e.g., [90, Theorem 6.18]).

3.2 Cheeger's L^2 -energy functional

In this subsection, we follow [5] to define Cheeger's L^2 -energy functional on metric measure spaces.

Let (Z, d_Z) be a complete separable metric space and $I \subset \mathbb{R}$ be a non-trivial interval. A curve $I \ni t \mapsto z_t \in Z$ is absolutely continuous if there exists a function $f \in L^1(I, dt)$ such that (dt denotes the Lebesgue measure on I)

$$d_Z(z_t, z_s) \leq \int_t^s f(r) dr \quad \forall t, s \in I, \quad t < s. \quad (3.2.1)$$

For an absolutely continuous curve z_t , the limit $\lim_{h \rightarrow 0} \frac{d_Z(z_{t+h}, z_t)}{|h|}$ exists for a.e. $t \in I$ and, this limit defines an $L^1(I, dt)$ function. We denote $\lim_{h \rightarrow 0} \frac{d_Z(z_{t+h}, z_t)}{|h|}$ by $|\dot{z}_t|$ called *the metric speed at t* . Note that the metric speed $|\dot{z}_t|$ is the minimal function in the a.e. sense among $L^1(I, dt)$ functions satisfying (3.2.1) (see [6]). We denote by $\text{AC}^p(I; Z)$ the set of all absolutely continuous curves with their metric derivatives in $L^p(I)$. Let $C(I; Z)$ denote the set of continuous functions from I to Z . Define a map

$$C(I; Z) \ni \gamma \mapsto \mathcal{E}_2[\gamma] = \begin{cases} \int_I |\dot{\gamma}_t|^2 dt, & \text{if } \gamma \in \text{AC}^2(I; Z), \\ +\infty & \text{otherwise.} \end{cases}$$

Given an absolutely continuous curve $\mu \in \text{AC}(I; (\mathcal{P}_2(Z), W_2))$, we denote by $|\dot{\mu}_t|$ its metric speed in the space $(\mathcal{P}_2(Z), W_2)$. Let $e_t : C(I; Z) \rightarrow Z$ be the evaluation map $e_t(\gamma) = \gamma_t$. If $\pi \in \mathcal{P}(C(I; Z))$ satisfies $(e_t)_\# \pi = \mu_t$ for any $t \in I$, it is easy to see that

$$\int_I |\dot{\mu}_t| dt \leq \int \mathcal{E}_2[\gamma] d\pi(\gamma).$$

Let (X, d, m) be a metric measure space. We now recall notions of *test plan*, *weak upper gradient*, *Sobolev class* and *Cheeger energy* on (X, d, m) .

Definition 3.2.1 ([4]) (test plan)

We say that $\pi \in \mathcal{P}(C([0, 1]; X))$ is a *test plan* if there exists a constant $c > 0$ such that

$$(e_t)_\# \pi \leq cm \quad \text{for every } t \in [0, 1], \quad \int \mathcal{E}_2[\gamma] d\pi(\gamma) < \infty.$$

Definition 3.2.2 ([4]) (weak upper gradient, Sobolev class and Cheeger energy)

(i) For a Borel function $f : X \rightarrow \mathbb{R}$, we say that $G \in L^2(X, m)$ is a *weak upper gradient* of f if the following inequality holds:

$$\int |f(\gamma_1) - f(\gamma_0)| d\pi(\gamma) \leq \int \int_0^1 G(\gamma_t) |\dot{\gamma}_t| dt d\pi(\gamma), \quad (3.2.2)$$

for every test plan π .

(ii) We say that f belongs to the Sobolev class $S^2(X, d, m)$ if there exists $G \in L^2(X, m)$ satisfying (3.2.2). For $f \in S^2(X, d, m)$, it turns out that there exists a minimal (in the m -a.e. sense) weak upper gradient G and we denote it by $|\nabla f|_w$.

(iii) For $f \in L^2(X, m)$, the *Cheeger energy* $\text{Ch}(f)$ is defined by

$$\text{Ch}(f) = \begin{cases} \frac{1}{2} \int |\nabla f|_w^2 dm, & \text{if } f \in L^2(X, m) \cap S^2(X, d, m), \\ +\infty, & \text{otherwise.} \end{cases} \quad (3.2.3)$$

Note that $\text{Ch} : L^2(X, m) \rightarrow [0, +\infty]$ is a lower semi-continuous and convex functional but not necessarily quadratic form. Let $W^{1,2}(X, d, m) = L^2(X, m) \cap S^2(X, d, m)$ endowed with the following norm:

$$\|f\|_{W^{1,2}} = \sqrt{\|f\|_{L^2}^2 + 2\text{Ch}(f)}.$$

Note that $(W^{1,2}(X, d, m), \|\cdot\|_{W^{1,2}})$ is a Banach space, but not necessarily a Hilbert space.

The Cheeger energy Ch can be defined also as the limit of the integral of local Lipschitz constants. Let $\text{Lip}(X)$ denote the set of real-valued Lipschitz continuous functions on X . For $f \in \text{Lip}(X)$, the local Lipschitz constant $|\nabla f| : X \rightarrow \mathbb{R}$ is defined as follows:

$$|\nabla f|(x) = \begin{cases} \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{d(y, x)} & \text{if } x \text{ is not isolated,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $f \in W^{1,2}(X, d, m)$, we have (see [4])

$$\text{Ch}(f) = \frac{1}{2} \inf \left\{ \liminf_{n \rightarrow \infty} \int |\nabla f_n|^2 dm : f_n \in \text{Lip}(X), \int_X |f_n - f|^2 dm \rightarrow 0 \right\}.$$

3.3 $\text{RCD}^*(K, N)$ condition

In this subsection, we recall the definition of metric measure spaces satisfying the $\text{RCD}^*(K, N)$ condition following Erbar–Kuwada–Sturm [30]. We also recall several properties satisfied by $\text{RCD}^*(K, N)$ spaces.

For each $\theta \in [0, \infty)$, we set

$$\Theta_\kappa(\theta) = \begin{cases} \frac{\sin(\sqrt{\kappa}\theta)}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ \theta & \text{if } \kappa = 0, \\ \frac{\sinh(\sqrt{-\kappa}\theta)}{\sqrt{-\kappa}} & \text{if } \kappa < 0, \end{cases}$$

and set for $t \in [0, 1]$,

$$\sigma_\kappa^{(t)}(\theta) = \begin{cases} \frac{\Theta_\kappa(t\theta)}{\Theta_\kappa(\theta)} & \text{if } \kappa\theta^2 \neq 0 \text{ and } \kappa\theta^2 < \pi^2, \\ t & \text{if } \kappa\theta^2 = 0, \\ +\infty & \text{if } \kappa\theta^2 \geq \pi^2. \end{cases}$$

Let (X, d, m) be a metric measure space. Let $\mathcal{P}_2(X, d, m)$ be the subset of $\mathcal{P}_2(X)$ consisting of $\mu \in \mathcal{P}_2(X)$ which is absolutely continuous with respect to m . Let $\mathcal{P}_\infty(X, d, m)$ be the subset of $\mathcal{P}_2(X, d, m)$ consisting of $\mu \in \mathcal{P}_2(X, d, m)$ which has bounded support.

Definition 3.3.1 ([9, 30, 7]) ($\text{CD}^*(K, N)$ and $\text{RCD}^*(K, N)$)

- (i) We say that (X, d, m) satisfies *the reduced curvature-dimension condition* $\text{CD}^*(K, N)$ for $K, N \in \mathbb{R}$ with $N > 1$ if, for each pair $\mu_0 = \rho_0 m$ and $\mu_1 = \rho_1 m$ in $P_\infty(X, d, m)$, there exists an optimal coupling q of μ_0 and μ_1 and a geodesic $\mu_t = \rho_t m$ ($t \in [0, 1]$) in $(P_\infty(X, d, m), W_2)$ connecting μ_0 and μ_1 such that, for all $t \in [0, 1]$ and $N' \geq N$, we have

$$\int \rho_t^{-\frac{1}{N'}} d\mu_t \geq \int_{X \times X} \left[\sigma_{K/N'}^{(1-t)}(d(x_0, x_1)) \rho_0^{-1/N'}(x_0) + \sigma_{K/N'}^{(t)}(d(x_0, x_1)) \rho_1^{-1/N'}(x_1) \right] dq(x_0, x_1). \quad (3.3.1)$$

- (ii) We say that (X, d, m) satisfies *the Riemannian curvature-dimension condition* $\text{RCD}^*(K, N)$ if the following two conditions hold:

- (a) $\text{CD}^*(K, N)$
- (b) the infinitesimal Hilbertian, that is the Cheeger energy Ch is quadratic:

$$2\text{Ch}(u) + 2\text{Ch}(v) = \text{Ch}(u + v) + \text{Ch}(u - v), \quad (3.3.2)$$

$$\forall u, v \in W^{1,2}(X, d, m).$$

When (X, d, m) satisfies the $\text{RCD}^*(K, N)$, we define the Dirichlet form (i.e., symmetric closed Markovian bilinear form) $(\mathcal{E}, \mathcal{F})$ induced by the Cheeger energy Ch as follows:

$$\mathcal{E}(u, v) = \frac{1}{4}(\text{Ch}(u + v) - \text{Ch}(u - v)) \quad u, v \in \mathcal{F} = W^{1,2}(X, d, m). \quad (3.3.3)$$

By [4], the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is strongly local and regular.

Example 3.3.2 We give several examples satisfying the $\text{RCD}^*(K, N)$.

- (A) (Ricci limit spaces) Let $M_n = (M_n, d_{g_n}, m_{g_n})$ be N -dimensional complete Riemannian manifolds with $\text{Ricci} \geq K$ where g_n is a Riemannian metric, d_{g_n} is the distance function induced by g_n , and m_{g_n} is the Riemannian measure induced by g_n and satisfies $m_{g_n} \in \mathcal{P}_2(M_n, d_{g_n})$. Let $\mathcal{X} = (X, d, m)$ be a metric measure space satisfying $\mathbb{D}(M_n, \mathcal{X}) \rightarrow 0$ as $n \rightarrow \infty$. Then $\mathcal{X}$ satisfies the $\text{RCD}^*(K, N)$ condition (see [30]).

- (B) (Alexandrov spaces)

Let $\mathcal{X} = (X, d, \overline{m}_d)$ be an N -dimensional Alexandrov space with $\text{Curv} \geq K$ where $\overline{m}_d$ denotes the normalized Hausdorff measure induced by d (see e.g., [20] for details). By [72, 91], $\mathcal{X}$ satisfies $\text{CD}^*((N-1)K, N)$. Moreover, by [61], $\mathcal{X}$ satisfies the infinitesimal Hilbertian condition, and as a result, $\mathcal{X}$ satisfies $\text{RCD}^*((N-1)K, N)$.

(C) (Weighted spaces) [[30, Proposition 3.3]]

For a functional $S : X \rightarrow [-\infty, +\infty]$ on a complete separable metric space (X, d) , let $D(S) = \{x \in X : S(x) < \infty\} \subset X$. For $N \in (0, \infty)$, define the functional $U_N : X \rightarrow [0, +\infty]$ as follows:

$$U_N(x) = \exp\left(-\frac{1}{N}S(x)\right).$$

Let $K \in \mathbb{R}$ and $N \in (0, \infty)$. A functional $S : X \rightarrow [-\infty, +\infty]$ is said to be *strongly* (K, N) -convex if, for each $x_0, x_1 \in D(S)$ and every geodesic $\gamma : [0, 1] \rightarrow X$ connecting x_0 and x_1 , the following holds: for all $t \in [0, 1]$

$$U_N(\gamma_t) \geq \sigma_{K/N}^{(1-t)}(d(\gamma_0, \gamma_1)) \cdot U_N(\gamma_0) + \sigma_{K/N}^{(t)}(d(\gamma_0, \gamma_1)) \cdot U_N(\gamma_1).$$

Let (X, d, m) be a $\text{RCD}^*(K, N)$ space. Let $V : X \rightarrow \mathbb{R}$ be continuous, bounded below and strongly (K', N') -convex function with $\int e^{-V} dm < \infty$. Then

$$(X, d, e^{-V} dm) \text{ satisfies the } \text{RCD}^*(K + K', N + N').$$

(D) (Cones) [[57]]

Let us define, for $t \geq 0$,

$$\cos_K(t) = \begin{cases} \cos(\sqrt{K}t) & \text{if } K > 0, \\ \cosh(\sqrt{-K}t) & \text{if } K < 0, \end{cases}$$

and

$$\sin_K(t) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}t) & \text{if } K > 0, \\ t & \text{if } K = 0, \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}t) & \text{if } K < 0. \end{cases}$$

For a metric measure space (X, d, m) , the (K, N) -cone is a metric measure space defined as follows:

- A set $\text{Con}_K(X)$ is defined as follows:

$$\text{Con}_K(X) = \begin{cases} X \times [0, \pi/\sqrt{K}]/(X \times \{0, \pi/\sqrt{K}\}) & \text{if } K > 0, \\ X \times [0, \infty)/(X \times \{0\}) & \text{if } K \leq 0. \end{cases}$$

- A distance d_{Con_K} is defined as follows: for $(x, t), (y, s) \in \text{Con}_K(X)$,

$$d_{Con_K}((x, t), (y, s)) = \begin{cases} \cos_K^{-1}(\cos_K(s) \cos_K(t) + K \sin_K(s) \sin_K(t) \cos(d(x, y) \wedge \pi)) & \text{if } K \neq 0, \\ \sqrt{s^2 + t^2 - 2st \cos(d(x, y) \wedge \pi)} & \text{if } K = 0. \end{cases}$$

- A measure $m_{Con_K}^N$ is defined as follows:

$$m_{Con_K}^N = \sin_K^N t dt \otimes m.$$

Let $K \geq 0$ and $N \geq 1$. Then, by [57, Corollary 1.3], we have that $(\text{Con}_K(X), d_{Con_K}, m_{Con_K})$ satisfies the $\text{RCD}^*(KN, N+1)$ if and only if (X, d, m) satisfies the $\text{RCD}^*(N-1, N)$ and $\text{Diam}(X) \leq \pi$.

We list below several properties of metric measure spaces satisfying the $\text{RCD}^*(K, N)$.

Fact 3.3.3 Assume that (X_n, d_n, m_n) ($n \in \mathbb{N}$) and (X, d, m) satisfy the $\text{RCD}^*(K, N)$ condition. Then the following statements hold.

- (i) (Stability under the $\mathbb{D}$ -convergence) ([30, Theorem 3.22])

If $(X_n, d_n, m_n) \xrightarrow{\mathbb{D}} (X_\infty, d_\infty, m_\infty)$ with $m_n \in \mathcal{P}_2(X_n, d_n)$, then

$$(X_\infty, d_\infty, m_\infty) \text{ satisfies the } \text{RCD}^*(K, N).$$

- (ii) (Tensorization) ([30, Theorem 3.23])

The product space $(X_1 \times X_2, d_1 \otimes d_2, m_1 \otimes m_2)$, defined by

$$d_1 \otimes d_2((x, y), (x', y'))^2 = d_1(x, x')^2 + d_2(y, y')^2,$$

also satisfies the $\text{RCD}^*(K, 2N)$.

- (iii) (Generalized Bishop–Gromov inequality) ([30, Proposition 3.6])

Fix $x_0 \in \text{supp}[m]$. Let us set

$$v(r) = m(\overline{B_r(x_0)}), \quad \text{and} \quad s(r) = \limsup_{\delta \rightarrow 0} \frac{1}{\delta} m(\overline{B_{r+\delta}(x_0)} \setminus B_r(x_0)),$$

where $\overline{A}$ means the closure of a subset $A \subset X$. Then, for each $x_0 \in X$ and $0 < r < R < \pi\sqrt{N/(K \vee 0)}$, the following inequalities hold:

$$\frac{v(r)}{v(R)} \geq \frac{\int_0^r \Theta_{K/N}(t)^N dt}{\int_0^R \Theta_{K/N}(t)^N dt}, \quad \frac{s(r)}{s(R)} \geq \left(\frac{\Theta_{K/N}(r)}{\Theta_{K/N}(R)} \right)^N. \quad (3.3.4)$$

- (iv) (local uniform volume doubling property) (implied by (3.3.4))
 For each open ball $B_R(x_0) \subset \text{supp}[m]$, there exists a positive constant $C_V = C_V(N, K, R) > 0$ depending only on N, K, R such that, for any $B_r(x)$ such that $B_{2r}(x) \subset B_R(x_0)$,

$$m(B_{2r}(x)) \leq C_V m(B_r(x)).$$

- (v) (local uniform weak (2,2)-Poincaré inequality) ([38, 74, 75]) For each open ball $B_R(x_0) \subset \text{supp}[m]$, there exists a positive constant $C_P = C_P(N, K, R) > 0$ depending only on N, K, R such that, for any $B_r(x)$ such that $B_{2r}(x) \subset B_R(x_0)$, and all $u \in W^{1,2}(B_{2r}(x))$,

$$\frac{1}{m(B_r(x))} \int_{B_r(x)} |u - u_{B_r}|^2 dm \leq \frac{C_P r^2}{m(B_{2r}(x))} \int_{B_{2r}(x)} |\nabla u|_w^2 dm, \quad (3.3.5)$$

where $W^{1,2}(B_r(x)) = W^{1,2}(B_r(x), d|_{B_r(x)}, m|_{B_r(x)}) \subset W^{1,2}(X, d, m)$ defined in Section 3.2 and

$$u_{B_r}(x) = \frac{1}{m(B_r(x))} \int_{B_r(x)} u dm.$$

- (vi) (The intrinsic distance coincides with d) ([5, Theorem 6.10])
 By [5], $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form. Let $d_{\mathcal{E}}$ denote the intrinsic distance defined by $(\mathcal{E}, \mathcal{F})$:

$$d_{\mathcal{E}}(x, y) = \sup\{u(x) - u(y) : u \in \mathcal{F}_{\text{loc}} \cap C_0(X), |\nabla u|_w \leq 1 \text{ } m\text{-a.e.}\},$$

where $\mathcal{F}_{\text{loc}}$ is defined as follows:

$$\mathcal{F}_{\text{loc}} = \{u \in L^2_{\text{loc}}(X, m) : \text{for all } K \subset X \text{ relative compact,} \\ \text{there exists } v \in \mathcal{F} \text{ s.t. } u = v \text{ on } K \text{ a.e.}\}.$$

Then we have

$$d_{\mathcal{E}} = d.$$

- (vii) (Parabolic Harnack inequality) (follows from (iv), (v) and (vi) (see [80, Theorem 3.5]))

Let A be the non-negative self-adjoint operator associated with the Cheeger energy Ch . For each open ball $B_R(x_0) \subset \text{supp}[m]$, there

exists a positive constant $C_H = C_H(N, K, R) > 0$ such that, for any ball $B_r(x) \subset A$ satisfying $B_{2r}(x) \subset B_R(x_0)$,

$$\sup_{(s,y) \in Q^-} u(s, y) \leq C_H \inf_{(s,y) \in Q^+} u(s, y), \quad (3.3.6)$$

whenever u is a nonnegative local solution of the parabolic equation $\frac{\partial}{\partial t} u = -Au$ on $Q = (t - 4r^2, t) \times B_{2r}(x)$. Here $Q^- = (t - 3r^2, t - 2r^2) \times B_r(x)$ and $Q^+ = (t - r^2, t) \times B_r(x)$.

(viii) (Hölder continuity of the local solutions) (follows from (vii) (see [80, Proposition 3.1])) Let A be the non-negative self-adjoint operator associated with the Cheeger energy Ch . For each open ball $B_R(x_0) \subset \text{supp}[m]$, there exists positive constants $\alpha = \alpha(N, K, R) \in (0, 1)$ and $C = C(N, K, R) > 0$ depending only on N, K, R such that for all $T > 0$ and all balls $B_r(x)$ satisfying $B_{2r}(x) \subset B_R(x_0)$, it holds that, for all $(s, y), (t, z) \in Q_1 = (T - r^2, T) \times B_r(x)$,

$$|u(s, y) - u(t, z)| \leq C \sup_{Q_2} |u| \left(\frac{|s - t|^{1/2} + d(y, z)}{r} \right)^\alpha, \quad (3.3.7)$$

whenever u is a local solution of the parabolic equation $\frac{\partial}{\partial t} u = -Au$ on $Q_2 = (T - 4r^2, T) \times B_{2r}(x)$.

Chapter 4

Convergence of Brownian motions on $\mathrm{RCD}^*(K, N)$ spaces

In this chapter, we show one of the main results of this theses, that is, we show in Theorem 4.4.1 that the weak convergence of the laws of Brownian motions is *equivalent* to the measured Gromov–Hausdorff convergence of the underlying metric measure spaces under the following assumption:

Assumption 4.0.4 Let N, K and D be constants with $1 < N < \infty$, $K \in \mathbb{R}$ and $0 < D < \infty$. For $n \in \bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, let $\mathcal{X}_n = (X_n, d_n, m_n)$ be a metric measure space satisfying the $\mathrm{RCD}^*(K, N)$ condition with $\mathrm{Diam}(X_n) \leq D$ and $m_n(X_n) = 1$.

In this chapter, we always assume the above assumption.

4.1 Brownian motions on $\mathrm{RCD}^*(K, N)$ spaces

Let $\{T_t\}_{t>0}$ be the semigroup on $L^2(X, m)$ associated with the Cheeger energy Ch . We say that a jointly measurable function $p(t, x, y)$ in $(0, \infty) \times X \times X$ is a *heat kernel* if

$$T_t f(x) = \int_X p(t, x, y) f(y) m(dy), \quad f \in L^2(X, m), \quad m\text{-a.e. } x \in X.$$

By [81, Theorem 7.4 & Proposition 7.5], the global parabolic Harnack inequality implies that there exists a heat kernel $p(t, x, y)$ which is locally Hölder continuous in $(t, x, y) \in (0, \infty) \times X \times X$ satisfying

(a) (Strong Feller property)

For any $f \in \mathcal{B}_b(X)$,

$$T_t f = \int_X f(y) p(t, \cdot, y) m(dx) \in C(X) \quad (\forall t > 0). \quad (4.1.1)$$

(b) (Gaussian estimate)

There exist positive constants $C_1 = C_1(N, K, D)$, $C'_1 = C'_1(N, K, D)$, $C_2 = C_2(N, K, D)$ and $C'_2 = C'_2(N, K, D)$ depending only on N, K, D such that

$$\frac{C'_1}{m(B_{\sqrt{t}}(x))} \exp\left\{-C'_2 \frac{d(x, y)^2}{t}\right\} \leq p(t, x, y) \leq \frac{C_1}{m(B_{\sqrt{t}}(x))} \exp\left\{-C_2 \frac{d(x, y)^2}{t}\right\}, \quad (4.1.2)$$

for all $x, y \in X$ and $0 < t \leq D^2$.

By (a) and (b), we know that $\{T_t\}_{t>0}$ is a Feller semigroup, that is, the following conditions hold:

(F-1) For any $f \in C(X)$, $T_t f \in C(X)$ for any $t > 0$.

(F-2) For any $f \in C(X)$, $\|T_t f - f\|_\infty \rightarrow 0$ as $t \downarrow 0$.

(Note that the strong Feller property is not actually stronger than the Feller property because the strong Feller property does not necessarily imply (F-2). The word *strong* is just a conventional use.)

In fact, (F-1) follows directly from the strong Feller property (a).

We show the condition (F-2). By the generalized Bishop–Gromov inequality (3.3.4), we have the following volume growth estimate: there exist positive constants $\nu = \nu(N, K, D) > 0$ and $c = c(N, K, D) > 0$ such that, for all $n \in \mathbb{N}$

$$m_n(B_r(x)) \geq cr^{2\nu} \quad (0 \leq r \leq 1 \wedge D). \quad (4.1.3)$$

In fact, taking $B = B_D(x_0)$ in (3.3.4) for some $x_0 \in X$, we have

$$m_n(B_r(x)) \geq \frac{\int_0^r \Theta_{K/N}(t)^N dt}{\int_0^D \Theta_{K/N}(t)^N dt} m_n(B_D(x)) = c(N, K, D) \int_0^r \Theta_{K/N}(t)^N dt.$$

Here we used $m_n(X_n) = 1$ and $c(N, K, D) = \frac{1}{\int_0^D \Theta_{K/N}(t)^N dt}$. Thus we have (4.1.3). Combining with the Gaussian heat kernel estimate (4.1.2), we have the following upper heat kernel estimate:

$$p(t, x, y) \leq \frac{C_1}{ct^\nu} \exp\left\{-C_2 \frac{d(x, y)^2}{t}\right\}, \quad (4.1.4)$$

for all $x, y \in X$ and $0 < t \leq D^2$.

For given $\varepsilon > 0$, take $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$. By the Gaussian estimate (4.1.4), we can choose a positive number T such that $p(t, x, y) < \varepsilon$ for any $0 < t < T$ and any $x, y \in X$ satisfying $d(x, y) \geq \delta$. Then we have that, for any $x \in X$

$$\begin{aligned} |T_t f(x) - f(x)| &= \left| \int_X p(t, x, y) f(y) m(dy) - f(x) \right| \\ &\leq \int_X p(t, x, y) |f(y) - f(x)| m(dy) \\ &= \int_{B_x(\delta)} p(t, x, y) |f(y) - f(x)| m(dy) + \int_{X \setminus B_x(\delta)} p(t, x, y) |f(y) - f(x)| m(dy) \\ &\leq \varepsilon + 2\varepsilon \|f\|_\infty. \end{aligned}$$

Thus we have shown that (F-2) holds.

By the Feller property of $\{T_t\}_{t>0}$, there exists the Hunt process $(\{\mathbb{P}^x\}_{x \in X}, \{B_t\}_{t \geq 0})$ satisfying (see e.g., [15, §I Theorem 9.4])

$$\mathbb{E}^x(f(B(t))) = T_t f(x)$$

for all $f \in \mathcal{B}_b(X) \cap L^2(X, m)$, all $t > 0$ and all $x \in X$. We call $(\{\mathbb{P}^x\}_{x \in X}, \{B_t\}_{t \geq 0})$ the *Brownian motion* on (X, d, m) . Since $(\mathcal{E}, \mathcal{F})$ is strongly local by [5], $B(\cdot)$ has continuous paths without inside killing almost surely with respect to $\mathbb{P}^x$ for all $x \in X$. See [34] for details.

4.2 Sturm's $\mathbb{D}$ -distance and measured Gromov–Hausdorff convergence

In this subsection, following [35, 82], we recall two notions of convergences, *Sturm's $\mathbb{D}$ -convergence* and *the measured Gromov–Hausdorff convergence*, and state when two notions are equivalent.

Let (X, d, m) be a normalized metric measure space, that is,

- (i) (X, d, m) is a metric measure space;
- (ii) $m(X) = 1$.

Two metric measure spaces (X_1, d_1, m_1) and (X_2, d_2, m_2) are said to be *isomorphic* if there exists an isometry $\iota : \text{supp}[m_1] \rightarrow \text{supp}[m_2]$ such that

$$\iota_\# m_1 = m_2.$$

4.2.1 Sturm's $\mathbb{D}$ -distance

The variance of (X, d, m) is defined as follows:

$$\mathbb{V}ar(X, d, m) = \inf \int_{X'} d'^2(z, x) dm'(x),$$

where the infimum is taken over all metric measure spaces (X', d', m') isomorphic to (X, d, m) and over all $z \in X'$. Note that (X, d, m) has a finite variance if and only if

$$\int_X d^2(z, x) dm(x) < \infty,$$

for some (hence all) $z \in X$.

Let $\mathbb{X}_1$ be the set of isomorphism classes of normalized metric measure spaces with finite variances. Now we equip $\mathbb{X}_1$ with a metric called *Sturm's $\mathbb{D}$ -distance* ([82]):

Definition 4.2.1 (See [82, Definition 3.2]) For $(X_1, d_1, m_1), (X_2, d_2, m_2) \in \mathbb{X}_1$,

$$\begin{aligned} \mathbb{D}((X_1, d_1, m_1), (X_2, d_2, m_2)) = \inf \left\{ \left(\int_{X_1 \times X_2} \hat{d}(x, y)^2 dq(x, y) \right)^{1/2} : \right. \\ \hat{d} \text{ is a coupling of } d_1 \text{ and } d_2, \\ \left. q \text{ is a coupling of } m_1 \text{ and } m_2 \right\}, \end{aligned}$$

where a pseudo metric $\hat{d}$ on the disjoint union $X_1 \sqcup X_2$ is called a *coupling of d_1 and d_2* if

$$\hat{d}(x, y) = d_i(x, y), \quad x, y \in \text{supp}[m_i] \quad (i = 1, 2).$$

Here we mean that $\hat{d}$ is a *pseudo metric* if $\hat{d}$ satisfies all the conditions of metric except *non-degeneracy*, i.e., $\hat{d}(x, y) = 0$ does not necessarily imply $x = y$.

We say that a sequence $\mathcal{X}_n = (X_n, d_n, m_n)$ is $\mathbb{D}$ -convergent to $\mathcal{X}_\infty = (X_\infty, d_\infty, m_\infty)$ if $\mathbb{D}(\mathcal{X}_n, \mathcal{X}_\infty) \rightarrow 0$ as $n \rightarrow \infty$.

It is known that $(\mathbb{X}_1, \mathbb{D})$ becomes a complete separable metric space (see [82]).

We know the following equivalent statement:

Proposition 4.2.2 (See [5, Proposition 2.7]) *Let $(X_n, d_n, m_n) \in \mathbb{X}_1$ for $n \in \mathbb{N}$. Then the following are equivalent:*

- (i) $(X_n, d_n, m_n) \xrightarrow{\mathbb{D}} (X_\infty, d_\infty, m_\infty)$ as $n \rightarrow \infty$;
- (ii) There exists a complete separable metric space (X, d) and isometric embeddings $\iota_n : \text{supp}[m_n] \rightarrow X$ for $n \in \bar{\mathbb{N}}$ such that

$$W_2(\iota_n\#m_n, \iota_\infty\#m_\infty) \rightarrow 0.$$

4.2.2 mGH-convergence

Now we recall *the measured Gromov–Hausdorff convergence (mGH-convergence)*. Since we only consider compact metric measure spaces in this chapter, we give the definition only for compact metric measure spaces. (For the non-compact case, see e.g., [35, Definition 3.24] for the definition of *the pointed measured Gromov–Hausdorff convergence*.)

Definition 4.2.3 ([33]) Let (X_n, d_n, m_n) be a sequence of compact metric measure spaces for $n \in \bar{\mathbb{N}}$. We say that (X_n, d_n, m_n) converges to $(X_\infty, d_\infty, m_\infty)$ in the sense of *the measured Gromov–Hausdorff* (mGH for short) if there exist $\varepsilon_n \rightarrow 0$ ($n \rightarrow \infty$) and Borel measurable maps $f_n : X_n \rightarrow X_\infty$ for each $n \in \bar{\mathbb{N}}$ such that

- (i) $\sup_{x, y \in X_n} |d_n(x, y) - d_\infty(f_n(x), f_n(y))| \leq \varepsilon_n$;
- (ii) $X_\infty \subset B_{\varepsilon_n}(f_n(X_n))$;
- (iii) for any $\phi \in C_b(X_\infty)$, it holds that

$$\lim_{n \rightarrow \infty} \int_{X_n} \phi \circ f_n \, dm_n \rightarrow \int_{X_\infty} \phi \, dm_\infty.$$

The map f_n is called *an ε_n -approximation* (an ε_n -isometry is also a standard name).

4.2.3 Relation between two convergences

In general, the mGH-convergence is stronger than the $\mathbb{D}$ -convergence and pmG-convergence. However, if a sequence (X_n, d_n, m_n) has *c-doubling property* for a positive constant $c > 0$ and $\text{supp}[m_\infty] = X_\infty$, these three notions of convergences are equivalent. Here we mean that, for a positive constant $c > 0$, a metric measure space (X, d, m) satisfies *c-doubling property* if

$$m(B_{2r}(x)) \leq cm(B_r(x)) \quad \forall x \in X, \quad \forall r > 0. \quad (4.2.1)$$

Note that the condition (4.2.1) implies that $\text{supp}[m_n] = X_n$ for all $n \in \bar{\mathbb{N}}$.

Proposition 4.2.4 (See [35, Proposition 3.33]) *Let (X_n, d_n, m_n) be a sequence of normalized metric measure spaces. Under the following two conditions, the mGH -convergence and the $\mathbb{D}$ -convergence are equivalent:*

- (i) (X_n, d_n, m_n) has c -doubling property for some $c > 0$ where c is independent of each n ;
- (ii) $\text{supp}[m_\infty] = X_\infty$.

In the setting of Assumption 4.0.4, we can check both of (i) and (ii) of Proposition 4.2.4. In fact, as we will state in Fact 3.3.3, we have the uniform global volume doubling property when $\text{Diam}(X_n) \leq D$. This implies (i) of Proposition 4.2.4. Since the $\text{RCD}^*(K, N)$ is stable under $\mathbb{D}$ -convergence (see (i) of Fact 3.3.3), we have that $(X_\infty, d_\infty, m_\infty)$ also satisfies the $\text{RCD}^*(K, N)$ with $\text{Diam}(X_\infty) \leq D$. This implies that $(X_\infty, d_\infty, m_\infty)$ satisfies the global volume doubling property, and we have $\text{supp}[m_\infty] = X_\infty$.

We give a useful equivalent statement for later use.

Proposition 4.2.5 *For $n \in \overline{\mathbb{N}}$, assume that (X_n, d_n, m_n) are metric measure spaces satisfying that each (X_n, d_n) are embedded isometrically into a complete separable metric space (X, d) , and*

- (i) $\sup_{n \in \overline{\mathbb{N}}} \text{Diam}(X_n) < D < \infty$;
- (ii) *the uniform volume doubling property: there exists $C > 0$ independent of $n \in \overline{\mathbb{N}}$ such that $m_n(B_{2r}(x)) \leq C m_n(B_r(x))$ for any $x \in X_n$ and $0 < r < D$;*
- (iii) m_n converges weakly to m_∞ ;
- (iv) $m_n(X_n) = 1$ for $n \in \overline{\mathbb{N}}$.

Then X_n converges to X_∞ in the Hausdorff sense in (X, d) .

Proof. Note that, by the doubling property (ii), we have $\text{supp}[m_n] = X_n$ for all $n \in \overline{\mathbb{N}}$. It suffices to show

- (1) for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, we have

$$X_\infty \subset X_n^\varepsilon;$$

- (2) for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, we have

$$X_\infty^\varepsilon \supset X_n.$$

The statement (1) follows only from the condition (iii) as follows: We show (1) by contradiction. Assume (1) does not hold. Then there exists $\varepsilon_0 > 0$ such that, for any $n \in \mathbb{N}$, we have $X_\infty \not\subseteq X_n^{\varepsilon_0}$. Then there exists $x_n \in X_\infty$ such that $d(x_n, X_n) := \inf_{z \in X_n} d(x_n, z) > \varepsilon_0$. By the compactness of X_∞ , we can take a subsequence (also denoted by x_n) converging to $x_\infty \in X_\infty$. Take n_0 such that, for any $n \geq n_0$, $d(x_n, x_\infty) < \varepsilon_0/2$. Then, by the triangle inequality, we have

$$d(x_\infty, X_n) \geq d(x_n, X_n) - d(x_\infty, x_n) > \frac{\varepsilon_0}{2}.$$

Thus we have $B(x_\infty, \varepsilon_0/2) \cap X_n = \emptyset$ for any $n \in \mathbb{N}$. Since $\text{supp}[m_\infty] = X_\infty$, we have

$$\liminf_{n \rightarrow \infty} m_n(B(x_\infty, \varepsilon_0/2)) = 0 < m_\infty(B(x_\infty, \varepsilon_0/2)).$$

This contradicts the condition (iii). We finish the proof of (1).

Now we show (2) by contradiction. Assume that (2) does not hold. Then there exists $\varepsilon_0 > 0$ such that, for any $n \in \mathbb{N}$, we have $X_\infty^{\varepsilon_0} \not\subseteq X_n$. Then there exists $x_n \in X_n$ such that $d(x_n, X_\infty) := \inf_{z \in X_\infty} d(x_n, z) > \varepsilon_0$. By (ii) and (iv), we have

$$1 = m_n(B(x_n, 2^{1+\log_2 \frac{D}{\varepsilon_0}} \varepsilon_0/2)) \leq C^{1+\log_2 \frac{D}{\varepsilon_0}} m_n(B(x_n, \varepsilon_0/2)).$$

Since ε_0, D, C are independent of n , we have

$$\inf_{n \in \mathbb{N}} m_n(B(x_n, \varepsilon_0/2)) \geq C^{-1-\log_2 \frac{D}{\varepsilon_0}} > 0. \quad (4.2.2)$$

Let $A = \overline{\cup_{n \in \mathbb{N}} B(x_n, \varepsilon_0/2)}$ denote the closure of $\cup_{n \in \mathbb{N}} B(x_n, \varepsilon_0/2)$. Then we have $A \cap X_\infty = \emptyset$ for any $n \in \mathbb{N}$ and, by and (4.2.2), we have

$$\limsup_{n \rightarrow \infty} m_n(A) > 0 = m_\infty(A).$$

This contradicts (iii). We finish the proof. $\square$

4.3 Mosco convergence of Cheeger energies

In Gigli–Mondino–Savaré [35], they introduced L^2 -convergences on varying metric measure spaces and showed a Mosco convergence of the Cheeger energies. We recall their results briefly.

Definition 4.3.1 (See [35, Definition 6.1]) Let (X_n, d_n, m_n) be normalized metric measure spaces. Assume that (X_n, d_n, m_n) converges to $(X_\infty, d_\infty, m_\infty)$ in the $\mathbb{D}$ -distance, or pmG-sense. Let (X, d) be a complete separable metric space and $\iota_n : \text{supp}[m_n] \rightarrow X$ be isometries as in Proposition 4.2.2. We identify (X_n, d_n, m_n) with $(\iota_n(X_n), d, \iota_{n\#}m_n)$ and omit ι_n .

- (i) We say that $u_n \in L^2(X, m_n)$ converges weakly to $u_\infty \in L^2(X, m_\infty)$ if the following hold:

$$\sup_{n \in \mathbb{N}} \int |u_n|^2 dm_n < \infty \quad \text{and} \quad \int \phi u_n dm_n \rightarrow \int \phi u_\infty dm_\infty \quad \forall \phi \in C_{bs}(X),$$

where recall that $C_{bs}(X)$ denotes the set of bounded continuous functions with bounded support.

- (ii) We say that $u_n \in L^2(X, m_n)$ converges strongly to $u_\infty \in L^2(X, m_\infty)$ if u_n converges weakly to u_∞ and the following holds:

$$\limsup_{n \rightarrow \infty} \int |u_n|^2 dm_n \leq \int |u_\infty|^2 dm_\infty.$$

Theorem 4.3.2 (See [35, Theorem 6.8]) Let (X_n, d_n, m_n) a sequence of normalized compact metric measure spaces satisfying the $\text{CD}(K, \infty)$ for all $n \in \mathbb{N}$. Assume $(X_n, d_n, m_n) \rightarrow (X_\infty, d_\infty, m_\infty)$ in the sense of $\mathbb{D}$ -convergence or pmG-convergence. Let (X, d) be a complete separable metric space as in Proposition 4.2.2. Let Ch_n be the Cheeger energy on $L^2(X, m_n)$ for $n \in \mathbb{N}$. Then Ch_n Mosco-converges to Ch_∞ , that is, the following two statements hold:

- (M1) for every $u_n \in L^2(X, m_n)$ converges weakly to some $u_\infty \in L^2(X, m_\infty)$, the following holds:

$$\liminf_{n \rightarrow \infty} \text{Ch}_n(u_n) \geq \text{Ch}_\infty(u_\infty).$$

- (M2) for every $u_\infty \in L^2(X, m_\infty)$, there exists a sequence $u_n \in L^2(X, m_n)$ such that u_n converges strongly to u_∞ and the following holds:

$$\limsup_{n \rightarrow \infty} \text{Ch}_n(u_n) \leq \text{Ch}_\infty(u_\infty).$$

Note that, under the infinitesimal Hilbertian condition (3.3.2), we have $\text{CD}^*(K, N) \iff \text{CD}^e(K, N) \implies \text{CD}(K, \infty)$ (see [30, Theorem 7 & Lemma 3.2]). Thus Theorem 4.3.2 holds for $\text{RCD}^*(K, N)$ spaces.

The Mosco convergence of the Cheeger energies implies the convergence of the Heat semigroups. Assume the same conditions as in Theorem 4.3.2 and Ch_n are quadratic (3.3.2) for any $n \in \mathbb{N}$. Let $\{T_t^n\}_{t>0}$ be the $L^2(X, m_n)$ -semigroup corresponding to the Cheeger energy Ch_n .

Theorem 4.3.3 (See [35, Theorem 6.11]) *Assume the same conditions as in Theorem 4.3.2 and Ch_n are quadratic in the sense of (3.3.2) for any $n \in \mathbb{N}$. Then, for any $u_n \in L^2(X, m_n)$ converging strongly to $u_\infty \in L^2(X, m_\infty)$, we have*

$$T_t^n u_n \text{ converges strongly to } T_t^\infty u_\infty \quad (\forall t > 0).$$

Note that, in [35, Theorem 6.11], the above Theorem 4.3.3 was stated without the condition of the infinitesimal Hilbertian. In this case, $\{T_t^n\}_{t>0}$ means the L^2 -gradient flow of Ch_n (see e.g., [35, §5.1.4]).

4.4 Equivalence of convergence of Brownian motions and mGH-convergence of the underlying spaces

We state our first main result in this thesis precisely:

Theorem 4.4.1 *Suppose that Assumption 4.0.4 holds. Then the following statements are equivalent:*

(i) (**$\mathbb{D}$ -convergence of the underlying spaces**)

$\mathcal{X}_n$ converges to $\mathcal{X}_\infty$ in the Sturm's $\mathbb{D}$ -distance.

(ii) (**Weak convergence of the laws of embedded Brownian motions**)

There exist

$$\begin{cases} \text{a compact metric space } (X, d) \\ \text{isometric embeddings } \iota_n : X_n \rightarrow X \text{ } (n \in \overline{\mathbb{N}}) \\ x_n \in X_n \text{ } (n \in \overline{\mathbb{N}}) \end{cases}$$

such that

$$\iota_n(B^n)_\# \mathbb{P}_n^{x_n} \rightarrow \iota_\infty(B^\infty)_\# \mathbb{P}_\infty^{x_\infty} \quad \text{weakly in } \mathcal{P}(C([0, \infty); X)).$$

We note that, under Assumption 4.0.4, Sturm's $\mathbb{D}$ -convergence is equivalent to the measured Gromov–Hausdorff convergence (Sturm [82], which will be stated in Proposition 4.2.4).

As a corollary of Theorem 4.4.1, the following holds:

Corollary 4.4.2 *Suppose that Assumption 4.0.4 holds. Then the following (i) implies (ii):*

(i) (**$\mathbb{D}$ -convergence of underlying spaces**)

(X_n, d_n, m_n) converges to $(X_\infty, d_\infty, m_\infty)$ in the Sturm's $\mathbb{D}$ -distance;

(ii) (**$\mathbb{D}$ -convergence of continuous path spaces with the laws of Brownian motions**)

$(C([0, \infty); X_n), \delta_n, B_n^\# \mathbb{P}_n^{x_n})$ converges to $(C([0, \infty); X_\infty), \delta_\infty, B_\infty^\# \mathbb{P}_\infty^{x_\infty})$ in the Sturm's $\mathbb{D}$ -distance for some sequence $x_n \in X_n$.

Remark 4.4.3 We give comments to several related works.

- (i) In [69], Ogura studied the weak convergence of the laws of the Brownian motions on Riemannian manifolds *by a different approach from ours*. He assumed uniform upper bounds for heat kernels, and the Kasue–Kumura spectral convergence of the underlying manifolds M_n . He push-forward each Brownian motions on M_n to the Kasue–Kumura spectral limit space M_∞ with respect to ε_n -isometry $f_n : M_n \rightarrow M_\infty$, and show the convergence in law on the càdlàg space of the push-forwarded Brownian motions on M_∞ with *time-discretization*. In his approach, since ε_n -isometry f_n is only ensured to be measurable, the time-discretization is necessary for each push-forwarded Brownian motions to belong to the càdlàg space.
- (ii) In [1], Albeverio and Kusuoka considered diffusion processes associated with SDEs on thin tubes in $\mathbb{R}^d$ shrinking to one-dimensional spider graphs. They studied the weak convergence of these diffusions to one-dimensional diffusions on the limit graphs. We note that their setting does not satisfy the $\text{RCD}^*(K, N)$ condition because Ricci curvatures are not bounded below at points of conjunctions in spider graphs.

4.5 Proof of Theorem 4.4.1 and Corollary 4.4.2

4.5.1 Sketch of proof of Theorem 4.4.1

Sketch of (i) $\implies$ (ii). It is enough to show *tightness* (Lemma 4.5.1) & *convergence of finite-dimensional distributions* (CFD) (Lemma 4.5.2). Tightness follows from the upper Gaussian heat kernel estimate (4.1.4), which follows

from Sturm [81] with the Poincaré inequality (Rajala [75]) and the Bishop–Gromov inequality (Sturm [82]). The CFD follows from the following three steps:

- (1) Mosco convergence of Cheeger energy (Gigli–Mondino–Savaré [35, Theorem 6.8]).
- (2) We extend the semigroups $T_t^n f$ on each X_n to the whole space X preserving Hölder continuity. We denote the extended semigroups by $\widetilde{T_t^n f}$ ((viii) of Fact 3.3.3 & Lemma 4.5.4 & Lemma 4.5.5).
- (3) We show stability of the extension operator $\sim$ under the Hausdorff convergence of X_n to X_∞ ((c) in the proof of Lemma 4.5.5).

Sketch of (ii) $\implies$ (i). We get information about the underlying measure m_n from the Brownian motion $\{B_t^n\}_{t \geq 0}$ as equilibrium states as $t \rightarrow \infty$ by the ergodic theorem (Lemma 4.5.6). The key point is to exchange $\liminf_{n \rightarrow \infty}$ and $\lim_{t \rightarrow \infty}$. We show this by the spectral gap estimate $\inf_{n \in \mathbb{N}} \lambda_n^1 > 0$, which follows immediately from the Poincaré inequality (Rajala [75]).

4.5.2 Proof of Theorem 4.4.1

Proof of (i) $\implies$ (ii) in Theorem 4.4.1

By Proposition 4.2.2, there exist a complete separable metric space (X, d) and a family of isometric embeddings $\iota_n : X_n \rightarrow X$ such that

$$W_2(\iota_n \# m_n, \iota_\infty \# m_\infty) \rightarrow 0. \quad (4.5.1)$$

By Proposition 4.2.5, we have that $\iota_n(X_n)$ converges to $\iota_\infty(X_\infty)$ in the Hausdorff sense in (X, d) . Since each X_n are compact with $\text{Diam}(X_n) \leq D$, we can take X as a compact set (e.g., see the proof of [5, Proposition 2.7]).

Let $x_n \in X_n$ be a sequence satisfying $\iota_n(x_n) \rightarrow \iota_\infty(x_\infty)$ in (X, d) (such sequence always exists because of the Hausdorff convergence of $\iota_n(X_n)$). For $n \in \overline{\mathbb{N}}$, let

$$\mathbb{B}_n := \iota_n(B_\cdot^n) \# \mathbb{P}_n^{x_n},$$

which is a sequence of probability measures on $\mathcal{P}(C([0, \infty); X))$.

Hereafter we identify $\iota_n(X_n)$ with X_n , and we omit ι_n .

To show $\mathbb{B}_n \rightarrow \mathbb{B}_\infty$ weakly, it is enough to show (see e.g., [14, §6])

(A) $\{\mathbb{B}_n\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(C([0, \infty), X))$ with respect to the weak topology;

(B) Convergence of the finite-dimensional distributions: For any $k \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \dots < t_k < \infty$ and $g_1, g_2, \dots, g_k \in C_b(X)$, the following holds:

$$\mathbb{E}^{x_n}[g_1(B_{t_1}^n) \cdots g_k(B_{t_k}^n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}^{x_\infty}[g_1(B_{t_1}^\infty) \cdots g_k(B_{t_k}^\infty)]. \quad (4.5.2)$$

We first show (A), that is, the following statement holds:

Lemma 4.5.1 $\{\mathbb{B}_n\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(C([0, \infty), X))$.

Proof. Since x_n converges to x_∞ in (X, d) , the laws of the initial distributions $\{B_0^n \# P_n^{x_n}\}_{n \in \mathbb{N}} = \{\delta_{x_n}\}_{n \in \mathbb{N}}$ is clearly tight in $\mathcal{P}(X)$. Thus it suffices for (A) to show the following (see [14, Theorem 12.3]): for each $T > 0$, there exist $\beta > 0$, $C > 0$ and $\theta > 1$ such that, for all $n \in \mathbb{N}$

$$\mathbb{E}^{x_n}[\tilde{d}^\beta(B_t^n, B_{t+h}^n)] \leq Ch^\theta, \quad (0 \leq t \leq T \text{ and } 0 \leq h \leq 1), \quad (4.5.3)$$

where $\tilde{d}(x, y) := d(x, y) \wedge 1$. Take $\beta > 0$ such that $\beta/2 - \nu > 1$, and set $\theta = \beta/2 - \nu$. By the Markov property, we have

$$\begin{aligned} & \text{L.H.S. of (4.5.3)} \\ &= \int_{X_n \times X_n} p_n(t, x_n, y) p_n(h, y, z) \tilde{d}^\beta(y, z) m_n(dy) m_n(dz). \\ &\leq \int_{X_n \times X_n} p_n(t, x_n, y) p_n(h, y, z) d^\beta(y, z) m_n(dy) m_n(dz). \end{aligned} \quad (4.5.4)$$

By the Gaussian heat kernel estimate (4.1.4), we have

$$\begin{aligned} & \int_{X_n} p_n(s, y, z) d^\beta(y, z) m_n(dz) \\ &\leq \frac{C_1}{cs^\nu} \int_{X_n} \exp\left(-C_2 \frac{d_n(y, z)^2}{s}\right) d^\beta(y, z) m_n(dz) \\ &= \frac{C_1}{cs^\nu} \int_{X_n} \exp\left(-C_2 \frac{d_n(y, z)^2}{s}\right) d_n^\beta(y, z) m_n(dz) \\ &\leq C_1 c^{-1} C_2^{2/\beta} s^{\beta/2 - \nu} m_n(X_n) \sup_{y, z \in X_n} \left\{ \left(C_2 \frac{d_n(y, z)^2}{s} \right)^{\beta/2} \exp\left(-C_2 \frac{d_n(y, z)^2}{s}\right) \right\} \\ &\leq C_1 c^{-1} C_2^{2/\beta} M_\beta s^{\beta/2 - \nu} \\ &= C_4 s^{\beta/2 - \nu}, \end{aligned} \quad (4.5.5)$$

where $M_\beta := \sup_{t \geq 0} t^{\beta/2} \exp(-t)$ and $C_4 = C_4(N, K, D, \beta) = C_1 c^{-1} C_2^{2/\beta} M_\beta > 0$ is a constant dependent only on N, K, D (independent of n). Note that, in the fifth line in (4.5.5), we used $m_n(X_n) = 1$ for all $n \in \bar{\mathbb{N}}$.

By (4.5.5), we have

$$\begin{aligned} \text{R.H.S. of (4.5.4)} &\leq C_4 h^{\beta/2-\nu} \int_{X_n} p_n(t, x_n, y) m_n(dy) \\ &\leq C_4 h^{\beta/2-\nu}. \end{aligned} \quad (4.5.6)$$

Thus we finish the proof. $\square$

Now we show (B), that is, the following statement holds:

Lemma 4.5.2 *For any $k \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \dots < t_k < \infty$ and $g_1, g_2, \dots, g_k \in C_b(X)$, the following holds:*

$$\mathbb{E}^{x_n}[g_1(B_{t_1}^n) \cdots g_k(B_{t_k}^n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}^{x_\infty}[g_1(B_{t_1}^\infty) \cdots g_k(B_{t_k}^\infty)]. \quad (4.5.7)$$

Proof. Recall in §4.1 that we have

$$T_t^n f(x) = \mathbb{E}_n^x[f(B_n(t))],$$

for all $x \in X_n$, and $f \in L^2(X_n, m_n) \cap \mathcal{B}_b(X_n)$. For $g \in C_b(X)$, we set

$$g^{(n)} := g|_{X_n} := \begin{cases} g & \text{on } X_n, \\ 0 & \text{otherwise.} \end{cases}$$

By $m_n(X_n) = 1$, we have $g^{(n)} \in L^2(X, m_n)$ for all $n \in \bar{\mathbb{N}}$. Now we show $g^{(n)} \rightarrow g^{(\infty)}$ strongly in the sense of Definition 4.3.1.

Lemma 4.5.3 *For any $g \in C_b(X)$, it holds that $g^{(n)}$ converges strongly to $g^{(\infty)}$ in the sense of Definition 4.3.1.*

Proof. Since g is bounded and $m_n(X_n) = 1$ for all $n \in \mathbb{N}$, we have

$$\sup_{n \in \mathbb{N}} \|g^{(n)}\|_{L^2(m_n)} < \infty. \quad (4.5.8)$$

The remainder to show are the following:

$$\int_X g^{(n)} \phi dm_n \rightarrow \int_X g^{(\infty)} \phi dm_\infty \quad \text{for any } \phi \in C_{bs}(X), \quad (4.5.9)$$

and

$$\int_X |g^{(n)}|^2 dm_n \rightarrow \int_X |g^{(\infty)}|^2 dm_\infty. \quad (4.5.10)$$

Noting, for any $n \in \bar{\mathbb{N}}$,

$$\begin{aligned} \int_X g^{(n)} \phi dm_n &= \int_X g \phi dm_n \\ \text{and} \\ \int_X |g^{(n)}|^2 dm_n &= \int_X |g|^2 dm_n, \end{aligned}$$

and $g\phi$ and $|g|^2$ are both in $C_b(X)$, we obtain the desired result by using $m_n \rightarrow m_\infty$ weakly in $\mathcal{P}(X)$.

□

Now we resume the proof of Lemma 4.5.2.

Proof of Lemma 4.5.2: For $g \in C_b(X)$, we have

$$\mathbb{E}_n^{x_n}[g(B_t^n)] = \mathbb{E}_n^{x_n}[g^{(n)}(B_t^n)] = T_t^n g^{(n)}(x_n).$$

By using the Markov property, for all $n \in \bar{\mathbb{N}}$, we have

$$\begin{aligned} \mathbb{E}^{x_n}[g_1(B_{t_1}^n) \cdots g_k(B_{t_k}^n)] \\ = T_{t_1-t_0}^n \left(g_1^{(n)} T_{t_2-t_1}^n \left(g_2^{(n)} \cdots T_{t_k-t_{k-1}}^n g_k^{(n)} \right) \right) (x_n) \\ =: \mathcal{T}_k^n(x_n). \end{aligned} \quad (4.5.11)$$

By the Hölder continuity of the local solutions (viii) of Fact 3.3.3, we have that $T_t^n g$ is Hölder continuous on (X_n, d_n) ($T_t^n g$ is a global solution of $\frac{\partial}{\partial t} u = Lu$ with $u_0 = g$), and

$$\begin{aligned} \sup_{n \in \bar{\mathbb{N}}} (\text{Hölder constant of } T_t^n g) &\leq \sup_{n \in \bar{\mathbb{N}}} \frac{C}{r^\alpha} \|T_t^n g\|_\infty \\ &\leq \sup_{n \in \bar{\mathbb{N}}} \frac{C}{r^\alpha} \|g\|_\infty < \infty \quad (0 < r < D), \end{aligned}$$

where $0 < \alpha < 1$ and $C > 0$ are constant depending only on N, K, D . Thus we have

$$\begin{aligned} \sup_{n \in \bar{\mathbb{N}}} (\text{Hölder constant of } \mathcal{T}_k^n) &\leq \sup_{n \in \bar{\mathbb{N}}} \frac{C}{r^\alpha} \|\mathcal{T}_k^n\|_\infty \\ &\leq \sup_{n \in \bar{\mathbb{N}}} \frac{C}{r^\alpha} \|g_k \mathcal{T}_{k-1}^n\|_\infty \\ &\leq \sup_{n \in \bar{\mathbb{N}}} \frac{C}{r^\alpha} \prod_{i=1}^k \|g_i\|_\infty =: H < \infty. \end{aligned} \quad (4.5.12)$$

Our objective is to show the uniform convergence of $\mathcal{T}_k^n$ to $\mathcal{T}_k^\infty$ in the whole space X , but $\mathcal{T}_k^n$ are defined only on each X_n . Thus now we extend $\mathcal{T}_k^n$ to the whole space X preserving its Hölder regularity. Let $\tilde{\mathcal{T}}_k^n$ be the following function on the whole space X

$$\tilde{\mathcal{T}}_k^n(x) := \sup_{a \in X_n} \{\mathcal{T}_k^n(a) - Hd(a, x)^\alpha\} \quad x \in X, \quad (4.5.13)$$

where H and α are the same Hölder constant and exponents as those of the original function $\mathcal{T}_k^n$. Then we have that $\tilde{\mathcal{T}}_k^n$ is a α -Hölder continuous function on the whole space X with its Hölder constant H such that $\tilde{\mathcal{T}}_k^n = \mathcal{T}_k^n$ on X_n :

Lemma 4.5.4 (Hölder extension) *$\tilde{\mathcal{T}}_k^n$ is a α -Hölder continuous function on X with its Hölder constant H such that $\tilde{\mathcal{T}}_k^n = \mathcal{T}_k^n$ on X_n . Here α and H are the same Hölder exponent and constant as those of $\mathcal{T}_k^n$.*

Proof. For $x \in X_n$, the supremum of (4.5.13) is attained at x because of the Hölder continuity of $\mathcal{T}_k^n$ on (X_n, d_n) . Thus $\tilde{\mathcal{T}}_k^n = \mathcal{T}_k^n$ on X_n .

We now show the α -Hölder continuity of $\tilde{\mathcal{T}}_k^n$. We may assume $\tilde{\mathcal{T}}_k^n(x) - \tilde{\mathcal{T}}_k^n(y) \geq 0$ (in the case of $\tilde{\mathcal{T}}_k^n(x) - \tilde{\mathcal{T}}_k^n(y) < 0$, we can do the same proof). Then we have

$$\begin{aligned} \tilde{\mathcal{T}}_k^n(x) - \tilde{\mathcal{T}}_k^n(y) &= \sup_{a \in X_n} \{\mathcal{T}_k^n(a) - Hd(a, x)^\alpha\} - \sup_{b \in X_n} \{\mathcal{T}_k^n(b) - Hd(b, y)^\alpha\} \\ &\leq \sup_{a \in X_n} \{\mathcal{T}_k^n(a) - Hd(a, x)^\alpha - (\mathcal{T}_k^n(a) - Hd(a, y)^\alpha)\} \\ &= H \sup_{a \in X_n} \{d(a, y)^\alpha - d(a, x)^\alpha\} \\ &\leq Hd(x, y)^\alpha, \end{aligned}$$

where in the last inequality, we used the triangle inequality with $(u + v)^\alpha \leq u^\alpha + v^\alpha$ for $u, v > 0$ and $0 < \alpha < 1$.

Thus we finish the proof. $\square$

Now we resume the proof of Lemma 4.5.2.

Proof of Lemma 4.5.2: It suffices for the desired result to show $\tilde{\mathcal{T}}_k^n \rightarrow \tilde{\mathcal{T}}_k^\infty$ uniformly. In fact, we have

$$\begin{aligned} &\left| \mathbb{E}^{x_n} [g_1(B_{t_1}^n) \cdots g_k(B_{t_k}^n)] - \mathbb{E}^{x_\infty} [g_1(B_{t_1}^\infty) \cdots g_k(B_{t_k}^\infty)] \right| \\ &= |\mathcal{T}_k^n(x_n) - \mathcal{T}_k^\infty(x_\infty)| \\ &\leq |\mathcal{T}_k^n(x_n) - \tilde{\mathcal{T}}_k^\infty(x_n)| + |\tilde{\mathcal{T}}_k^\infty(x_n) - \mathcal{T}_k^\infty(x_\infty)| \\ &=: (\text{I})_n + (\text{II})_n. \end{aligned}$$

The quantity $(I)_n$ goes to zero as $n \rightarrow \infty$ because of $\mathcal{T}_k^n(x_n) = \tilde{\mathcal{T}}_k^n(x_n)$ (by $x_n \in X_n$) and the uniform convergence $\tilde{\mathcal{T}}_k^n \rightarrow \tilde{\mathcal{T}}_k^\infty$.

The quantity $(II)_n$ goes to zero as $n \rightarrow \infty$ because of $\mathcal{T}_k^\infty(x_\infty) = \tilde{\mathcal{T}}_k^\infty(x_\infty)$ and the continuity of $\tilde{\mathcal{T}}_k^\infty$.

Thus we now show $\tilde{\mathcal{T}}_k^n \rightarrow \tilde{\mathcal{T}}_k^\infty$ uniformly.

Lemma 4.5.5 $\tilde{\mathcal{T}}_k^n \rightarrow \tilde{\mathcal{T}}_k^\infty$ uniformly.

Proof. It suffices to show

- (a) $\mathcal{T}_k^n \rightarrow \mathcal{T}_k^\infty$ strongly in the Definition 4.3.1;
- (b) $\{\tilde{\mathcal{T}}_k^n\}_{n \in \mathbb{N}}$ is a family of uniform Hölder continuous functions (i.e. Hölder exponents/constants are independent of n), and $\sup_{n \in \mathbb{N}} \|\mathcal{T}_k^n\|_\infty < \infty$.
- (c) If a subsequence $\tilde{\mathcal{T}}_k^{n_1}$ converges uniformly to some function F_1 as $n_1 \rightarrow \infty$, then

$$\widetilde{F_1|_{X_\infty}} = F_1. \quad (4.5.14)$$

Here the extension $\sim$ is taken with respect to the same Hölder exponents/constants as those of $\{\tilde{\mathcal{T}}_k^n\}_{n \in \mathbb{N}}$.

In fact, by (b), using Ascoli–Arzelá theorem, $\{\tilde{\mathcal{T}}_k^n\}_{n \in \mathbb{N}}$ has a converging subsequence with respect to the uniform topology and limit functions are Hölder continuous with the same Hölder exponents/constants as those of $\{\tilde{\mathcal{T}}_k^n\}_{n \in \mathbb{N}}$. Let $\{\tilde{\mathcal{T}}_k^{n_1}\}_{n_1}$ and $\{\tilde{\mathcal{T}}_k^{n_2}\}_{n_2}$ be two subsequences with these uniform limits F_1 and F_2 as $n_1 \rightarrow \infty$ and $n_2 \rightarrow \infty$, respectively. By using (a) and continuity of the limit functions F_1 and F_2 , we have

$$F_1|_{X_\infty}(x) = F_2|_{X_\infty}(x) = \mathcal{T}_k^\infty(x) \quad \forall x \in X_\infty. \quad (4.5.15)$$

By (4.5.14) and (4.5.15), we have $F_1 = F_2 = \tilde{\mathcal{T}}_k^\infty$ on the whole space X and thus every subsequence of $\{\tilde{\mathcal{T}}_k^n\}_{n \in \mathbb{N}}$ converges to the same limit $\tilde{\mathcal{T}}_k^\infty$.

We first show (a) by induction in k . By Lemma 4.5.3, we have $g_1^{(n)} \rightarrow g_1^{(\infty)}$ strongly. Thus by Theorem 4.3.3, the statement (a) is true for $k = 1$. Assume that (a) is true when $k = l$. By noting (with Theorem 4.3.3)

$$\mathcal{T}_{l+1}^n = T_{t_{l+1}-t_l}^n(g_{l+1}^{(n)} \mathcal{T}_l^n),$$

it suffices to show $g_{l+1}^{(n)} \mathcal{T}_l^n \rightarrow g_{l+1}^{(\infty)} \mathcal{T}_l^\infty$ strongly. This is easy to show because $\mathcal{T}_l^n \rightarrow \mathcal{T}_l^\infty$ strongly (the assumption of the induction), $g_{l+1}^{(n)} \rightarrow g_{l+1}^{(\infty)}$ strongly

(by Lemma 4.5.3), and $\mathcal{T}_l^n$ and $g_{l+1}^{(n)}$ are bounded uniformly in n . Thus (a) is true for any $k \in \mathbb{N}$.

We show (b). By $\|T_t^n g\|_\infty \leq \|g\|_\infty$, we have

$$\sup_{n \in \mathbb{N}} \|\mathcal{T}_k^n\|_\infty \leq \prod_{i=1}^k \|g_i\|_\infty < \infty.$$

The uniform Hölder continuity follows from (4.5.12) and Lemma 4.5.4.

Now we show (c). Let $\tilde{\mathcal{T}}_k^{n_1}$ be a subsequence converging uniformly to F_1 as $n_1 \rightarrow \infty$. It suffices for (c) to show $\widetilde{\mathcal{T}_k^{n_1}}(x)$ converges to $\widetilde{F_1|_{X_\infty}}(x)$ for all $x \in X$. We have

$$\begin{aligned} & \left| \widetilde{\mathcal{T}_k^{n_1}}(x) - \widetilde{F_1|_{X_\infty}}(x) \right| \\ & \leq \left| \widetilde{\mathcal{T}_k^{n_1}}(x) - \widetilde{F_1|_{X_{n_1}}}(x) \right| + \left| \widetilde{F_1|_{X_{n_1}}}(x) - \widetilde{F_1|_{X_\infty}}(x) \right| \\ & = \left| \sup_{a \in X_{n_1}} \{ \mathcal{T}_k^{n_1}(a) - Hd(a, x)^\alpha \} - \sup_{b \in X_{n_1}} \{ F_1|_{X_{n_1}}(b) - Hd(b, x)^\alpha \} \right| \\ & \quad + \left| \sup_{c \in X_{n_1}} \{ F_1|_{X_{n_1}}(c) - Hd(c, x)^\alpha \} - \sup_{d \in X_\infty} \{ F_1|_{X_\infty}(d) - Hd(d, x)^\alpha \} \right| \\ & = \left| \sup_{a \in X_{n_1}} \{ \mathcal{T}_k^{n_1}(a) - Hd(a, x)^\alpha \} - \sup_{b \in X_{n_1}} \{ F_1(b) - Hd(b, x)^\alpha \} \right| \\ & \quad + \left| \sup_{c \in X_{n_1}} \{ F_1(c) - Hd(c, x)^\alpha \} - \sup_{d \in X_\infty} \{ F_1(d) - Hd(d, x)^\alpha \} \right| \\ & =: |(\text{I})_{n_1}| + |(\text{II})_{n_1}|. \end{aligned}$$

Since we have

$$\begin{aligned} & - \sup_{b \in X_{n_1}} \{ |F_1(b) - \mathcal{T}_k^{n_1}(b)| \} \leq - \sup_{b \in X_{n_1}} \{ F_1(b) - \mathcal{T}_k^{n_1}(b) \} \\ & \leq \sup_{a \in X_{n_1}} \{ \mathcal{T}_k^{n_1}(a) - Hd(a, x)^\alpha \} - \sup_{b \in X_{n_1}} \{ F_1(b) - Hd(b, x)^\alpha \} \\ & \leq \sup_{a \in X_{n_1}} \{ \mathcal{T}_k^{n_1}(a) - F_1(a) \} \\ & \leq \sup_{a \in X_{n_1}} \{ | \mathcal{T}_k^{n_1}(a) - F_1(a) | \}, \end{aligned}$$

the quantity $|(\text{I})_{n_1}|$ goes to zero because $\tilde{\mathcal{T}}_k^{n_1}$ converges uniformly to F_1 .

We show that the quantity $|(\text{II})_{n_1}|$ goes to zero as $n_1 \rightarrow \infty$. Let

$$L(\cdot) := F_1(\cdot) - Hd(\cdot, x)^\alpha.$$

Let $c_{n_1}^* \in X_{n_1}$ and $d^* \in X_\infty$ such that

$$L(c_{n_1}^*) = \sup_{c \in X_{n_1}} L(c), \quad L(d^*) = \sup_{d \in X_\infty} L(d).$$

Since X_∞ is a closed set (by the compactness of X_∞), there exists $z_{n_1} \in X_\infty$ such that $d(c_{n_1}^*, z_{n_1}) = d(c_{n_1}^*, X_\infty)$. Since X_{n_1} converges to X_∞ in (X, d) in the Hausdorff sense by Proposition 4.2.5, we have $d(c_{n_1}^*, z_{n_1}) = d(c_{n_1}^*, X_\infty) \rightarrow 0$ as $n_1 \rightarrow \infty$. Thus, by the uniform continuity of $L(\cdot)$ (implied by the compactness of X and the continuity of $L(\cdot)$ on X), we have

$$(II)_{n_1} = L(c_{n_1}^*) - L(d^*) \leq L(c_{n_1}^*) - L(z_{n_1}) \rightarrow 0. \quad (4.5.16)$$

On the other hand, by the same argument, there exists $w_{n_1} \in X_{n_1}$ such that $d(d^*, w_{n_1}) = d(d^*, X_{n_1})$. By the Hausdorff convergence of X_n to X_∞ , we have $d(d^*, w_{n_1}) = d(d^*, X_{n_1}) \rightarrow 0$. Thus we have

$$(II)_{n_1} = L(c_{n_1}^*) - L(d^*) \geq L(w_{n_1}) - L(d^*) \rightarrow 0. \quad (4.5.17)$$

By (4.5.16) and (4.5.17), we have $|(II)_{n_1}| \rightarrow 0$ as $n_1 \rightarrow \infty$, and we completed the proof. $\square$

Now we resume the proof of Lemma 4.5.2.

Proof of Lemma 4.5.2: By Lemma 4.5.5, we completed the proof of Lemma 4.5.2. $\square$

Now we resume the proof of (i) $\Rightarrow$ (ii) in Theorem 4.4.1.

Proof of (i) $\Rightarrow$ (ii) in Theorem 4.4.1:

By Lemma 4.5.1 and Lemma 4.5.2, we have completed the proof of (i) $\Rightarrow$ (ii) in Theorem 4.4.1. $\square$

Proof of (ii) $\Rightarrow$ (i) in Theorem 4.4.1:

Recall that we identified $\iota_{n\#}m_n$ with m_n , and we consider m_n as measures on X . Since X is compact, $m_n \rightarrow m_\infty$ weakly if and only if $m_n \rightarrow m_\infty$ in the W_2 -distance. Thus, by Proposition 4.2.2, it suffices for the desired result to show $m_n \rightarrow m_\infty$ weakly. To show $m_n \rightarrow m_\infty$ weakly is equivalent to show

$$\liminf_{n \rightarrow \infty} m_n(G) \geq m_\infty(G) \quad \forall G \subset X \text{ open}. \quad (4.5.18)$$

The key point for the proof is how to get information about m_n from the Brownian motions B_t^n . The following ergodic theorem gives us information about m_n as a equilibrium state of B_t^n as $t \rightarrow \infty$.

Lemma 4.5.6 For any open set $G \subset X$,

$$\mathbb{E}_n^{x_n}(\mathbf{1}_G(B_t^n)) \xrightarrow{t \rightarrow \infty} \int_X \mathbf{1}_G dm_n, \quad (4.5.19)$$

where $\mathbf{1}_G$ denotes the indicator function on G .

Proof. By the ergodic theorem of Markov processes (see e.g., [34, Theorem 4.7.3 & Exercise 4.7.2]), it suffices for the desired result to check that the Brownian motion $(\{\mathbb{P}_n^x\}_{x \in X_n}, \{B_t^n\}_{t \geq 0})$ is irreducible and recurrent.

Irreducibility:

We show that, for any T_t^n -invariant set $A \subset X_n$, it holds either $m_n(A) = 0$, or $m_n(X_n \setminus A) = 0$. Let A be a T_t^n -invariant set, that is, (see [34, Lemma 1.6.1])

$$T_t^n(\mathbf{1}_A f) = 0, \quad m_n\text{-a.e. on } X_n \setminus A,$$

for any $f \in L^2(X_n, m_n)$ and any $t > 0$. Assume that neither $m_n(A) = 0$ and $m_n(X_n \setminus A) = 0$ hold. Then, taking $f = \mathbf{1}_A$, by the Gaussian estimate (4.1.2), we have, for each $x \in X_n \setminus A$,

$$\begin{aligned} T_t^n(\mathbf{1}_A f)(x) &= \int_A p_n(t, x, y) m_n(dy) \\ &\geq \frac{C'_1}{m_n(B_{\sqrt{t}}(x))} \int_A \exp\left\{-C'_2 \frac{d(x, y)^2}{t}\right\} m_n(dy) \\ &> 0. \end{aligned} \quad (4.5.20)$$

This contradicts the T_t -invariance of A , and we finish the proof of the irreducibility.

Recurrence: Let $f : X_n \rightarrow \mathbb{R}$ be a non-negative measurable function such that $m_n(\{x : f(x) > 0\}) > 0$ and $\int_{X_n} |f| dm_n < \infty$. By [34, Lemma 1.6.4], it suffices to show

$$Gf(x) := \lim_{t \rightarrow \infty} \int_0^t T_t^n f(x) dt = \infty \quad m_n\text{-a.e. } x. \quad (4.5.21)$$

Since $m_n(B_{\sqrt{t}}(x)) = m_n(X_n) = 1$ for any $t > D^2$, we have

$$Gf(x) > \lim_{t \rightarrow \infty} \int_0^t \frac{C'_1}{m_n(B_{\sqrt{t}}(x))} \int_{X_n} \exp\left\{-C'_2 \frac{d(x, y)^2}{t}\right\} f(y) m_n(dy) = \infty.$$

Thus we have (4.5.21) and the desired result is obtained.

Thus we finish the proof. $\square$

Now we resume the proof of **(ii)** $\Rightarrow$ **(i)**.

Proof of (ii) $\Rightarrow$ (i): By **(ii)**, we have the convergence of the finite-dimensional distributions of B_t^n , and thus the following holds

$$\liminf_{n \rightarrow \infty} \mathbb{P}_n^{x_n}(B_t^n \in G) \geq \mathbb{P}_\infty^{x_\infty}(B_t^\infty \in G). \quad (4.5.22)$$

It suffices for the proof to show that $\liminf_{n \rightarrow \infty}$ and $\lim_{t \rightarrow \infty}$ are exchangeable:

$$\liminf_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P}_n^{x_n}(B_t^n \in G) = \lim_{t \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P}_n^{x_n}(B_t^n \in G). \quad (4.5.23)$$

In fact, by (4.5.18)-(4.5.23),

$$\begin{aligned} \liminf_{n \rightarrow \infty} m_n(G) &= \liminf_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P}_n^{x_n}(B_t^n \in G) = \lim_{t \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P}_n^{x_n}(B_t^n \in G) \\ &\geq \lim_{t \rightarrow \infty} \mathbb{P}_\infty^{x_\infty}(B_t^\infty \in G) = m_\infty(G). \end{aligned}$$

Thus we now show (4.5.23). To show (4.5.23), it suffices to show that, for each Brownian motions, rates of convergences to the equilibrium states are controlled uniformly in n . This is done by an uniform estimate of the spectral gaps because rates of convergences to the equilibrium states are controlled by the spectral gaps. We now show it concretely.

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |\mathbb{P}_n^x(B_t^n \in G) - m_n(G)| &= |\mathbb{E}_n^x(\mathbf{1}_G(B_t^n)) - \int_X \mathbf{1}_G dm_n| \\ &\leq \int_X |p_n(t, x, y) - \mathbf{1}| \mathbf{1}_G dm_n \\ &\leq m_n(G)^{1/2} \|p_n(t, x, \cdot) - \mathbf{1}\|_{L^2(m_n)} \\ &\leq \|p_n(t, x, \cdot) - \mathbf{1}\|_{L^2(m_n)}. \end{aligned} \quad (4.5.24)$$

Let λ_n^1 be the spectral gap of Ch_n :

$$\lambda_n^1 := \inf \left\{ \frac{\text{Ch}_n(f)}{\|f\|_{L^2(m_n)}^2} : f \in \text{Lip}(X_n) \setminus \{0\}, \int_{X_n} f dm_n = 0 \right\}. \quad (4.5.25)$$

The following is a well-known fact (easy to obtain by using the spectral resolution):

$$\|T_t^n f - m_n(f)\|_{L^2(m_n)} \leq e^{-\lambda_n^1 t} \|f - m_n(f)\|_{L^2(m_n)}, \quad (4.5.26)$$

for $f \in L^2(X_n, m_n)$ and any $t > 0$. Here we mean $m_n(f) := \int_{X_n} f dm_n$.

By (4.5.26) and the Gaussian estimate (4.1.4),

$$\begin{aligned}
\|p_n(t, x, \cdot) - \mathbf{1}\|_{L^2(m_n)} &= \|(T_s^n - m_n(\cdot))p_n(t-s, x, \cdot)\|_{L^2(m_n)} \\
&\leq e^{-\lambda_n^1 s} \|p_n(t-s, x, \cdot) - m_n(p_n(t-s, x, \cdot))\|_{L^2(m_n)} \\
&= e^{-\lambda_n^1 s} \|p_n(t-s, x, \cdot) - \mathbf{1}\|_{L^2(m_n)} \quad (0 < s < t, \quad \varepsilon := t-s) \\
&< C(N, K, D) e^{-\lambda_n^1(t-\varepsilon)}, \tag{4.5.27}
\end{aligned}$$

where $C(N, K, D) > 0$ is a positive constant depending only on N, K, D . To show the desired result (4.5.23), it suffices to show

$$\limsup_{n \rightarrow \infty} e^{-\lambda_n^1 t} \rightarrow 0 \quad (t \rightarrow \infty). \tag{4.5.28}$$

In fact, by the ergodic theorem, we have

$$\begin{aligned}
&\lim_{t \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbb{P}_n^{x_n}(B_t^n \in G) - \liminf_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P}_n^{x_n}(B_t^n \in G) \\
&= \lim_{t \rightarrow \infty} (\liminf_{n \rightarrow \infty} \mathbb{P}_n^{x_n}(B_t^n \in G) - \liminf_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P}_n^{x_n}(B_t^n \in G)) \\
&= \lim_{t \rightarrow \infty} (\liminf_{n \rightarrow \infty} \mathbb{P}_n^{x_n}(B_t^n \in G) - \liminf_{n \rightarrow \infty} m_n(B_t^n \in G)) \\
&=: \lim_{t \rightarrow \infty} (I)_t.
\end{aligned}$$

By (4.5.24) and (4.5.27) with the sub-linearity of $\limsup_{n \rightarrow \infty}$ (or the super-linearity of $\liminf_{n \rightarrow \infty}$), we have

$$\begin{aligned}
(I)_t &= \limsup_{n \rightarrow \infty} (-\mathbb{P}_n^{x_n}(B_t^n \in G)) - \limsup_{n \rightarrow \infty} (-m_n(B_t^n \in G)) \\
&\leq \limsup_{n \rightarrow \infty} (-\mathbb{P}_n^{x_n}(B_t^n \in G) + m_n(B_t^n \in G)) \\
&\leq C(N, K, D) \limsup_{n \rightarrow \infty} e^{-\lambda_n^1(t-\varepsilon)},
\end{aligned}$$

and

$$\begin{aligned}
(I)_t &\geq \liminf_{n \rightarrow \infty} (\mathbb{P}_n^{x_n}(B_t^n \in G) - m_n(B_t^n \in G)) \\
&\geq C(N, K, D) \liminf_{n \rightarrow \infty} (-e^{-\lambda_n^1(t-\varepsilon)}) \\
&= -C(N, K, D) \limsup_{n \rightarrow \infty} e^{-\lambda_n^1(t-\varepsilon)}.
\end{aligned}$$

Thus the statement (4.5.28) implies $\lim_{t \rightarrow \infty} (I)_t = 0$, that is, the desired result (4.5.23).

Thus we now show (4.5.28). For (4.5.28), it suffices to show $\liminf_{n \rightarrow \infty} \lambda_n^1 > 0$, which follows immediately from the uniform Poincaré inequality (v) in Fact 3.3.3 (in fact, we have $\inf_{n \in \mathbb{N}} \lambda_n^1 > 0$. See [47] for detailed estimates of λ_n^1).

We finish the proof of (ii) $\Rightarrow$ (i) in Theorem 4.4.1.

Thus we finish the proof of Theorem 4.4.1. $\square$

4.5.3 Proof of Corollary 4.4.2

Now we prove Corollary 4.4.2

Proof of Corollary 4.4.2: Assume (i) in Corollary 4.4.2. By Theorem 4.4.1, there exists a compact metric space (X, d) satisfying

$$\iota_n(B^n)_{\#} \mathbb{P}_n^{x_n} \rightarrow \iota_{\infty}(B^{\infty})_{\#} \mathbb{P}_{\infty}^{x_{\infty}} \quad \text{weakly in } \mathcal{P}(C([0, \infty); X)),$$

for some x_n . Let $\mathbb{B}_n := \iota_n(B^n)_{\#} \mathbb{P}_n^{x_n}$ for any $n \in \overline{\mathbb{N}}$. By Proposition 4.2.2, it suffices to show

$$W_2(\mathbb{B}_n, \mathbb{B}_{\infty}) \rightarrow 0.$$

Since $\text{Diam}(X_n) < D$ and X_n Hausdorff-converges to X_{∞} in (X, d) , we have $\text{Diam}(C([0, \infty), X)) < \infty$ with respect to the local uniform distance δ . It is known that the W_2 -convergence is equivalent to the weak convergence and the convergence of the second moment (see [90, Theorem 6.9]):

$$\int_{C([0, \infty), X)} \delta^2(x, x_0) d\mathbb{B}_n(x) \rightarrow \int_{C([0, \infty), X)} \delta^2(x, x_0) d\mathbb{B}_{\infty}(x) \quad (4.5.29)$$

for some (thus any) $x_0 \in X$. By $\text{Diam}(C([0, \infty), X)) < \infty$, the function $\delta(\cdot, x_0)$ is a bounded continuous function and thus the weak convergence of $\mathbb{B}_n$ implies (4.5.29). We finish the proof. $\square$

Chapter 5

Convergence of continuous stochastic processes on compact metric spaces converging in the Lipschitz distance

In this chapter, as one of the main results in this thesis, we formulate a new notion of convergence of stochastic processes on varying metric spaces in the Lipschitz sense, which is called *Lipschitz–Prokhorov distance* in Section 5.2, and study topological properties of this new topology in Section 5.3.

5.1 Lipschitz distance

Recall the Lipschitz distance. We say that a map $f : X \rightarrow Y$ between two metric spaces is *an isometry* if f is surjective and distance preserving. Let $\mathcal{M}$ denote the set of isometry classes of compact metric spaces. Let X and Y be in $\mathcal{M}$. For a bi-Lipschitz homeomorphism $f : X \rightarrow Y$, the *dilation* of f is defined to be the smallest Lipschitz constant of f :

$$\text{dil}(f) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}.$$

For $\varepsilon \geq 0$, a bi-Lipschitz homeomorphism $f : X \rightarrow Y$ is said to be *an ε -isometry* if

$$|\log \text{dil}(f)| + |\log \text{dil}(f^{-1})| \leq \varepsilon.$$

By definition, 0-isometry is an isometry. *The Lipschitz distance* $d_L(X, Y)$ between X and Y is defined to be the infimum of $\varepsilon \geq 0$ such that an ε -isometry between X and Y exists:

$$d_L(X, Y) = \inf\{\varepsilon \geq 0 : \exists f : X \rightarrow Y \text{ } \varepsilon\text{-isometry}\}.$$

If no bi-Lipschitz homeomorphism exists between X and Y , we define $d_L(X, Y) = \infty$. We say that a sequence X_i in $\mathcal{M}$ *Lipschitz converges to* X if

$$d_L(X_i, X) \rightarrow 0 \quad (i \rightarrow \infty).$$

We note that $(\mathcal{M}, d_L)$ is a complete metric space. This fact may be known, but we do not know any references and, for the readers' convenience, we will prove the completeness of $(\mathcal{M}, d_L)$ in Proposition 5.1.1. Note that $(\mathcal{M}, d_L)$ is not separable because the Hausdorff dimensions of X and Y must coincide if $d_L(X, Y) < \infty$. See Remark 5.1.2. We refer the reader to e.g., [37, 20] for details of the Lipschitz convergence.

Proposition 5.1.1 *$(\mathcal{M}, d_L)$ is a complete metric space.*

Proof. Let $\{X_i : i \in \mathbb{N}\}$ be a d_L -Cauchy sequence in $\mathcal{M}$. It suffices to show that there are a compact metric space $X \in \mathcal{M}$ and ε_i -isometries $f_i : X_i \rightarrow X$ with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$.

The construction of X : Let $f_{ij} : X_i \rightarrow X_j$ be an ε_{ij} -isometry for $i < j$ where $\varepsilon_{ij} \rightarrow 0$ as $i, j \rightarrow \infty$. Take a subsequence such that $\varepsilon_{i, i+1} < 1/2^i$. Let $\tilde{f}_{ij} : X_i \rightarrow X_j$ be defined by

$$\tilde{f}_{ij} = f_{j-1, j} \circ f_{j-2, j-1} \circ \cdots \circ f_{i, i+1} \quad (i < j), \quad (5.1.1)$$

and $\tilde{\varepsilon}_{ij} = \sum_{l=i}^{j-1} \varepsilon_{l, l+1}$. Then $\tilde{f}_{ij}$ is an $\tilde{\varepsilon}_{ij}$ -isometry and $\tilde{\varepsilon}_{ij} \rightarrow 0$ as $i, j \rightarrow \infty$. Since every compact metric space is separable, there is a countable dense subset $\{x_\alpha^1 : \alpha \in \mathbb{N}\} \subset X_1$. We define, for any $i > 1$,

$$x_\alpha^i = \tilde{f}_{1i}(x_\alpha^1).$$

Since $\tilde{f}_{1i}$ is a homeomorphism, the subset $\{x_\alpha^i : \alpha \in \mathbb{N}\}$ is dense in X_i for each i . Fix $\alpha, \beta \in \mathbb{N}$, and consider the sequence of the real numbers

$$\{d(x_\alpha^i, x_\beta^i) : i \in \mathbb{N}\}. \quad (5.1.2)$$

Since $\{\tilde{f}_{1i} : i \in \mathbb{N}\}$ has a bounded Lipschitz constant and the compact metric space X_1 is bounded, we have that (5.1.2) is a bounded sequence:

$$d(x_\alpha^i, x_\beta^i) \leq \sup_i \text{dil}(\tilde{f}_{1i}) d(x_\alpha^1, x_\beta^1) < \infty.$$

Thus we can take a subsequence of (5.1.2) converging to some real number, write $r(\alpha, \beta)$. We can check that r becomes a metric on $\{\alpha : \alpha \in \mathbb{N}\}$. In fact, if $r(\alpha, \beta) = 0$, we have $\alpha = \beta$ because

$$0 < \frac{1}{\sup_i \text{dil}(\tilde{f}_{1i}^{-1})} d(x_\alpha^1, x_\beta^1) \leq d(x_\alpha^i, x_\beta^i). \quad (5.1.3)$$

By definition, $r(\alpha, \alpha) = 0$ and r is symmetric and non-negative. It is easy to see the triangle inequality. Let (X, d) be the completion of the metric space $(\{\alpha : \alpha \in \mathbb{N}\}, r)$. The compactness of (X, d) will be shown later in this proof.

The construction of ε_i -isometries f_i : We define a map $f_i : \{x_\alpha^i : \alpha \in \mathbb{N}\} \rightarrow X$ by

$$f_i(x_\alpha^i) = \alpha.$$

Now we extend the map f_i to the whole space X_i . Since $\text{dil}(\tilde{f}_{ij})$ is bounded, we have

$$\begin{aligned} d(f_i(x_\alpha^i), f_i(x_\beta^i)) &= d(\alpha, \beta) = \lim_{j \rightarrow \infty} d(x_\alpha^j, x_\beta^j) = \lim_{j \rightarrow \infty} d(\tilde{f}_{ij}(x_\alpha^i), \tilde{f}_{ij}(x_\beta^i)) \\ &\leq \sup_j \text{dil}(\tilde{f}_{ij}) d(x_\alpha^i, x_\beta^i) < \infty. \end{aligned} \quad (5.1.4)$$

Let $x_{\alpha(n)}^i \rightarrow x^i \in X_i$ as $n \rightarrow \infty$. By the inequality (5.1.4), we have that $\lim_{n \rightarrow \infty} f_i(x_{\alpha(n)}^i)$ exists. This limit does not depend on the way of taking sequences converging to x^i (use the triangle inequality to check it). Thus we define

$$f_i(x^i) = \lim_{n \rightarrow \infty} f_i(x_{\alpha(n)}^i),$$

and this is well-defined. Thus we have extended the map f_i to the whole space X_i .

Now we check that f_i is bi-Lipschitz. We have

$$\begin{aligned} d(f_i(x_\alpha^i), f_i(x_\beta^i)) &= d(\alpha, \beta) = \lim_{j \rightarrow \infty} d(x_\alpha^j, x_\beta^j) = \lim_{j \rightarrow \infty} d(\tilde{f}_{ij}(x_\alpha^i), \tilde{f}_{ij}(x_\beta^i)) \\ &\geq \frac{1}{\sup_j \text{dil}(\tilde{f}_{ij}^{-1})} d(x_\alpha^i, x_\beta^i). \end{aligned} \quad (5.1.5)$$

Note that $0 < \sup_j \text{dil}(\tilde{f}_{ij}^{-1}) < \infty$. By the inequality (5.1.5), we see that f_i is bijective. By the inequality (5.1.4) and (5.1.5), we have f_i is bi-Lipschitz. Since f_i is a homeomorphism and X_i is compact, we see that $X = f_i(X_i)$ is compact. Thus $X \in \mathcal{M}$.

Finally we check that f_i is an ε_i -isometry for some $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$. We set

$$\varepsilon_i = \max\{|\log(\sup_j \text{dil}(\tilde{f}_{ij}^{-1}))|, |\log(\sup_j \text{dil}(\tilde{f}_{ij}))|\}.$$

Then, by the inequality (5.1.4) and (5.1.5), we can see that $f_i : X_i \rightarrow X$ is an ε_i -isometry with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$. Thus we have shown that X is the d_L -limit of X_i . We have completed the proof. $\square$

Note that, in the above proof, we have that

$$f_j \circ \tilde{f}_{ij} = f_i. \quad (5.1.6)$$

We use (5.1.6) in the proof of Theorem 5.2.7.

Remark 5.1.2 Note that $(\mathcal{M}, d_L)$ is not separable. This is because of the following two facts:

- (a) if $d_L(X, Y) < \infty$, the Hausdorff dimensions of X and Y must coincide;
- (b) for any non-negative real number d , there is a compact metric space X whose Hausdorff dimension is equal to d .

See, e.g., [20, Proposition 1.7.19] for (a) and [76] for (b). Let $X \in \mathcal{M}$ and $\mathcal{M}_X = \{Y \in \mathcal{M} : d_L(X, Y) < \infty\}$. We also note that there is a $X \in \mathcal{M}$ such that even when we restrict d_L to $\mathcal{M}_X$, the metric space $(\mathcal{M}_X, d_L)$ is not separable. See [89].

We consider the case of Riemannian manifolds. For a positive integer n and positive constants $K, V, D > 0$, let $\mathcal{R}(n, K, V, D)$ denote the set of isometry classes of n -dimensional connected compact Riemannian manifolds M satisfying

$$|\text{sec}(M)| \leq K, \quad \text{Vol}(M) \geq V, \quad \text{and} \quad \text{Diam}(M) \leq D. \quad (5.1.7)$$

Here $\text{sec}(M)$, $\text{Vol}(M)$ and $\text{Diam}(M)$ denote the sectional curvature, the Riemannian volume and the diameter of M , respectively. We write $\mathcal{R}$ shortly for $\mathcal{R}(n, K, V, D)$. Note that, by the Bishop inequality, there is a constant V' such that $\text{Vol}(M) < V'$ for all $M \in \mathcal{R}$. Now we give a criterion for precompactness by Gromov [37, Theorem 8.19] and Katsuda [55] (see also [13, Theorem 383]).

Proposition 5.1.3 $\mathcal{R}(n, K, V, D)$ is precompact in $(\mathcal{M}, d_L)$.

5.2 Lipschitz–Prokhorov distance

In this section, we introduce a distance d_{LP} on $\mathcal{PM}$, called *the Lipschitz–Prokhorov distance* and show $(\mathcal{PM}, d_{LP})$ is a complete metric space.

Now we introduce *the Lipschitz–Prokhorov distance*. For a continuous map $f : X \rightarrow Y$, we define $\Phi_f : \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ by

$$\Phi_f(v)(t) = f(v(t)) \quad (v \in \mathcal{C}(X), t \in [0, T]).$$

Let (X, P) be a pair of a compact metric space X and a probability measure P on $\mathcal{C}(X)$. Note that

P is *not* a probability measure on X , but on $\mathcal{C}(X)$.

We say that two pairs of (X, P) and (Y, Q) are *isomorphic* if there is an isometry $f : X \rightarrow Y$ such that the push-forward measure $\Phi_{f\#}P$ is equal to Q . Note that $\Phi_{f\#}P = Q$ implies $\Phi_{f^{-1}\#}Q = P$ and thus the isomorphic relation becomes an equivalence relation. Let $\mathcal{PM}$ denote the set of isomorphism classes of pairs (X, P) . Let (X, P) and (Y, Q) be in $\mathcal{PM}$. Now we introduce a notion of an (ε, δ) -isomorphism, which is a kind of generalization of ε -isometry. A map $f : (X, P) \rightarrow (Y, Q)$ is called an (ε, δ) -isomorphism if the following hold:

- (i) $f : X \rightarrow Y$ is an ε -isometry;
- (ii) the following inequalities hold:

$$\begin{aligned} \Phi_{f\#}P(A) &\leq Q(A^{\delta e^\varepsilon}) + \delta e^\varepsilon, \quad Q(A) \leq \Phi_{f\#}P(A^{\delta e^\varepsilon}) + \delta e^\varepsilon, \\ \Phi_{f^{-1}\#}Q(B) &\leq P(B^{\delta e^\varepsilon}) + \delta e^\varepsilon, \quad P(B) \leq \Phi_{f^{-1}\#}Q(B^{\delta e^\varepsilon}) + \delta e^\varepsilon, \end{aligned} \quad (5.2.1)$$

for any Borel sets $A \subset \mathcal{C}(Y)$ and $B \subset \mathcal{C}(X)$.

We now define a distance between (X, P) and (Y, Q) in $\mathcal{PM}$, which is called *the Lipschitz–Prokhorov distance*.

Definition 5.2.1 Let (X, P) and (Y, Q) be in $\mathcal{PM}$. *The Lipschitz–Prokhorov distance* between (X, P) and (Y, Q) is defined to be the infimum of $\varepsilon + \delta \geq 0$ such that an (ε, δ) -isomorphism $f : (X, P) \rightarrow (Y, Q)$ exists:

$$d_{LP}((X, P), (Y, Q)) = \inf\{\varepsilon + \delta \geq 0 : \exists f : (X, P) \rightarrow (Y, Q) \text{ } (\varepsilon, \delta)\text{-isomorphism}\}.$$

If there is no (ε, δ) -isomorphism between (X, P) and (Y, Q) , we define

$$d_{LP}((X, P), (Y, Q)) = \infty.$$

Remark 5.2.2 If we replace e^ε to 1 in the inequalities (5.2.1), then the triangle inequality fails for d_{LP} .

Remark 5.2.3 If we start at metric measure spaces, it may seem to be more natural than (X, P) to consider triplets $(\mathcal{C}(X), d_{\mathcal{C}}, P)$ where $X \in \mathcal{M}$ and P is a probability measure on $\mathcal{C}(X)$. It is, however, not suitable for our motivation because the Lipschitz convergence of $\mathcal{C}(X_i)$ does not imply the Lipschitz convergence of X_i in general.

It is clear by definition that d_{LP} is well-defined in $\mathcal{PM}$, that is, if (X, P) is isomorphic to (X', P') and (Y, Q) is isomorphic to (Y', Q') , then

$$d_{LP}((X, P), (Y, Q)) = d_{LP}((X', P'), (Y', Q')).$$

It is also clear by definition that $d_{LP}((X, P), (X, P)) = 0$, and d_{LP} is non-negative and symmetric. To show that d_{LP} is a metric on $\mathcal{PM}$, it is enough to show that d_{LP} satisfies the triangle inequality and that (X, P) and (Y, Q) are isomorphic if $d_{LP}((X, P), (Y, Q)) = 0$. Before the proof, we utilize the following lemma:

Lemma 5.2.4 *Let $f : X \rightarrow Y$ be an ε -isometry. Then $\Phi_f : \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ is also an ε -isometry with respect to the uniform metric $d_{\mathcal{C}}$. As a byproduct, for any $a \geq 0$ and Borel set $A \subset \mathcal{C}(Y)$, we have*

$$\Phi_f^{-1}(A^a) \subset \Phi_f^{-1}(A)^{ae^\varepsilon} \quad \text{and} \quad \Phi_f^{-1}(A)^a \subset \Phi_f^{-1}(A^{ae^\varepsilon}),$$

and, for any Borel set $B \subset \mathcal{C}(X)$, we have

$$\Phi_f(B^a) \subset \Phi_f(B)^{ae^\varepsilon} \quad \text{and} \quad \Phi_f(B)^a \subset \Phi_f(B^{ae^\varepsilon}).$$

Proof. We first show that $\Phi_f : \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ is an ε -isometry. Since f is homeomorphic, it is clear that Φ_f is a homeomorphism. It is enough to show that $\text{dil}(\Phi_f) = \text{dil}(f)$ and $\text{dil}(\Phi_f^{-1}) = \text{dil}(f^{-1})$. Let $v, w \in \mathcal{C}(X)$. By the compactness of $[0, T]$ and the continuity of v, w and f , there are $t_0 \in [0, T]$ and $s_0 \in [0, T]$ such that

$$d_{\mathcal{C}}(v, w) = d(v(t_0), w(t_0)) \quad \text{and} \quad d_{\mathcal{C}}(\Phi_f(v), \Phi_f(w)) = d(f(v(s_0)), f(w(s_0))).$$

Then we have

$$\begin{aligned} \frac{d_{\mathcal{C}}(\Phi_f(v), \Phi_f(w))}{d_{\mathcal{C}}(v, w)} &= \frac{d(f(v(s_0)), f(w(s_0)))}{d(v(t_0), w(t_0))} \\ &\leq \text{dil}(f) \frac{d(v(s_0), w(s_0))}{d(v(t_0), w(t_0))} \\ &\leq \text{dil}(f). \end{aligned}$$

Thus we have $\text{dil}(\Phi_f) \leq \text{dil}(f)$. For $x \in X$, let $c_x \in \mathcal{C}(X)$ denote the constant path on x , that is, $c_x(t) = x$ for all $t \in [0, T]$. Then we have

$$\frac{d(f(x), f(y))}{d(x, y)} = \frac{d_{\mathcal{C}}(\Phi_f(c_x), \Phi_f(c_y))}{d_{\mathcal{C}}(c_x, c_y)} \leq \text{dil}(\Phi_f).$$

Thus we have $\text{dil}(\Phi_f) = \text{dil}(f)$. By the same argument, we also have $\text{dil}(\Phi_f^{-1}) = \text{dil}(f^{-1})$. These imply that Φ_f is an ε -isometry.

We second show the inclusions in the statement. It is enough to show one of the inclusions, say, $\Phi_f(B^a) \subset \Phi_f(B)^{ae^\varepsilon}$ for $a \geq 0$ and any Borel sets $B \subset \mathcal{C}(X)$. Let $x \in \Phi_f(B^a)$ and $y \in B^a$ such that $\Phi_f(y) = x$. Since Φ_f is an ε -isometry, we have

$$d_{\mathcal{C}}(x, \Phi_f(B)) \leq \text{dil}(\Phi_f)d_{\mathcal{C}}(y, B) \leq e^\varepsilon d_{\mathcal{C}}(y, B) \leq ae^\varepsilon.$$

Thus we have $x \in \Phi_f(B)^{ae^\varepsilon}$ and finish the proof. $\square$

Now we show that d_{LP} is a metric on $\mathcal{PM}$.

Theorem 5.2.5 *d_{LP} is a metric on $\mathcal{PM}$.*

Proof. It is enough to show the following two statements:

- (i) d_{LP} satisfies the triangle inequality;
- (ii) $d_{LP}((X, P), (Y, Q)) = 0$ implies that (X, P) and (Y, Q) are isomorphic.

We first show the statement (i). Let $(X, P), (Y, Q)$ and $(Z, R) \in \mathcal{PM}$ such that there are $(\varepsilon_1, \delta_1)$ -isomorphism $f_1 : X \rightarrow Y$ and $(\varepsilon_2, \delta_2)$ -isomorphism $f_2 : Y \rightarrow Z$. It suffices to show that $f_2 \circ f_1 : X \rightarrow Z$ is an $(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2)$ -isomorphism. In fact, this implies

$$d_{LP}((X, P), (Z, R)) < \varepsilon_1 + \varepsilon_2 + \delta_1 + \delta_2.$$

By taking the infimum of $\varepsilon_1 + \delta_1$ and $\varepsilon_2 + \delta_2$, we have the triangle inequality.

Thus we now show that $f_2 \circ f_1 : X \rightarrow Z$ is an $(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2)$ -isomorphism. We know that $f_2 \circ f_1$ is an $(\varepsilon_1 + \varepsilon_2)$ -isometry (see e.g., [20, Theorem 7.2.4]). For any Borel set $A \subset \mathcal{C}(Z)$, we have

$$\begin{aligned} \Phi_{f_2 \circ f_1 \#} P(A) &= (\Phi_{f_1 \#} P)(\Phi_{f_2}^{-1}(A)) \\ &\leq Q(\Phi_{f_2}^{-1}(A)^{\delta_1 e^{\varepsilon_1}}) + \delta_1 e^{\varepsilon_1} \\ &\leq Q(\Phi_{f_2}^{-1}(A^{\delta_1 e^{\varepsilon_1 + \varepsilon_2}})) + \delta_1 e^{\varepsilon_1} \\ &\leq R(A^{\delta_1 e^{\varepsilon_1 + \varepsilon_2} + \delta_2 e^{\varepsilon_2}}) + \delta_1 e^{\varepsilon_1} + \delta_2 e^{\varepsilon_2} \\ &\leq R(A^{(\delta_1 + \delta_2) e^{\varepsilon_1 + \varepsilon_2}}) + (\delta_1 + \delta_2) e^{\varepsilon_1 + \varepsilon_2}. \end{aligned}$$

The inequality of the third line follows from Lemma 5.2.4. The other directions of (5.2.1) can be shown by the same argument. Thus we have that $f_2 \circ f_1$ is an $(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2)$ -isomorphism. We have the triangle inequality.

We second show (ii). We show that there is an isometry $\iota : X \rightarrow Y$ such that $\Phi_\iota : (\mathcal{C}(X), P) \rightarrow (\mathcal{C}(Y), Q)$ is a measure-preserving map, that is, for any real-valued uniformly continuous and bounded function u on $\mathcal{C}(Y)$, we have

$$\int_{\mathcal{C}(X)} u \circ \Phi_\iota dP = \int_{\mathcal{C}(Y)} u dQ. \quad (5.2.2)$$

Let $f_i : X \rightarrow Y$ be an $(\varepsilon_i, \delta_i)$ -isomorphism with $\varepsilon_i, \delta_i \rightarrow 0$ as $i \rightarrow \infty$. Since the dilations of $\{f_i : i \in \mathbb{N}\}$ are uniformly bounded, by the Ascoli–Arzelà theorem, we can take a subsequence from $\{f_i : i \in \mathbb{N}\}$ converging uniformly to a continuous function ι . Since $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$, we have that ι is an isometry from X to Y , and, for any $\varepsilon > 0$, there is an i_0 such that, for any $i_0 \leq i$, we have $d(\iota(x), f_i(x)) < \varepsilon$ for all $x \in X$. See e.g., [20, Theorem 7.2.4] for details. By this fact, we have

$$d_{\mathcal{C}}(\Phi_\iota(v), \Phi_{f_i}(v)) \leq \varepsilon \quad (\forall i \geq i_0, \forall v \in \mathcal{C}(X)).$$

By the uniform continuity of u , we have

$$\left| \int_{\mathcal{C}(X)} u \circ \Phi_\iota dP - \int_{\mathcal{C}(X)} u \circ \Phi_{f_i} dP \right| \leq \varepsilon' P(\mathcal{C}(X)) \rightarrow 0 \quad (i \rightarrow \infty). \quad (5.2.3)$$

Since $d_P(\Phi_{f_i\#}P, Q) \rightarrow 0$ as $i \rightarrow \infty$, we know that $\Phi_{f_i\#}P$ converges weakly to Q as $i \rightarrow \infty$ (see e.g. [14, §6]):

$$\int_{\mathcal{C}(X)} u \circ \Phi_{f_i} dP = \int_{\mathcal{C}(Y)} u d(\Phi_{f_i\#}P) \rightarrow \int_{\mathcal{C}(Y)} u dQ \quad (i \rightarrow \infty). \quad (5.2.4)$$

By (5.2.3) and (5.2.4), we have the equality (5.2.2) and we finish the proof.

□

Remark 5.2.6 When we take $X = Y$, by definition, we have $d_P(P, Q) \geq d_{LP}((X, P), (X, Q))$. The relation between d_P and d_{LP} is as follows:

$$d_{LP}((X, P), (X, Q)) = \inf_{\substack{f: X \rightarrow X \\ \text{isometry}}} d_P(\Phi_{f\#}P, Q). \quad (5.2.5)$$

The following example shows that $d_{LP}((X, P), (X, Q)) = 0$ does not imply $d_P(P, Q) = 0$: Let $X = S^1$ with the metric d where d is the restriction of

the Euclidean metric in $\mathbb{R}^2$. Let $x, y \in S^1$ with $x \neq y$. Let $c_x, c_y \in \mathcal{C}(S^1)$ denote the constant paths on x and y , that is, $c_x(t) = x$ and $c_y(t) = y$ for all $t \in [0, T]$. Let δ_{c_x} and δ_{c_y} be the Dirac measures on c_x and c_y . Let $f : S^1 \rightarrow S^1$ be the rotation which rotate x to y . Then, by (5.2.5), we have

$$d_{LP}((S^1, \delta_{c_x}), (S^1, \delta_{c_y})) \leq d_P(\Phi_{f\#} \delta_{c_x}, \delta_{c_y}) = 0.$$

We see, however, that $d_P(\delta_{c_x}, \delta_{c_y}) = d(x, y) \neq 0$. See also Figure 5.1 below.

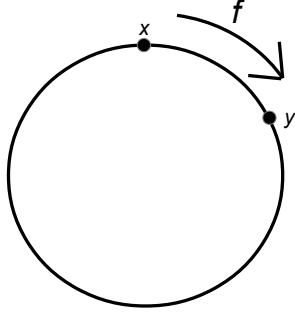


Figure 5.1: $d_{LP} = 0$ does not imply $d_P = 0$.

Now we show that the metric space $(\mathcal{PM}, d_{LP})$ is complete.

Theorem 5.2.7 *The metric space $(\mathcal{PM}, d_{LP})$ is complete.*

Proof. Let $\{(X_i, P_i) : i \in \mathbb{N}\}$ be a d_{LP} -Cauchy sequence in $\mathcal{PM}$. It is enough to show that there are a pair $(X, P) \in \mathcal{PM}$ and a family of $(\varepsilon_i, \delta_i)$ -isomorphisms $f_i : (X_i, P_i) \rightarrow (X, P)$ with $\varepsilon_i, \delta_i \rightarrow 0$ as $i \rightarrow \infty$.

The existence of X : The existence of X follows directly from the completeness of $(\mathcal{M}, d_L)$. In fact, since $\{(X_i, P_i) : i \in \mathbb{N}\}$ is a d_{LP} -Cauchy sequence, the sequence $\{X_i : i \in \mathbb{N}\}$ is a d_L -Cauchy sequence in $\mathcal{M}$. By the completeness of $(\mathcal{M}, d_L)$ (see Proposition 5.1.1), there is a compact metric space X such that

$$d_L(X_i, X) \rightarrow 0 \quad (i \rightarrow \infty). \quad (5.2.6)$$

The existence of P and f_i : Since $\{(X_i, P_i) : i \in \mathbb{N}\}$ is a d_{LP} -Cauchy sequence, there is a family of $(\varepsilon_{ij}, \delta_{ij})$ -isomorphisms $f_{ij} : X_i \rightarrow X_j$ for $i < j$

with $\varepsilon_{ij} \rightarrow 0$ and $\delta_{ij} \rightarrow 0$ as $i, j \rightarrow \infty$. Take a subsequence such that $\varepsilon_{i,i+1} + \delta_{i,i+1} < 1/2^i$. Let $\tilde{f}_{ij} : X_i \rightarrow X_j$ be defined by

$$\tilde{f}_{ij} = f_{j-1,j} \circ f_{j-2,j-1} \circ \cdots \circ f_{i,i+1} \quad (i < j), \quad (5.2.7)$$

and $\tilde{\varepsilon}_{ij} = \sum_{l=i}^{j-1} \varepsilon_{l,l+1}$, and $\tilde{\delta}_{ij} = \sum_{l=i}^{j-1} \delta_{l,l+1}$. By the proof of Theorem 5.2.5, we see that $\tilde{f}_{ij}$ is an $(\tilde{\varepsilon}_{ij}, \tilde{\delta}_{ij})$ -isomorphism and $\tilde{\varepsilon}_{ij}, \tilde{\delta}_{ij} \rightarrow 0$ as $i, j \rightarrow \infty$. By the proof of Proposition 5.1.1 (see also the equality (5.1.6)), there is a family of ε_i -isometries $f_i : X_i \rightarrow X$ such that $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$ and

$$f_i \circ \tilde{f}_{ji} = f_j.$$

This implies that

$$\Phi_{f_i} \circ \Phi_{\tilde{f}_{ji}} = \Phi_{f_j}. \quad (5.2.8)$$

We now show that $\{\Phi_{f_i} P_i : i \in \mathbb{N}\}$ is a d_P -Cauchy sequence in $\mathcal{P}(\mathcal{C}(X))$. Let us set

$$\varepsilon = \varepsilon(i, j) = (\tilde{\delta}_{ij} + \tilde{\delta}_{ji}) e^{\tilde{\varepsilon}_{ij} + \tilde{\varepsilon}_{ji} + \varepsilon_i + \varepsilon_j}.$$

Note that $\varepsilon \rightarrow 0$ as $i, j \rightarrow \infty$. It suffices to show that, for any Borel set $A \subset \mathcal{C}(X)$,

$$\Phi_{f_i} P_i(A) - \Phi_{f_j} P_j(A^\varepsilon) \leq \varepsilon, \quad \Phi_{f_i} P_i(A^\varepsilon) - \Phi_{f_j} P_j(A) \leq \varepsilon.$$

We only show the left-hand side of the above inequalities (the right-hand side can be shown by the same argument). For any Borel set $A \subset \mathcal{C}(X)$, we have

$$\begin{aligned} & \Phi_{f_i} P_i(A) - \Phi_{f_j} P_j(A^\varepsilon) \\ &= (\Phi_{f_i} P_i(A) - \Phi_{f_i \circ \tilde{f}_{ji}} P_j(A^\varepsilon)) + (\Phi_{f_i \circ \tilde{f}_{ji}} P_j(A^\varepsilon) - \Phi_{f_j \circ \tilde{f}_{ij}} P_i(A)) \\ & \quad + (\Phi_{f_j \circ \tilde{f}_{ij}} P_i(A) - \Phi_{f_j} P_j(A^\varepsilon)) \\ &=: \text{(I)} + \text{(II)} + \text{(III)}. \end{aligned}$$

Since $\tilde{f}_{ji} : X_j \rightarrow X_i$ is the $(\tilde{\varepsilon}_{ji}, \tilde{\delta}_{ji})$ -isomorphism, we have

$$\text{(I)} = (P_i \Phi_{f_i}^{-1}(A) - (\Phi_{\tilde{f}_{ji}} P_j) \Phi_{f_i}^{-1}(A^\varepsilon)) \leq \tilde{\delta}_{ji} e^{\tilde{\varepsilon}_{ji}} \leq \varepsilon.$$

By the same argument, we also have $(\text{III}) \leq \varepsilon$. For the estimate of (II), by Lemma 5.2.4 and (5.2.8), we have

$$\begin{aligned}
\Phi_{f_j \circ \tilde{f}_{ij} \#} P_i(A) &= (\Phi_{\tilde{f}_{ij} \#} P_i)(\Phi_{f_j}^{-1}(A)) \leq P_j(\Phi_{f_j}^{-1}(A)^{\tilde{\delta}_{ij} e^{\tilde{\varepsilon}_{ij}}}) + \tilde{\delta}_{ij} e^{\tilde{\varepsilon}_{ij}} \\
&\leq P_j(\Phi_{f_j}^{-1}(A^{\tilde{\delta}_{ij} e^{\tilde{\varepsilon}_{ij} + \varepsilon_j}})) + \tilde{\delta}_{ij} e^{\tilde{\varepsilon}_{ij}} \\
&= P_j(\Phi_{f_{ji}}^{-1} \circ \Phi_{f_i}^{-1}(A^{\tilde{\delta}_{ij} e^{\tilde{\varepsilon}_{ij} + \varepsilon_j}})) + \tilde{\delta}_{ij} e^{\tilde{\varepsilon}_{ij}} \\
&= \Phi_{f_i \circ \tilde{f}_{ji} \#} P_j(A^\varepsilon) + \varepsilon.
\end{aligned}$$

We used Lemma 5.2.4 in the second line and (5.2.8) in the third line. We have $(\text{II}) \leq \varepsilon$. Thus $\{\Phi_{f_i \#} P_i : i \in \mathbb{N}\}$ is a d_P -Cauchy sequence. By the completeness of $(\mathcal{P}(\mathcal{C}(X)), d_P)$, there exists a probability measure P on $\mathcal{C}(X)$ such that

$$d_P(\Phi_{f_i \#} P_i, P) \rightarrow 0 \quad (i \rightarrow \infty). \quad (5.2.9)$$

We finally show that $f_i : (X_i, P_i) \rightarrow (X, P)$ is an $(\varepsilon_i, \delta_i)$ -isomorphism for some sequence $\delta_i \rightarrow 0$ as $i \rightarrow \infty$. By (5.2.9), there is a sequence $\delta_i \rightarrow 0$ as $i \rightarrow \infty$ such that

$$\Phi_{f_i \#} P_i(A) \leq P(A^{\delta_i}) + \delta_i \quad \text{and} \quad P(A) \leq \Phi_{f_i \#} P_i(A^{\delta_i}) + \delta_i, \quad (5.2.10)$$

for any Borel set $A \subset \mathcal{C}(X)$. By the inequalities (5.2.10) and Lemma 5.2.4, we have

$$\begin{aligned}
\Phi_{f_i^{-1} \#} P(B) &= P(\Phi_{f_i}^{-1}(B)) \leq \Phi_{f_i \#} P_i(\Phi_{f_i}^{-1}(B)^{\delta_i}) + \delta_i \\
&\leq P_i(\Phi_{f_i}^{-1} \circ \Phi_{f_i^{-1}}^{-1}(B^{\delta_i e^{\varepsilon_i}})) + \delta_i e^{\varepsilon_i} \\
&\leq P_i(B^{\delta_i e^{\varepsilon_i}}) + \delta_i e^{\varepsilon_i} \quad (\forall B \subset \mathcal{C}(X_i) : \text{Borel}).
\end{aligned} \quad (5.2.11)$$

By the same argument, we also have

$$P_i(B) \leq \Phi_{f_i^{-1} \#} P(B^{\delta_i e^{\varepsilon_i}}) + \delta_i e^{\varepsilon_i} \quad (\forall B \subset \mathcal{C}(X_i) : \text{Borel}). \quad (5.2.12)$$

Note in (5.2.10) that

$$P(A^{\delta_i}) + \delta_i \leq P(A^{\delta_i e^{\varepsilon_i}}) + \delta_i e^{\varepsilon_i} \quad \text{and} \quad \Phi_{f_i \#} P_i(A^{\delta_i}) + \delta_i \leq \Phi_{f_i \#} P_i(A^{\delta_i e^{\varepsilon_i}}) + \delta_i e^{\varepsilon_i}. \quad (5.2.13)$$

Thus, by (5.2.10), (5.2.11), (5.2.12) and (5.2.13), we have

$$\begin{aligned}\Phi_{f_i\#}P_i(A) &\leq P(A^{\delta_i e^{\varepsilon_i}}) + \delta_i e^{\varepsilon_i}, \quad P(A) \leq \Phi_{f_i\#}P_i(A^{\delta_i e^{\varepsilon_i}}) + \delta_i e^{\varepsilon_i}, \\ \Phi_{f_i^{-1}\#}P(B) &\leq P_i(B^{\delta_i e^{\varepsilon_i}}) + \delta_i e^{\varepsilon_i}, \quad P_i(B) \leq \Phi_{f_i^{-1}\#}P_i(B^{\delta_i e^{\varepsilon_i}}) + \delta_i e^{\varepsilon_i},\end{aligned}\tag{5.2.14}$$

for any Borel sets $A \subset \mathcal{C}(X)$ and $B \subset \mathcal{C}(X_i)$. Since $f_i : X_i \rightarrow X$ is an ε_i -isometry with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$, the inequalities (5.2.14) means that $f_i : (X_i, P_i) \rightarrow (X, P)$ is an $(\varepsilon_i, \delta_i)$ -isomorphism with $\varepsilon_i, \delta_i \rightarrow 0$ as $i \rightarrow \infty$. We therefore have the desired result. $\square$

Remark 5.2.8 The metric space $(\mathcal{PM}, d_{LP})$ is not separable. This is because $(\mathcal{M}, d_L)$ is not separable as in Remark 5.1.2. Let $X \in \mathcal{M}$ and $\mathcal{M}_X = \{Y \in \mathcal{M} : d_L(X, Y) < \infty\}$. Let $\mathcal{PM}_X$ be the set of isomorphism classes of pairs (Y, P) where $Y \in \mathcal{M}_X$ and $P \in \mathcal{P}(\mathcal{C}(Y))$. By [89], there is a $X \in \mathcal{M}$ such that $(\mathcal{M}_X, d_L)$ is not separable. Thus, for such X , $(\mathcal{PM}_X, d_{LP})$ is not separable.

By the proof of Theorem 5.2.7, we have the following:

Corollary 5.2.9 *Let $(X_i, P_i), (X, P) \in \mathcal{PM}$ for all $i \in \mathbb{N}$. The sequence (X_i, P_i) converges to (X, P) in d_{LP} as $i \rightarrow \infty$ if and only if there is a family of ε_i -isometries $f_i : X_i \rightarrow X$ with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$ such that*

$$\Phi_{f_i\#}P_i \rightarrow P \quad \text{weakly} \quad (i \rightarrow \infty).$$

5.3 Relative compactness

In this section, we first give a sufficient condition for subsets in $\mathcal{PM}$ to be relative compact in the case of Markov processes on Riemannian manifolds. Roughly speaking, this sufficient condition consists of two conditions: one is a boundedness condition of Riemannian manifolds for sectional curvatures, diameters and volumes with a fixed dimension, and the other is an upper heat kernel bound for Markov processes uniformly in each manifolds. Second we give a sufficient condition for sequences in a relatively compact set to be convergent. This sufficient condition will be stated in terms of the generalized Mosco-convergence introduced by [62].

We explain Markov processes considered in this section. Let $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on $L^2(M, \text{Vol})$ and $\{T_t\}_{t \in (0, \infty)}$ be the

corresponding semigroup on $L^2(M, \text{Vol})$. We say that $\{p(t, x, y) : x, y \in M, t \in (0, \infty)\}$ is a *heat kernel* of $\{T_t\}_{t \in (0, \infty)}$ if $p(t, x, y)$ becomes an integral kernel of T_t : for $f \in L^2(M; \text{Vol})$

$$T_t f(x) = \int_M f(y) p(t, x, y) \text{Vol}(dy) \quad (\forall x \in M).$$

Assumption 5.3.1 The following conditions hold:

- (i) $(\mathcal{E}, \mathcal{F})$ is a strongly local regular symmetric Dirichlet form on $L^2(M, \text{Vol})$;
- (ii) the corresponding semigroup $\{T_t\}_{t \in (0, \infty)}$ is a Feller semigroup and has a jointly continuous heat kernel $p(t, x, y)$ on $(0, T] \times M \times M$.

See, e.g., [15, Theorem I.9.4] for the Feller property.

Remark 5.3.2 We assumed the joint-continuity and the Feller property of the heat kernel only for simplicity. The following arguments can be modified by excluding null-capacity sets with respect to $(\mathcal{E}, \mathcal{F})$ if we need to remove the Feller property assumption. The joint-continuity of a function ϕ which will be taken in (5.3.3) is assumed for the same reason.

By [15, Theorem I.9.4], there is a Hunt process

$$(\Omega, \mathcal{M}, \{\mathcal{M}(t)\}_{t \in [0, \infty)}, \{P^x\}_{x \in M}, \{X(t)\}_{t \in [0, \infty)}) \quad (5.3.1)$$

such that, for any bounded Borel function f in $L^2(M, \text{Vol})$, we have

$$E^x(f(X(t))) = T_t f(x) \quad (\forall t \in (0, \infty), \forall x \in M).$$

By the locality of $(\mathcal{E}, \mathcal{F})$, we know that $X(\cdot)$ has continuous paths almost surely. By the strong locality and the compactness of M , we know that $X(t) \in M$ for all $t \in [0, \infty)$ almost surely. Thus we see that $X : \Omega \rightarrow \mathcal{C}(M)$ almost surely and the law of X lives on $\mathcal{C}(M)$. We refer the reader to e.g., [34] for details of Dirichlet forms and Hunt processes.

Let μ be a probability measure on M . Let P^μ denote the probability measure with the initial distribution μ :

$$P^\mu(A) = \int_A P^x \mu(dx) \quad (\forall A \subset M : \text{Borel}). \quad (5.3.2)$$

Now we introduce a main object in this section, a subset $\mathcal{P}_\phi\mathcal{R}(n, K, V, D)$ of $\mathcal{PM}$ determined by a certain function ϕ . Let $\phi : (0, T] \times [0, D] \rightarrow [0, \infty)$ be a jointly continuous function satisfying that, for any $\varepsilon > 0$,

$$\lim_{\lambda \rightarrow 0} \sup_{r > \varepsilon, \xi \in (0, \lambda]} \phi(\xi, r) = 0, \quad (5.3.3)$$

where $D > 0$ is the uniform bound of diameters of elements in $\mathcal{R} = \mathcal{R}(n, K, V, D)$.

Definition 5.3.3 For a function ϕ satisfying the above conditions, the set $\mathcal{P}_\phi\mathcal{R}(n, K, V, D)$ is defined to be the set of isomorphism classes of pairs (M, P) where $M \in \mathcal{R}$ and P is the law of P^μ for an initial distribution μ and a Hunt process on M associated with $(\mathcal{E}, \mathcal{F})$ satisfying Assumption 5.3.1 and that the heat kernel $p(t, x, y)$ is dominated by ϕ in the following sense: there exists a $\tau > 0$ such that for all $t \in (0, \tau \wedge T]$, all $x, y \in M$ and all $M \in \mathcal{R}$,

$$p(t, x, y) \leq \phi(t, d_M(x, y)). \quad (5.3.4)$$

We also write $\mathcal{P}_\phi\mathcal{R}$ shortly for $\mathcal{P}_\phi\mathcal{R}(n, K, V, D)$.

Then we have the main theorem of this section:

Theorem 5.3.4 *The set $\mathcal{P}_\phi\mathcal{R}$ is relatively compact in $(\mathcal{PM}, d_{LP})$.*

Remark 5.3.5 Let $\overline{\mathcal{P}_\phi\mathcal{R}}$ be the completion of $\mathcal{P}_\phi\mathcal{R}$ with respect to d_{LP} . As a byproduct of Theorem 5.3.4, we see

$$(\overline{\mathcal{P}_\phi\mathcal{R}}, d_{LP}) \text{ is a compact metric space.}$$

Let $\overline{\mathcal{R}}$ be the completion of $\mathcal{R}$ with respect to d_L . In general, $M \in \overline{\mathcal{R}}$ has a $C^{1,\alpha}$ -Riemannian structure for any $0 < \alpha < 1$. See, e.g., [13, Theorem 384].

Proof of Theorem 5.3.4: Since any metric spaces satisfy the first axiom of countability, it is enough to show that any sequence $(M_i, P_i) \in \mathcal{P}_\phi\mathcal{R}$ has a subsequence converging to some $(M, P) \in \mathcal{PM}$. Since $\mathcal{R}$ is relatively compact with respect to d_L (see Proposition 5.1.3), the sequence $M_i \in \mathcal{R}$ has a converging subsequence (write also M_i) to a compact metric space M with respect to d_L . Thus there is a family of ε_i -isometries $f_i : M_i \rightarrow M$ with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$.

By Corollary 5.2.9, the proof is completed if we show $\{\Phi_{f_i\#}P_i : i \in \mathbb{N}\}$ is relatively compact in $(\mathcal{P}(\mathcal{C}(M)), d_P)$. By [14, §5], it is equivalent to show

that $\{\Phi_{f_i \#} P_i : i \in \mathbb{N}\}$ is tight. That is, for any $\varepsilon > 0$, there is a compact set $K \subset \mathcal{C}(M)$ such that

$$\Phi_{f_i \#} P_i(K) > 1 - \varepsilon \quad (\forall i \in \mathbb{N}).$$

Let $(\mathcal{E}_i, \mathcal{F}_i)$ and μ_i be a strongly local regular Dirichlet form and its initial distribution associated with P_i . Let a Hunt process induced by $(\mathcal{E}_i, \mathcal{F}_i)$ be denoted by

$$(\{\mathcal{M}_i(t)\}_{t \in [0, \infty)}, \{P_i^x\}_{x \in M_i}, \{X_i(t)\}_{t \in [0, \infty)}).$$

By [2, Theorem 1] (see also [26, Proposition 4.3]), it suffices for tightness of $\{\Phi_{f_i \#} P_i : i \in \mathbb{N}\}$ to show the following two statements:

- (i) for some fixed $m \in M$, it holds that, for each positive η , there are $a \geq 0$ and $i_0 \in \mathbb{N}$ such that

$$\Phi_{f_i \#} P_i(v : d(v(0), m) \geq a) \leq \eta \quad (\forall i \geq i_0);$$

- (ii) for any positive constants γ and ζ , there is $\lambda > 0$ such that

$$\limsup_{i \rightarrow \infty} \Phi_{f_i \#} P_i(v : \sup_{s: t \leq s \leq t+\lambda} d(v(s), v(t)) > \gamma) < \zeta \quad (\forall t \in [0, T - \lambda]). \quad (5.3.5)$$

We show the statement (i). Since f_i is an ε_i -isometry, we have that, for any $a > 0$,

$$\Phi_{f_i \#} P_i(v : d(v(0), m) \geq a) \leq P_i(v : d_i(v(0), f_i^{-1}(m)) \geq ae^{-\varepsilon_i}).$$

Since $\text{Diam}(M) < D$ for any $M \in \mathcal{R}$, if we take a sufficiently large a such that $\sup_i ae^{-\varepsilon_i} > D$, we have

$$P_i(v : d_i(v(0), f_i^{-1}(m)) \geq ae^{-\varepsilon_i}) = 0.$$

This concludes (i).

We show the statement (ii). Since Φ_{f_i} is an ε_i -isometry, we have

$$\Phi_{f_i \#} P_i(v : \sup_{s: t \leq s \leq t+\lambda} d(v(s), v(t)) > \gamma) \leq P_i(v : \sup_{s: t \leq s \leq t+\lambda} d_i(v(s), v(t)) > \gamma e^{-\varepsilon_i}). \quad (5.3.6)$$

By the Markov property, we have, for any $t \in [0, T - \lambda]$,

$$\begin{aligned}
& P_i^x \left(\sup_{s:t \leq s \leq t+\lambda} d_i(X_i(s), X_i(t)) > \gamma e^{-\varepsilon_i} \right) \\
&= E_i^x \left(P_i^{X_i(t)} \left(\sup_{s:t \leq s \leq t+\lambda} d_i(X_i(s-t), X_i(0)) > \gamma e^{-\varepsilon_i} \right) \right) \\
&\leq \sup_{x \in M_i} P_i^x \left(\sup_{s:t \leq s \leq t+\lambda} d_i(X_i(s-t), x) > \gamma e^{-\varepsilon_i} \right).
\end{aligned} \tag{5.3.7}$$

Define a $\{\mathcal{M}_i(t) : t \in [0, T]\}$ -stopping time S_i as

$$S_i = 1 \wedge \inf\{t \in [0, 1] : d_i(X_i(t), X_i(0)) > \gamma e^{-\varepsilon_i}\}.$$

By the strong Markov property, we have

$$\begin{aligned}
& P_i^x \left(d_i(X_i(\lambda), X_i(S_i)) > \frac{\gamma e^{-\varepsilon_i}}{2}, S_i \leq \lambda \right) \\
&= E_i^x \left(\mathbf{1}_{S_i \leq \lambda} P_i^{X_i(S_i)} \left(d_i(X_i(\lambda - s), X_i(0)) > \frac{\gamma e^{-\varepsilon_i}}{2} \right) \Big|_{s=S_i} \right).
\end{aligned} \tag{5.3.8}$$

By (5.3.8), we have, for any $t \in [0, T - \lambda]$,

$$\begin{aligned}
& P_i^x \left(\sup_{s:t \leq s \leq t+\lambda} d_i(X_i(s-t), x) > \gamma e^{-\varepsilon_i} \right) = P_i^x (S_i \leq \lambda) \\
&\leq P_i^x \left(d_i(X_i(\lambda), X_i(0)) \geq \frac{\gamma e^{-\varepsilon_i}}{2} \right) + P_i^x \left(d_i(X_i(\lambda), X_i(0)) < \frac{\gamma e^{-\varepsilon_i}}{2}, S_i \leq \lambda \right) \\
&\leq P_i^x \left(d_i(X_i(\lambda), X_i(0)) \geq \frac{\gamma e^{-\varepsilon_i}}{2} \right) + P_i^x \left(d_i(X_i(\lambda), X_i(S_i)) > \frac{\gamma e^{-\varepsilon_i}}{2}, S_i \leq \lambda \right) \\
&\leq P_i^x \left(d_i(X_i(\lambda), X_i(0)) \geq \frac{\gamma e^{-\varepsilon_i}}{2} \right) + E_i^x \left(\mathbf{1}_{S_i \leq \lambda} P_i^{X_i(S_i)} \left(d_i(X_i(\lambda - s), X_i(0)) > \frac{\gamma e^{-\varepsilon_i}}{2} \right) \Big|_{s=S_i} \right) \\
&\leq 2 \sup_{x \in M_i, \xi \in (0, \lambda]} P_i^x \left(d_i(X_i(\xi), X_i(0)) \geq \frac{\gamma e^{-\varepsilon_i}}{2} \right) \\
&\leq 2 \sup_{x \in M_i, \xi \in (0, \lambda]} \int_{B(x, \gamma e^{-\varepsilon_i}/2)^c} p_i(\xi, x, y) \text{Vol}(dy).
\end{aligned} \tag{5.3.9}$$

Since the Riemannian volumes of $M \in \mathcal{R}$ are bounded by V' , by (5.3.7), (5.3.9) and (5.3.4) of Definition 5.3.3, for any $t \in [0, \tau \wedge T - \lambda]$ taking λ

sufficiently small as $0 < \lambda < \tau \wedge T$, we have

$$\begin{aligned}
& P_i \left(v : \sup_{s:t \leq s \leq t+\lambda} d_i(v(s), v(t)) > \gamma e^{-\varepsilon_i} \right) \\
&= \int_{M_i} P_i^x \left(\sup_{s:t \leq s \leq t+\lambda} d_i(X_i(s), X_i(t)) > \gamma e^{-\varepsilon_i} \right) \mu_i(dx) \\
&\leq 2 \sup_{x \in M_i, \xi \in (0, \lambda]} \int_{B(x, \gamma e^{-\varepsilon_i}/2)^c} p_i(\xi, x, y) \text{Vol}_{M_i}(dy) \\
&\leq 2 \sup_{x \in M_i, \xi \in (0, \lambda]} \int_{B(x, \gamma e^{-\varepsilon_i}/2)^c} \phi(\xi, d_i(x, y)) \text{Vol}_{M_i}(dy) \\
&\leq 2 \sup_{\substack{x, y \in M_i, \xi \in (0, \lambda], \\ d_i(x, y) \geq \gamma e^{-\varepsilon_i}/2}} \phi(\xi, d_i(x, y)) \text{Vol}_{M_i}(B(x, \gamma e^{-\varepsilon_i}/2)^c) \\
&\leq 2V' \sup_{\substack{x, y \in M_i, \xi \in (0, \lambda], \\ d_i(x, y) > \gamma' > 0}} \phi(\xi, d_i(x, y)) < \infty.
\end{aligned}$$

The constant γ' is taken to be $\gamma' = \liminf_{i \rightarrow \infty} \gamma e^{-\varepsilon_i}/2$ in the last line. The last inequality follows from the boundedness of ϕ on $(0, \lambda] \times [\gamma', D]$, which follows from the joint-continuity of ϕ and the condition (5.3.3). By (5.3.3), we have

$$\begin{aligned}
& \lim_{\lambda \rightarrow 0} \limsup_{i \rightarrow \infty} \sup_{\substack{x, y \in M_i, \xi \in (0, \lambda], \\ d_i(x, y) > \gamma' > 0}} \phi(\xi, d_i(x, y)) \\
&\leq \lim_{\lambda \rightarrow 0} \sup_{\xi \in (0, \lambda], r > \gamma' > 0} \phi(\xi, r) = 0.
\end{aligned}$$

This implies (5.3.5) and we complete the proof. $\square$

We start to consider the second objective in this section, that is, a sufficient condition for sequences in $\mathcal{P}_\phi \mathcal{R}$ to be convergent. Let $(M_i, P_i) \in \mathcal{P}_\phi \mathcal{R}$. By Theorem 5.3.4, we know that there is a subsequence $(M_{i'}, P_{i'})$ converging to some (M, P) in the completion $\overline{\mathcal{P}_\phi \mathcal{R}}^{d_{LP}}$ with respect to d_{LP} . Hereafter we consider under what conditions, the whole sequence (M_i, P_i) converges to (M, P) .

Let (M_i, g_i) be a sequence of Riemannian manifolds with Riemannian metrics g_i . Assume that M_i converges to some $M \in \mathcal{M}$ in the Lipschitz distance d_L with ε_i -isometries $f_i : M_i \rightarrow M$. We know that the limit space M has a structure of the n -dimensional $C^{1, \alpha}$ -Riemannian manifold for any $0 < \alpha < 1$. That is, M is a n -dimensional C^∞ -manifold with a $C^{1, \alpha}$ -Riemannian metric g . See, e.g., [13, Theorem 384]. Let Vol_i and Vol be Riemannian volumes induced by g_i and g .

Let $(\mathcal{E}_i, \mathcal{F}_i)$ be a sequence of Dirichlet forms on $L^2(M_i; \text{Vol}_i)$ and $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(M; \text{Vol})$ both satisfying Assumption 5.3.1. We consider a convergence of Dirichlet forms $(\mathcal{E}_i, \mathcal{F}_i)$ to $(\mathcal{E}, \mathcal{F})$, which is a special case of [62, Definition 2.11]. Let us set $\mathcal{E}_i(u) := \mathcal{E}_i(u, u)$ for $u \in \mathcal{F}_i$ and $\mathcal{E}_i(u) := \infty$ if $u \in L^2(M_i; \text{Vol}) \setminus \mathcal{F}_i$ (we also treat $(\mathcal{E}, \mathcal{F})$ in the same manner). For $u \in L^2(M_i)$, we define the *push-forward* $f_{i\#}u \in L^2(M)$ by $f_{i\#}u(x) = u \circ f_i^{-1}(x)$ for $x \in M$. We can check $f_{i\#}u \in L^2(M)$ because of the following inequality:

$$e^{-n\varepsilon_i} \text{Vol} \leq f_{i\#} \text{Vol}_i \leq e^{n\varepsilon_i} \text{Vol}. \quad (5.3.10)$$

The above inequality follows from the definition of an ε_i -isometry. By this inequality, we have that $f_{i\#} \text{Vol}_i$ are absolutely continuous with respect to Vol . Similarly, for $u \in L^2(M)$, we define the *pull-back* f_i^*u by $u \circ f_i(x)$ for any $x \in M_i$. We can also check $f_i^*u \in L^2(M_i)$ by the same argument of $f_{i\#}u \in L^2(M)$.

Now we define a convergence of Dirichlet forms, which is a special case of [62, Definition 2.11](see also [22, Definition 8.1]).

Definition 5.3.6 We say that $(\mathcal{E}_i, \mathcal{F}_i)$ *converges in the Mosco sense* to $(\mathcal{E}, \mathcal{F})$ if the following statement holds: there is a family of ε_i -isometries $f_i : M_i \rightarrow M$ with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$ satisfying

- (i) for any $u_i \in L^2(M_i)$ and $u \in L^2(M)$ satisfying $f_{i\#}u_i$ converges weakly to u in $L^2(M)$, we have

$$\liminf_{i \rightarrow \infty} \mathcal{E}_i(u_i) \geq \mathcal{E}(u);$$

- (ii) for any $u \in L^2(M)$, there exists a sequence $u_i \in L^2(M_i)$ satisfying that $f_{i\#}u_i$ converges to u in $L^2(M)$ and

$$\limsup_{i \rightarrow \infty} \mathcal{E}_i(u_i) \leq \mathcal{E}(u).$$

Note that the notion of the Mosco-convergence does not depend on a specific family of ε_i -isometries f_i in the following sense: if $(\mathcal{E}_i, \mathcal{F}_i)$ converges in the Mosco sense to another Dirichlet form $(\mathcal{E}', \mathcal{F}')$ with respect to another family of ε_i -isometries $g_i : M_i \rightarrow M'$, then there is an isometry $\iota : M \rightarrow M'$ satisfying

$$\mathcal{E}'(u, v) = \mathcal{E}(\iota^*u, \iota^*v) \quad (\forall u, v \in \mathbb{D}(\mathcal{E}')).$$

Let $\{G_i(\alpha)\}_{\alpha>0}$ and $\{G(\alpha)\}_{\alpha>0}$ be the resolvents corresponding to $(\mathcal{E}_i, \mathcal{F}_i)$ and $(\mathcal{E}, \mathcal{F})$, respectively. We have the following statement, which is a special case of [62, Theorem 2.4] (see also [22, Theorem 8.3]):

Proposition 5.3.7 *The following statements are equivalent:*

- (i) $(\mathcal{E}_i, \mathcal{F}_i)$ converges in the Mosco sense to $(\mathcal{E}, \mathcal{F})$;
- (ii) there is a family of ε_i -isometries $f_i : M_i \rightarrow M$ satisfying

$$f_{i\#}T_i(t)f_i^*u \rightarrow T(t)u \quad \text{in } L^2(M),$$

for any $u \in L^2(M)$ and the convergence is uniformly in $t \in [0, T]$.

- (iii) there is a family of ε_i -isometries $f_i : M_i \rightarrow M$ satisfying

$$f_{i\#}G_i(\alpha)f_i^*u \rightarrow G(\alpha)u \quad \text{in } L^2(M),$$

for any $\alpha > 0$ and any $u \in L^2(M)$.

Proof. Modify the proof of [22, Theorem 8.3] as $\pi_i u = f_i^* u$ for $u \in L^2(M)$ and $E_i u = f_{i\#} u$ for $u \in L^2(M_i)$. Noting the inequality (5.3.10), we can do the same argument in [22, Theorem 8.3] and have the desired result. $\square$

Assumption 5.3.8 For $i \in \mathbb{N}$, let $(M_i, P_i) \in \mathcal{P}_\phi \mathcal{R}$ for $(\mathcal{E}_i, \mathcal{F}_i)$ with an initial distribution $\mu_i = \varphi_i \text{Vol}_i$ where $\varphi_i \in L^2(M_i)$. Let (M, P) be an element in the completion $\overline{\mathcal{P}_\phi \mathcal{R}}$ with respect to d_{LP} and assume P to be the law of P^μ for a Hunt process associated with $(\mathcal{E}, \mathcal{F})$ satisfying Assumption 5.3.1 with an initial distribution $\mu = \varphi \text{Vol}$ with $\varphi \in L^2(M)$. Assume that there is a family of maps $f_i : M_i \rightarrow M$ satisfying the following conditions:

- (i) f_i is an ε_i -isometry with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$;
- (ii) f_i satisfies (i) and (ii) of Definition 5.3.6;
- (iii) $f_{i\#}\varphi_i$ converges to φ in $L^2(M)$.

Theorem 5.3.9 *If Assumption 5.3.8 holds, then (M_i, P_i) converges to (M, P) as $i \rightarrow \infty$ in the sense of d_{LP} .*

Proof of Theorem 5.3.9. By Corollary 5.2.9 and Theorem 5.3.4, it suffices to show that the finite-dimensional distributions of $\Phi_{f_{i\#}} P_i$ converge weakly to those of P .

Since M is compact, any bounded continuous functions on M are square-integrable. Thus it suffices to show that, for any $k \in \mathbb{N}$, any $0 = t_0 < t_1 <$

$t_2 < \dots < t_k \leq T$ and any bounded Borel measurable functions $g_1, g_2, \dots, g_k$ in $L^2(M)$,

$$\begin{aligned} & E_i(f_i^* g_1 \circ X_i(t_1) f_i^* g_2 \circ X_i(t_2) \cdots f_i^* g_k \circ X_i(t_k)) \\ & \rightarrow E(g_1 \circ X(t_1) g_2 \circ X(t_2) \cdots g_k \circ X(t_k)) \quad (i \rightarrow \infty). \end{aligned} \quad (5.3.11)$$

Let us set inductively

$$\begin{aligned} h_i^k &= g_k, \quad h_i^{k-1}(\cdot) = g_{k-1}(\cdot) f_{i\#} T_i(t_k - t_{k-1}) f_i^* h_i^k(\cdot), \dots, \\ h_i^1(\cdot) &= g_1(\cdot) f_{i\#} T_i(t_2 - t_1) f_i^* h_i^2(\cdot), \end{aligned}$$

and

$$\begin{aligned} h^k &= g_k, \quad h^{k-1}(\cdot) = g_{k-1}(\cdot) T(t_k - t_{k-1}) h^k(\cdot), \dots, \\ h^1(\cdot) &= g_1(\cdot) T(t_2 - t_1) h^2(\cdot). \end{aligned}$$

By Proposition 5.3.7 and boundedness of g_k , we have

$$\begin{aligned} \|h_i^{k-1} - h^{k-1}\|_{L^2(M)} &= \|g_{k-1}(\cdot) f_{i\#} T_i(t_k - t_{k-1}) f_i^* h_i^k(\cdot) - g_{k-1}(\cdot) T(t_k - t_{k-1}) h^k(\cdot)\|_{L^2(M)} \\ &\rightarrow 0 \quad (i \rightarrow \infty). \end{aligned} \quad (5.3.12)$$

Inductively, we have

$$\begin{aligned} & \|h_i^{k-2} - h^{k-2}\|_{L^2(M)} \\ &= \|g_{k-2}(\cdot) f_{i\#} T_i(t_{k-1} - t_{k-2}) f_i^* h_i^{k-1}(\cdot) - g_{k-2}(\cdot) T(t_{k-1} - t_{k-2}) h^{k-1}(\cdot)\|_{L^2(M)} \\ &\leq \|g_{k-2} f_{i\#} T_i(t_{k-1} - t_{k-2}) f_i^* h_i^{k-1} - g_{k-2} f_{i\#} T_i(t_{k-1} - t_{k-2}) f_i^* h^{k-1}\|_{L^2(M)} \\ &\quad + \|g_{k-2} f_{i\#} T_i(t_{k-1} - t_{k-2}) f_i^* h^{k-1} - g_{k-2} T(t_{k-1} - t_{k-2}) h^{k-1}\|_{L^2(M)} \\ &\leq \|g_{k-2}\|_\infty \|f_{i\#} T_i(t_{k-1} - t_{k-2})\|_{\text{op}} \|f_i^* h_i^{k-1} - f_i^* h^{k-1}\|_{L^2(M)} \\ &\quad + \|g_{k-2}\|_\infty \|f_{i\#} T_i(t_{k-1} - t_{k-2}) f_i^* h^{k-1} - T(t_{k-1} - t_{k-2}) h^{k-1}\|_{L^2(M)} \\ &=: (\text{I})_i + (\text{II})_i, \end{aligned}$$

where $\|f_{i\#} T_i(t_{k-1} - t_{k-2})\|_{\text{op}}$ means the operator norm of $f_{i\#} T_i(t_{k-1} - t_{k-2}) : L^2(M_i) \rightarrow L^2(M)$.

The quantity $(\text{II})_i$ converges to 0 as $i \rightarrow \infty$ by (ii) of Assumption 5.3.8 and Proposition 5.3.7.

We estimate $(\text{I})_i$. By the inequality (5.3.10) and the contraction property of the semigroup $\{T_i(t)\}_{t>0}$, we can check easily that there is a constant C independent to i satisfying

$$\|f_{i\#} T_i(t_{k-1} - t_{k-2})\|_{\text{op}} \leq C. \quad (5.3.13)$$

By (5.3.12) and the inequality (5.3.10), we have

$$\|f_i^* h_i^{k-1} - f_i^* h^{k-1}\|_{L^2(M)} \rightarrow 0 \quad (i \rightarrow \infty). \quad (5.3.14)$$

Thus we have $(I)_i \rightarrow 0$ as $i \rightarrow \infty$.

By using the above argument inductively and the Markov property, we have

$$\begin{aligned} & \|h_i^1 - h^1\|_{L^2(M)} \\ &= \left\| E_i^{f_i^{-1}(x)} (f_i^* g_1 \circ X_i(t_1) f_i^* g_2 \circ X_i(t_2) \cdots f_i^* g_k \circ X_i(t_k)) \right. \\ & \quad \left. - E^x (g_1 \circ X(t_1) g_2 \circ X(t_2) \cdots g_k \circ X(t_k)) \right\|_{L^2(M)} \\ & \rightarrow 0 \quad (i \rightarrow \infty). \end{aligned} \quad (5.3.15)$$

On the other hand, by the inequality (5.3.10), we have that

$$\frac{d(f_{i\#} \text{Vol}_i)}{d\text{Vol}} \rightarrow \mathbf{1}_M \quad \text{uniformly}, \quad (5.3.16)$$

where $\mathbf{1}_M$ means the indicator function on M . By the fact (5.3.16) and (iii) of Assumption 5.3.8, we have

$$\left\| \frac{d(f_{i\#}(\varphi_i \text{Vol}_i))}{d\text{Vol}} - \varphi \right\|_{L^2(M)} \rightarrow 0 \quad (i \rightarrow \infty). \quad (5.3.17)$$

Thus, by (5.3.15) and (5.3.17), using the Schwarz inequality, we have

$$\begin{aligned} & \left| E_i (f_i^* g_1 \circ X_i(t_1) f_i^* g_2 \circ X_i(t_2) \cdots f_i^* g_k \circ X_i(t_k)) \right. \\ & \quad \left. - E (g_1 \circ X(t_1) g_2 \circ X(t_2) \cdots g_k \circ X(t_k)) \right| \quad (i \rightarrow \infty) \\ &= \left| \int_M E_i^{f_i^{-1}(x)} (f_i^* g_1 \circ X_i(t_1) f_i^* g_2 \circ X_i(t_2) \cdots f_i^* g_k \circ X_i(t_k)) f_{i\#}(\varphi_i \text{Vol}_i)(dx) \right. \\ & \quad \left. - \int_M E^x (g_1 \circ X(t_1) g_2 \circ X(t_2) \cdots g_k \circ X(t_k)) \varphi(x) \text{Vol}(dx) \right| \\ &= \left| \int_M E_i^{f_i^{-1}(x)} (f_i^* g_1 \circ X_i(t_1) f_i^* g_2 \circ X_i(t_2) \cdots f_i^* g_k \circ X_i(t_k)) \frac{d(f_{i\#}(\varphi_i \text{Vol}_i))}{d\text{Vol}}(x) \text{Vol}(dx) \right. \\ & \quad \left. - \int_M E^x (g_1 \circ X(t_1) g_2 \circ X(t_2) \cdots g_k \circ X(t_k)) \varphi(x) \text{Vol}(dx) \right| \\ & \rightarrow 0 \quad (i \rightarrow \infty). \end{aligned}$$

We therefore have shown (5.3.11) and we have completed the proof. $\square$

5.4 Examples

5.4.1 Brownian motions on Riemannian manifolds

In this subsection, we consider the case when a state space M is in $\mathcal{R} = \mathcal{R}(n, K, V, D)$ and a probability measure P is the law of the Brownian motion, which is the Markov process induced by the Laplacian on M . In this case, the convergence of processes should follow only from the convergence of state spaces. We show that the convergence in d_{LP} follows only from the convergence in d_L .

Let (M, g) be in $\mathcal{R}$. Let ∇ denote the gradient operator induced by g . Let $(\mathcal{E}, \mathcal{F})$ be the smallest closed extension of the following bilinear form on $L^2(M; \text{Vol})$:

$$\mathcal{E}(u, v) = \frac{1}{2} \int_M g_x(\nabla u, \nabla v) \frac{1}{\text{Vol}(M)} \text{Vol}(dx) \quad (u, v \in C^\infty(M)). \quad (5.4.1)$$

We write $\overline{\text{Vol}}(dx) = \text{Vol}(dx)/\text{Vol}(M)$.

Definition 5.4.1 The set $\mathcal{LR}(n, K, V, D)$ is defined to be the set of isomorphism classes of pairs (M, P) where $M \in \mathcal{R}$ and P is the law of P^μ for a Markov process on a time interval $[0, T]$ associated with $(\mathcal{E}, \mathcal{F})$ defined in (5.4.1) with an initial probability measure $\mu = \overline{\text{Vol}}_M$. We denote $\mathcal{LR}$ shortly for $\mathcal{LR}(n, K, V, D)$.

We show the relative compactness of $\mathcal{LR}$.

Proposition 5.4.2 *The set $\mathcal{LR}$ is relatively compact in $(\mathcal{PM}, d_{LP})$.*

Before the proof, we recall the uniform heat kernel estimate of [52, §4]. Let $\overline{p}_M(t, x, y)$ be a heat kernel of the standard energy form $(\mathcal{E}, \mathcal{F})$ with respect to the normalized volume measure $\overline{\text{Vol}}$ for $M \in \mathcal{R}$. We know that $\overline{p}_M(t, x, y)$ is jointly continuous in (t, x, y) and has the Feller property (see, e.g., [36, Theorem 7.16 & 7.20]). By [52, §4], we have the following heat kernel estimate:

$$\overline{p}_M(t, x, y) \leq \frac{C}{t^{1+\nu}} \exp\left(-\frac{d_M(x, y)^2}{4t}\right), \quad (5.4.2)$$

for all $x, y \in M$ and $0 < t \leq D^2$, where $\nu = \nu(n, K, D) > 2$ and $C = C(n, K, D) > 0$ are positive constants depending only on n, K and D . The important point is that ν and C do not depend on each $M \in \mathcal{R}$. Note that if we have $|\text{Sec}(M)| \leq K$ for $K > 0$, we have $\text{Ric}(M) \geq -K$.

Now we show Proposition 5.4.2.

Proof of Proposition 5.4.2. Let $p_M(t, x, y)$ be the heat kernel of the standard energy form $(\mathcal{E}, \mathcal{F})$ with respect to the Riemannian volume measure Vol (not with respect to $\overline{\text{Vol}}$). Note that

$$\frac{\overline{p}_M(t, x, y)}{\text{Vol}(M)} = p_M(t, x, y). \quad (5.4.3)$$

It suffices to show that there is a jointly continuous function $\phi : (0, T] \times [0, D] \rightarrow [0, \infty)$ satisfying (5.3.3) and dominating $p_M(t, x, y)$ as (5.3.4). In fact, if we show this, we have

$$\mathcal{LR} \subset \mathcal{P}_\phi \mathcal{R}.$$

Since $\mathcal{P}_\phi \mathcal{R}$ is relatively compact by Theorem 5.3.4, we obtain the desired result. The existence of ϕ satisfying (5.3.3) and (5.3.4) follows from [52]. In fact, by (5.4.2), (5.4.3) and the lower-bounds V of the volumes, we have that there is a constant $C' = C'(n, K, V, D) > 0$ such that

$$p_M(t, x, y) \leq \frac{C'}{t^{1+\nu}} \exp\left(-\frac{d_M(x, y)^2}{4t}\right), \quad (5.4.4)$$

for all $M \in \mathcal{R}$, all $t \in (0, D^2]$ and all $x, y \in M$. Note that the constant C' does not depend on each $M \in \mathcal{R}$. Thus we have checked (5.3.4) with $\tau = D^2$. Let

$$\phi(\xi, r) = \frac{C'}{\xi^{1+\nu}} \exp\left(-\frac{r^2}{4\xi}\right).$$

Then we can check easily that ϕ satisfies (5.3.3). Thus we have completed the proof. $\square$

Let $(M_i, P_i) \in \mathcal{LR}$ and assume M_i converges to some $M \in \overline{\mathcal{R}}^{d_L}$ where $\overline{\mathcal{R}}^{d_L}$ denotes the completion of $\mathcal{R}$ with respect to d_L . As stated in Section 5.3, the limit space M has a $C^{1,\alpha}$ -Riemannian structure. Such manifolds are in the class of Lipschitz-Riemannian manifolds (see, e.g., [62, §3]). In this framework, we have a Riemannian volume Vol_M induced by a $C^{1,\alpha}$ -Riemannian metric g and the standard energy form $(\mathcal{E}, \mathcal{F})$ defined by a similar way to (5.4.1) with respect to the weak derivative. See the detail in [62, §3] and references therein.

Let P be a law of Markov process on M associated with the above $(\mathcal{E}, \mathcal{F})$ whose initial distribution is the Riemannian volume $\overline{\text{Vol}}_M$. Then we have the following:

Proposition 5.4.3 *If M_i converges to M in d_L , then (M_i, P_i) converges to (M, P) in d_{LP} .*

Proof. By Theorem 5.3.9, it is sufficient to check that Assumption 5.3.8 are satisfied. Since we assume that $M_i \rightarrow M$ in d_L , there is a family of ε_i -isometries $f_i : M_i \rightarrow M$ with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$. By the inequality (5.3.10), we can easily check that there is a function $h : [0, \infty) \rightarrow [0, \infty)$ with $\lim_{r \rightarrow 0} h(r) = 0$ satisfying the following inequalities (see [62, §3]): for any $u \in L^2(M)$ and $u_i \in L^2(M_i)$,

$$\begin{aligned} \left| \|f_i^* u\|_{L^2(M_i)} - \|u\|_{L^2(M)} \right| &\leq h(\varepsilon_i) \|u\|_{L^2(M)}, \\ |\mathcal{E}_i(f_i^* u) - \mathcal{E}(u)| &\leq h(\varepsilon_i) \mathcal{E}(u), \\ |\mathcal{E}_i(u_i) - \mathcal{E}(f_{i\#} u_i)| &\leq h(\varepsilon_i) \mathcal{E}_i(u_i). \end{aligned}$$

By these inequalities, we can check easily that the conditions of Definition 5.3.6 are satisfied with f_i (see [62, Proposition 3.1]). Since we take $\mu_i = \text{Vol}_{M_i}$ and $\mu = \text{Vol}_M$ in this section, there is nothing to check about (iii) of Assumption 5.3.8. Thus the conditions in Assumption 5.3.8 are satisfied and we finish the proof. $\square$

By Proposition 5.4.3, we know what is the completion $\overline{\mathcal{LR}}^{d_{LP}}$ of $\mathcal{LR}$ with respect to d_{LP} . We define the subset $\mathcal{LR}^{d_L}$ consisting of pairs (M, P) where $M \in \overline{\mathcal{R}}^{d_L}$ and P is the law of P^μ where P is the Brownian motion associated with the standard energy form $(\mathcal{E}, \mathcal{F})$ defined in (5.4.1) with the initial distribution $\mu = \overline{\text{Vol}}_M$.

Corollary 5.4.4 *We have the following:*

$$\overline{\mathcal{LR}}^{d_{LP}} = \mathcal{LR}^{d_L}.$$

Proof. We first show $\overline{\mathcal{LR}}^{d_{LP}} \subset \mathcal{LR}^{d_L}$. Let $(M, P) \in \overline{\mathcal{LR}}^{d_{LP}}$. Then we have a sequence $(M_i, P_i) \in \mathcal{LR}$ such that $(M_i, P_i) \rightarrow (M, P)$ in d_{LP} . Thus $M_i \in \mathcal{R}$ converges to M in d_L , and $M \in \overline{\mathcal{R}}^{d_L}$. By Proposition 5.4.3, we have that P is the law of the Brownian motion associated with the standard energy form on M with the initial distribution $\mu = \overline{\text{Vol}}_M$. Thus $(M, P) \in \mathcal{LR}^{d_L}$.

We second show $\overline{\mathcal{LR}}^{d_{LP}} \supset \mathcal{LR}^{d_L}$. Let $(M, P) \in \mathcal{LR}^{d_L}$. Then we have a sequence $M_i \rightarrow M$ in d_L . Let P_i be the law of the Brownian motion on M_i associated with the standard energy form with the initial distribution $\mu = \overline{\text{Vol}}_M$. Thus, by Proposition 5.4.3, we have that $(M_i, P_i) \rightarrow (M, P)$ in d_{LP} and thus $(M, P) \in \overline{\mathcal{LR}}^{d_{LP}}$. $\square$

5.4.2 Uniformly elliptic diffusions on Riemannian manifolds

In this subsection, we consider (M, P) where $(M, g) \in \mathcal{R}$ and P is a law of a Markov process associated with another smooth Riemannian metric h comparable to the given Riemannian metric g , that is, there is a constant $\Lambda > 1$ satisfying

$$\Lambda^{-1}g \leq h \leq \Lambda g. \quad (5.4.5)$$

The generator associated with h is a second order differential operator having smooth coefficients with the uniform elliptic condition in local coordinates.

To be precise, let ∇_h denote the gradient operator induced by h satisfying (5.4.5). Let Vol_h be the volume measure associated with h . Let $(\mathcal{E}^h, \mathcal{F}^h)$ be the smallest closed extension of the following bilinear form on $L^2(M; \text{Vol}_h)$:

$$\mathcal{E}^h(u, v) = \frac{1}{2} \int_M h_x(\nabla_h u, \nabla_h v) \frac{1}{\text{Vol}_h(M)} \text{Vol}_h(dx) \quad (u, v \in C^\infty(M)). \quad (5.4.6)$$

We write $\overline{\text{Vol}}_h(dx) = \text{Vol}_h(dx)/\text{Vol}_h(M)$.

Definition 5.4.5 For $\Lambda > 1$, the set $\mathcal{L}_\Lambda \mathcal{R}(n, K, V, D)$ is defined to be the set of isomorphism classes of pairs (M, P) where $(M, g) \in \mathcal{R}$ and P is the law of P^μ for a Markov process on $[0, T]$ associated with $(\mathcal{E}^h, \mathcal{F}^h)$ defined in (5.4.6) for h satisfying (5.4.5) with an initial probability measure $\mu = \overline{\text{Vol}}_h$. We denote $\mathcal{L}_\Lambda \mathcal{R}$ shortly for $\mathcal{L}_\Lambda \mathcal{R}(n, K, V, D)$.

We show the relative compactness of $\mathcal{L}_\Lambda \mathcal{R}$.

Proposition 5.4.6 *The set $\mathcal{L}_\Lambda \mathcal{R}$ is relatively compact in $(\mathcal{PM}, d_{LP})$.*

Proof. By [52, §4], we have the same heat kernel estimate as (5.4.2) for $(\mathcal{E}^h, \mathcal{F}^h)$. Note that, of course, the positive constant C in (5.4.2) depends also on Λ in this case. Thus the proof follows from the same argument of Proposition 5.4.2. $\square$

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