

A Context Sensitive Grammar generating the set of all primitive words *

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Abstract

It is still an open problem whether the set Q of all primitive words is a context free language or not. There are many related works to this problem. It is known that Q is not a deterministic context free language but it is a (deterministic) context sensitive language[1]. However, a context sensitive grammar generating Q have not explicitly given so far. We give such a context sensitive grammar here.

1 Introduction

Let X be a finite nonempty set of *letters*, called an alphabet and let X^* be the free monoid generated by X under the operation of catenation. An element of X^* is called a *word* over X . The identity of X^* , denoted by ε , is called the *empty word* and $X^* \setminus \{\varepsilon\}$ is denoted by X^+ . The catenation of two words x and y is denoted by xy . A word $x \in X^+$ is primitive if $y = f^n$ for some $f \in X^+$ implies $n = 1$, where $f^n = \overbrace{ff \cdots f}^n$.

A phrase-structure grammar G is a quadruple $G = (V, X, S, P)$ where V (resp. X) is a finite nonempty set of nonterminal (resp. terminal) symbols, $S \in V$ is the initial symbol, and $P \subset (V \cup X)^* V (V \cup X)^* \times (V \cup X)^*$ is a finite nonempty set of production rules. A production rule $(l, r) \in P$ is often written in the form $l \rightarrow r$. A relation $\rightarrow$ is extended on $(V \cup X)^*$ and defined by $xlry \rightarrow xryy$ where $(l, r) \in P$ and $x, y \in (V \cup X)^*$. The transitive closure and reflective and transitive closure of $\rightarrow$ is denoted by $\rightarrow^+$ and $\rightarrow^*$, respectively. The language generated by G is defined by $\{w \in X^* | S \rightarrow^* w\}$ and denoted by $L(G)$. The number of nonterminal symbols and terminal symbols occurred in $\alpha \in (V \cup X)^*$ is denoted by $|\alpha|$.

A phrase-structure grammar $G = (V, X, S, P)$ is said a context sensitive grammar (CSG for short) if every production rule $l \rightarrow r$ satisfies $|l| \leq |r|$. Then a language generated by a CSG is called a context sensitive language (CSL for short).

2 CSG generating the set of all primitive words over X

At first, we give the phrase-structure grammar G which generate the set Q of all primitive words. G is very close to be a CSL. After that, we transform the grammar into the context sensitive one by new special nonterminal symbols.

Lemma 2.1 Let $Q \subset X^+$ be the set of all primitive words. There is some phrase-structure grammar G which generates Q and whose maximum length on computation is $n + 6$, for each primitive word of the length n .

(Sketch of Proof) We define the phrase-structure grammar $G = (V, X, S, P)$ as follows: The set V of all nonterminal symbols is defined by

$$V = \{S, S_1, L, \bar{L}, R, \sigma_0, \sigma_1, \bar{\sigma}, \bar{\sigma}_a, \bar{\sigma}_R, \bar{\sigma}, \tau_0, \tau, \$, \& | a \in X\}.$$

The initial symbol is S . The production rules $P = \bigcup_{i=1}^8 P_i$:

(Produce an initial configuration)

$$P_1 = \{S \rightarrow \$LRS_1\sigma_0\tau_0\&, S_1 \rightarrow aS_1, S_1 \rightarrow a | a \in X\},$$

* This is an abstract and the paper will appear elsewhere.

(get ready to check mode again)

$$P_2 = \{\sigma_0 \rightarrow \sigma_0 a, R\sigma_0 \rightarrow \sigma_0 R, L\sigma_0 \rightarrow \overleftarrow{L}\sigma_1, \sigma_1 a \rightarrow a\sigma_1, a\tau_0 \rightarrow \tau_0 a | a \in X\}.$$

(Move R one letter to the right and start to check the multiplicity)

$$P_3 = \{\sigma_1 R\tau_0 a \rightarrow aR\overleftarrow{\sigma}\tau | a \in X\},$$

(Scanner σ goes to the left and memorize the next letter)

$$P_4 = \{a\overleftarrow{\sigma} \rightarrow \overleftarrow{\sigma} a, R\overleftarrow{\sigma} \rightarrow \overleftarrow{\sigma} R, L\overleftarrow{\sigma} a \rightarrow aL\overrightarrow{\sigma}_a, L\overleftarrow{\sigma} R \rightarrow \overleftarrow{L}R\overrightarrow{\sigma}_R | a \in X\},$$

(Scanner $\overrightarrow{\sigma}_a$ goes to the right and check the letter)

$$P_5 = \{\overrightarrow{\sigma}_a b \rightarrow b\overrightarrow{\sigma}_a, \overrightarrow{\sigma}_a R \rightarrow R\overrightarrow{\sigma}_a, \overrightarrow{\sigma}_a \tau a \rightarrow \overleftarrow{\sigma} a \tau, \overrightarrow{\sigma}_a \tau c \rightarrow \sigma_0 \tau_0 c, \overrightarrow{\sigma}_a \tau \& \rightarrow \sigma_0 \tau_0 \& | a, b, c \in X, a \neq c\},$$

(Scanner $\overrightarrow{\sigma}_R$ goes to the right and check the multiplicity)

$$P_6 = \{\overrightarrow{\sigma}_R a \rightarrow a\overrightarrow{\sigma}_R, \overrightarrow{\sigma}_R R \rightarrow R\overrightarrow{\sigma}_R, \overrightarrow{\sigma}_R \tau a \rightarrow \overleftarrow{\sigma} \tau a | a \in X\},$$

(Movement of the Left-mergin $\overleftarrow{L}$)

$$P_7 = \{a\overleftarrow{L} \rightarrow \overleftarrow{L} a, \$\overleftarrow{L} \rightarrow \$L | a \in X\},$$

(Clean up all 6 non-terminal symbols)

$$P_8 = \{\sigma_1 R\tau_0 \& \rightarrow E, aE \rightarrow Ea, \$LE \rightarrow \varepsilon | a \in X\},$$

Then the grammar G generates Q . It is easily shown that when G generates a primitive word x , the length of each configuration on its derivation is equal to or shorter than $|x| + 6$.

EXAMPLE 2.1 These are examples of derivations by the grammar G .

(1)

$$\begin{aligned} &S \\ &\rightarrow^* \$LRa\sigma_0\tau_0\& \\ &\rightarrow^* \$L\sigma_1 R\tau_0 a\& \\ &\rightarrow \$LaR\overleftarrow{\sigma}\tau\& \quad \text{start to check} \\ &\rightarrow^* \$aLR\overrightarrow{\sigma}_a\tau\& \\ &\rightarrow \$aLR\sigma_0\tau_0\& \\ &\rightarrow^* \$La\sigma_1 R\tau_0\& \quad \text{success} \\ &\rightarrow^* a. \end{aligned}$$

(2)

$$\begin{aligned} &S \\ &\rightarrow^* \$LRaa\sigma_0\tau_0\& \\ &\rightarrow^* \$LaR\overleftarrow{\sigma}\tau a\& \quad \text{start to check} \\ &\rightarrow^* \$aLR\overrightarrow{\sigma}_a\tau a\& \\ &\rightarrow \$aLR\overleftarrow{\sigma} a\tau\& \\ &\rightarrow^* \$aLRa\overrightarrow{\sigma}_R\tau\&. \quad \text{fail} \end{aligned}$$

This derivation cannot go any longer.

(3)

$$\begin{aligned} &S \\ &\rightarrow^* \$LRab\sigma_0\tau_0\& \\ &\rightarrow^* \$LaR\overleftarrow{\sigma}\tau b\& \quad \text{start to check} \\ &\rightarrow^* \$aLR\overrightarrow{\sigma}_a\tau b\& \\ &\rightarrow \$aLR\sigma_0\tau_0 b\& \\ &\rightarrow^* \$La\sigma_1 R\tau_0 b\& \\ &\rightarrow \$LabR\overleftarrow{\sigma}\tau\& \quad \text{start to check} \\ &\rightarrow^* \$aLbR\overrightarrow{\sigma}_a\tau\& \\ &\rightarrow \$aLbR\sigma_0\tau_0\& \\ &\rightarrow^* \$Lab\sigma_1 R\tau_0\& \quad \text{success} \\ &\rightarrow^* ab. \end{aligned}$$

(4)

$$\begin{aligned}
& S \\
& \rightarrow^* \$LRaab\sigma_0\tau_0\& \\
& \rightarrow^* \$LaR\overline{\sigma}\tau ab\& \quad \text{start to check} \\
& \rightarrow^* \$aLR\overline{\sigma}_a\tau ab\& \\
& \rightarrow \$aLR\overline{\sigma}a\tau b\& \\
& \rightarrow^* \$aLRa\overline{\sigma}_R\tau b\& \\
& \rightarrow \$aLRa\sigma_0\tau_0b\& \\
& \rightarrow^* \$LaR\overline{\sigma}\tau b\& \quad \text{start to check} \\
& \rightarrow^* \$aLaR\overline{\sigma}_a\tau b\& \\
& \rightarrow \$aLaR\sigma_0\tau_0b\& \\
& \rightarrow^* \$LaabR\overline{\sigma}\tau\& \quad \text{start to check} \\
& \rightarrow^* \$aLabR\overline{\sigma}_a\tau\& \\
& \rightarrow \$aLabR\sigma_0\tau_0\& \\
& \rightarrow^* \$Laab\sigma_1R\tau_0\& \quad \text{success} \\
& \rightarrow^* aab.
\end{aligned}$$

Lemma 2.2 Let $Q \subset X^+$ be the set of all primitive words. Then there is a CSG G which generates Q .

(Sketch of Proof) We define the following CSG $G = (V, X, S, P)$ as follows. Let V_0 be the set of the fundamental nonterminal symbols, V_1 be the set of the contextual nonterminal symbols.

$$\begin{aligned}
V_0 &= \{L, \overline{L}, R, \sigma_0, \sigma_1, \overline{\sigma}, \overline{\sigma}_a, \overline{\sigma}_R, \overline{\sigma}, \tau_0, \tau, \$, \& | a \in X\}, \\
V_1 &= \{[\alpha] | \alpha \in (V_0 \cup X)^3 \cup (V_0 \cup X)^5\}, \\
V &= \{S, S_1\} \cup V_0 \cup V_1,
\end{aligned}$$

where S is the initial symbol. The production rules $P = \cup_{i=1}^9 P_i$:

(Produce an initial configuration)

$$P_1 = \{S \rightarrow a, S \rightarrow ab, S \rightarrow \$LR[S_1[ac\sigma_0\tau_0\&], S_1 \rightarrow a, S_1 \rightarrow aS_1 | a, b, c \in X, a \neq b\}$$

Each rule from P_2 to P_7 is the same production rule as that in Lemma 2.1, respectively. Note that each rules $l \rightarrow r$ in P_2 to P_7 has the property of $|l| = |r|$.

(Rules for the contextual nonterminal symbols)

$$P_8 = \{[\gamma\alpha_1]\alpha_2 \rightarrow [\gamma\beta_1]\beta_2, \alpha_1[\alpha_2\gamma] \rightarrow \beta_1[\beta_2\gamma], [\gamma\alpha\delta] \rightarrow [\gamma\beta\delta], |\alpha \rightarrow \beta| \in \cup_{i=2}^7 P_i, \alpha = \alpha_1\alpha_2, \beta = \beta_1\beta_2, \gamma, \delta \in (V_0 \cup X)^*, |\alpha_1| = |\beta_1| > 0, |\alpha_2| = |\beta_2| > 0\}$$

(Eliminate all 6 nonterminal symbols)

$$P_9 = \{[\$La] \rightarrow a, [a\sigma_1R\tau_0\&] \rightarrow a | a \in X\}$$

References

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