

Isolated eigenvalues of the Perron-Frobenius operators for random beta-maps

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1 Introduction

Let $\beta > 0$. The beta-map $\tau_\beta : [0, 1] \rightarrow [0, 1]$ is defined by

$$\tau_\beta(x) = \beta x - [\beta x]$$

for $x \in [0, 1]$, where $[y]$ denotes the integer part of $y \in \mathbb{R}$. For $\beta > 1$ the beta-map is known as a map generating expansions of real numbers with the base β and has been investigated after [12] at the intersection of ergodic theory and number theory (e.g., [5], [11]). In this paper, we consider an i.i.d. random dynamical system generated by beta-maps and investigate isolated eigenvalues of its sample-averaged (or annealed) Perron-Frobenius operator. As a main result, we give a formula for a certain analytic function, each of whose zero is the inverse of an isolated eigenvalue of the sample-averaged Perron-Frobenius operator. This formula is a natural generalization of deterministic cases given in [4] and [8].

2 Deterministic beta-maps

2.1 Beta-expansions

For $\beta > 1$ it is well-known that the map τ_β gives an expansion of $x \in [0, 1]$ as follows. Since

$$x = \frac{[\beta x]}{\beta} + \frac{\tau_\beta(x)}{\beta}$$

for $x \in [0, 1]$, we have

$$\tau_\beta^n(x) = \frac{[\beta \tau_\beta^n(x)]}{\beta} + \frac{\tau_\beta^{n+1}(x)}{\beta}$$

for $n \geq 0$. Using the above equations inductively, we obtain

$$x = \sum_{n=1}^N \frac{[\beta \tau_\beta^{n-1}(x)]}{\beta^n} + \frac{\tau_\beta^N(x)}{\beta^N}$$

for $N \geq 1$. Set $a_n(\beta, x) = [\beta \tau_\beta^{n-1}(x)]$ for $n \geq 1$. Taking $N \rightarrow +\infty$ on the right side of the above equation provides the greedy expansion of x :

$$x = \sum_{n=1}^{\infty} \frac{a_n(\beta, x)}{\beta^n}.$$

2.2 Perron-Frobenius operators

Let us denote by l the Lebesgue measure on $[0, 1]$ and by $(L^1(l), \|\cdot\|_1)$ the Banach space of integrable functions with respect to the Lebesgue measure l . For a positive constant $\beta > 0$, we define the Perron-Frobenius operator for the corresponding map τ_β by

$$\mathcal{L}_\beta f(x) = \frac{1}{\beta} \sum_{y; x=\tau_\beta(y)} f(y)$$

for $x \in [0, 1]$ and $f \in L^1(l)$. This operator is linear and bounded on $L^1(l)$ with $\|\mathcal{L}_\beta f\|_1 = \|f\|_1$ for $f \in L^1(l)$. For a function $f : [0, 1] \rightarrow \mathbb{C}$ denote by $\bigvee f$ the total variation of f . We define

$$|f|_{BV} = \inf \left\{ \bigvee f^*; f^* \text{ is a version of } f \right\}.$$

Set $BV = \{f \in L^1(l) ; |f|_{BV} < \infty\}$ and

$$\|f\|_{BV} = \|f\|_1 + |f|_{BV}$$

for $f \in BV$. Then $(BV, \|\cdot\|_{BV})$ is a Banach space and the Perron-Frobenius operator is linear and bounded on $(BV, \|\cdot\|_{BV})$ (e.g., [1], [3], [9], [10]). If $\beta > 1$, we know that the map τ_β is uniformly expanding and the corresponding Perron-Frobenius operator $\mathcal{L}_\beta$ is quasi-compact, i.e., for any spectral value $\lambda \in \mathbb{C}$ whose modulus is greater than $1/\beta$ is an isolated eigenvalue with finite multiplicity, which plays an important role in investigating the ergodic properties of τ_β (e.g., [1], [3], [9], [10]).

Denote by BV^* the set of complex-valued linear functionals on BV . Let us define the dual operator of the Perron-Frobenius operator $\mathcal{L}_\beta^* : BV^* \rightarrow BV^*$ by

$$\mathcal{L}_\beta^*(\nu(f)) = \nu(\mathcal{L}_\beta f)$$

for $\nu \in BV^*$ and $f \in BV$.

2.3 Isolated eigenvalues of $\mathcal{L}_\beta$

Let $\beta > 1$. We define the power series $\phi_\beta(z)$ associated to the coefficient sequence $\{a_n(\beta, 1)\}_{n=1}^\infty$ by

$$\phi_\beta(z) = \sum_{n=1}^{\infty} \frac{a_n(\beta, 1)}{\beta^n} z^n.$$

We can easily see that the convergence radius of $\phi_\beta(z)$ is at least β . For the analytic function $1 - \phi_\beta(z)$ defined on $\{z \in \mathbb{C}; |z| < \beta\}$, we have the following:

Theorem 2.1. *A complex number λ with $1/\beta < |\lambda| \leq 1$ is an isolated eigenvalue of $\mathcal{L}_\beta$ if and only if λ^{-1} is a zero of $1 - \phi_\beta(z)$.*

The proof of the above theorem is deduced from an explicit formula for the analytic continuation of the dynamical zeta function of τ_β in [8] (see also [4]) and application of the results in [2] (or [6]) to our setting, which yields that each zero of its continuation is the inverse of an isolated eigenvalue of $\mathcal{L}_\beta$, and vice versa.

The main result of this paper is a generalization of the above theorem to random dynamical systems generated by beta-maps. In the setting of random dynamical systems, however, there are few results for dynamical zeta functions, including their analytic continuation. In this paper, we use a new approach derived from a formula for an eigenfunctional of the Perron-Frobenius operator, described in deterministic cases in the following. A key ingredient is the following equation:

Lemma 2.2. *Let $n \geq 0$ be a non-negative integer. Then we have*

$$\mathcal{L}_\beta \mathbf{1}_{[0, \tau_\beta^n(x)]} = \frac{a_{n+1}(\beta, x)}{\beta} \mathbf{1}_{[0, 1]} + \frac{1}{\beta} \mathbf{1}_{[0, \tau_\beta^{n+1}(x)]},$$

where $\mathbf{1}_A$ denote the indicator function of A .

The above lemma yields the following formula:

Theorem 2.3. *Let $\lambda \in \mathbb{C}$ be an isolated eigenvalue of the Perron-Frobenius operator $\mathcal{L}_\beta$ with $|\lambda| > 1/\beta$ and let $\nu \in BV^*$ be a non-zero eigenfunctional corresponding to λ . Then for $x \in [0, 1]$ we have*

$$\nu(\mathbf{1}_{[0, x]}) = \nu(\mathbf{1}_{[0, 1]}) \sum_{n=1}^{\infty} \frac{a_n(\beta, x)}{\beta^n \lambda^n}. \quad (2.1)$$

In addition, $\nu(\mathbf{1}_{[0, 1]}) \neq 0$ and the geometric multiplicity of $\mathcal{L}_\beta$ is 1.

By taking $x = 1$ in the equation (2.1), we obtain the following result.

Theorem 2.4. *Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1/\beta$ be an isolated eigenvalue of $\mathcal{L}_\beta$. Then*

$$\sum_{n=1}^{\infty} \frac{a_n(\beta, 1)}{\beta^n \lambda^n} = 1.$$

That is, λ^{-1} is a zero of an analytic function $1 - \phi_\beta(z)$.

The above theorem states that any isolated eigenvalues of $\mathcal{L}_\beta$ is the inverse of a zero of $1 - \phi_\beta(z)$. In fact, as in the construction of an eigenfunction of $\mathcal{L}_\beta$ in [13], we have that the inverse of any zero of $1 - \phi_\beta(z)$ is actually an isolated eigenvalue of $\mathcal{L}_\beta$.

Theorem 2.5. *Let λ^{-1} be a zero of $1 - \phi_\beta(z)$ with $1 \leq |\lambda^{-1}| < \beta$. Set*

$$h = C \sum_{m=0}^{\infty} \frac{\mathbf{1}_{[0, \tau_\beta^m(1)]}}{\beta^m \lambda^m},$$

where C is a non-zero constant. Then h is a non-zero function of bounded variation satisfying $\mathcal{L}_\beta h = \lambda h$. In particular, λ is an eigenvalue of $\mathcal{L}_\beta$.

By Theorem 2.4 and Theorem 2.5, we reprove Theorem 2.1 not using the theory of dynamical zeta functions. Our main result in this paper is a generalization of Theorem 2.1 to random beta-maps, whose proof is given by an analogue of Theorem 2.4 and 2.3 to random cases.

3 Random beta-maps

3.1 Setting

Let $\mathcal{A}$ be a set of finite positive integers $\{1, \dots, n\}$ or all positive integers $\{1, 2, \dots\}$. Denote by $2^{\mathcal{A}}$ the power set of $\mathcal{A}$. Let $\hat{\mathbb{P}}$ be a Bernoulli measure on $\mathcal{A}$. For the product space $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathcal{A}, 2^{\mathcal{A}}, \hat{\mathbb{P}})^{\mathbb{N}}$, let $\theta : \Omega \rightarrow \Omega$ be the left shift on it, defined by $\theta((\omega_i)_{i=1}^\infty) = (\omega_{i+1})_{i=1}^\infty$ for $(\omega_i)_{i=1}^\infty$. Let $\hat{\beta} : \mathcal{A} \rightarrow (0, \infty)$ be a measurable function and define the function $\beta : \Omega \rightarrow (0, \infty)$ by $\beta((\omega_i)_{i=1}^\infty) = \hat{\beta}(\omega_1)$ for $(\omega_i)_{i=1}^\infty \in \Omega$. Then the function β is a positive measurable function on $(\Omega, \mathcal{F})$ and the random variables $\{\beta \circ \theta^n\}_{n=0}^\infty$ are i.i.d. on $(\Omega, \mathcal{F}, \mathbb{P})$. We define the map $\tau : \Omega \times [0, 1] \rightarrow [0, 1]$ by $(\omega, x) \mapsto \tau_{\beta(\omega)}(x)$. Then the random dynamical system can be represented by the skew-product map $R(\omega, x) = (\theta\omega, \tau(\omega, x))$ for $(\omega, x) \in \Omega \times [0, 1]$. For simplicity, we write $\tau_\omega(\cdot) = \tau(\omega, \cdot)$ for $\omega \in \Omega$. By setting $\tau_\omega^0 = id$ and $\tau_\omega^n = \tau_{\theta^{n-1}\omega} \circ \dots \circ \tau_\omega$ for $\omega \in \Omega$ and $n \geq 1$, where id denotes the identity map on $[0, 1]$, we can write the n -th iteration of the skew product map as

$$R^n(\omega, x) = (\theta^n \omega, \tau_\omega^n(x))$$

for $(\omega, x) \in \Omega \times [0, 1]$.

3.2 Random beta-expansions

As a generalization of beta-expansions generated by a single beta-map, the random dynamical systems as defined above generate multiple base expansions of $x \in [0, 1]$ as follows. Set $\beta_\omega^{(0)} = 1$ and

$$\beta_\omega^{(n)} = \prod_{i=0}^{n-1} \beta(\theta^i \omega)$$

for $\omega \in \Omega$ and $n \geq 1$. We write $\beta_\omega = \beta_\omega^{(1)} = \beta(\omega)$ for simplicity. Define the digit function by $d_n(\omega, x) = [\beta_{\theta^{n-1}\omega} \tau_\omega^{n-1}(x)]$ for $(\omega, x) \in \Omega \times [0, 1]$ and $n \geq 1$. Then

by the definition of the map τ , we have

$$x = \frac{[\beta_\omega x]}{\beta_\omega} + \frac{\tau_\omega(x)}{\beta_\omega}$$

for $(\omega, x) \in \Omega \times [0, 1]$. By the fact that

$$\tau_\omega^n(x) = \frac{[\beta_{\theta^n \omega} \tau_\omega^n(x)]}{\beta_{\theta^n \omega}} + \frac{\tau_\omega^{n+1}(x)}{\beta_{\theta^n \omega}}$$

for $n \geq 0$, we obtain

$$\begin{aligned} x &= \frac{[\beta_\omega x]}{\beta_\omega} + \frac{\tau_\omega(x)}{\beta_\omega} \\ &= \frac{[\beta_\omega x]}{\beta_\omega} + \frac{[\beta_{\theta \omega} \tau_\omega(x)]}{\beta_\omega \beta_{\theta \omega}} + \frac{\tau_\omega^2(x)}{\beta_\omega \beta_{\theta \omega}} \\ &= \dots \\ &= \sum_{n=1}^N \frac{d_n(\omega, x)}{\beta_\omega^{(n)}} + \frac{\tau_\omega^N(x)}{\beta_\omega^{(N)}} \end{aligned}$$

for $N \geq 2$. Note that $\tau_\omega^n(x) \in [0, 1]$ for $n \geq 0$. Then under the assumption that

$$(\beta_\omega^{(n)})^{-1} \rightarrow 0$$

as $n \rightarrow \infty$, we have

$$x = \sum_{n=1}^{\infty} \frac{d_n(\omega, x)}{\beta_\omega^{(n)}}.$$

One of the sufficient conditions which guarantees $(\beta_\omega^{(n)})^{-1} \rightarrow 0$ as $n \rightarrow \infty$ for $\mathbb{P}$ -a.e. $\omega \in \Omega$ is the following.

Lemma 3.1. *If*

$$\int_{\Omega} \frac{d\mathbb{P}(\omega)}{\beta(\omega)} < 1, \quad (3.1)$$

then for $\mathbb{P}$ -a.e. $\omega \in \Omega$ we have

$$(\beta_\omega^{(n)})^{-1} \rightarrow 0$$

as $n \rightarrow \infty$.

3.3 Sample-averaged Perron-Frobenius operators

Let us define the sample averaged (or annealed) Perron-Frobenius operator $\mathcal{L} : L^1(l) \rightarrow L^1(l)$ by

$$\mathcal{L}f = \int_{\Omega} (\mathcal{L}_{\beta(\omega)} f) d\mathbb{P}$$

for $f \in L^1(l)$. We note that this operator is well-defined by the inequality

$$\|\mathcal{L}_{\beta(\omega)}f\|_1 \leq \int_0^1 \int_{\Omega} |\mathcal{L}_{\beta(\omega)}f| d\mathbb{P} dl \leq \int_{\Omega} \int_0^1 |\mathcal{L}_{\beta(\omega)}f| dl d\mathbb{P} \leq \|f\|_1$$

for $f \in L^1(l)$, which also ensures that the operator $\mathcal{L}$ is bounded. The fact that the operator $\mathcal{L}$ is linear follows from its definition. Assume that

$$\int_{\Omega} \frac{d\mathbb{P}(\omega)}{\beta(\omega)} = \int_{\hat{\Omega}} \frac{d\hat{\mathbb{P}}(a)}{\hat{\beta}(a)} < 1.$$

By applying Lemma 6.7 in [7] to our setting, we have that $\mathcal{L}$ is well-defined as a linear bounded operator on BV and it is quasi-compact, i.e., any spectral value $\lambda \in \mathbb{C}$ whose modulus is greater than $\int_{\Omega} d\mathbb{P}(\omega)/\beta(\omega)$ is an isolated eigenvalue with finite multiplicity. In addition, 1 is actually this eigenvalue of $\mathcal{L}$, which ensures the existence of an R -invariant probability measure μ absolutely continuous with respect to $\mathbb{P} \times l$. Let $\mathcal{U}$ be the set of all functions of the form $\sum_{i=1}^n a_i f_i$, where $n \geq 1$, $a_i \in \mathbb{C}$ and f_i is an indicator function of some interval in $[0, 1]$ for $1 \leq i \leq n$. Let $\mathcal{F}$ be the closure of $\mathcal{U}$ in the sense of the topology derived from the norm $\|\cdot\|_{BV}$ in BV . Then we can see that $\mathcal{L}\mathcal{F} \subset \mathcal{F}$ and $\mathcal{L}$ is a linear bounded operator, which is quasi-compact on $(\mathcal{F}, \|\cdot\|_{BV})$. Since $1_{[0,1]} \in \mathcal{F}$ we know that 1 is also an isolated eigenvalue of $\mathcal{L}$ on $\mathcal{F}$.

4 Main results

Let $\Phi(z)$ be the formal power series defined by

$$\Phi(z) = \sum_{n=1}^{\infty} \left(\int_{\Omega} \frac{d_n(\omega, 1)}{\beta_{\omega}^{(n)}} d\mathbb{P} \right) z^n.$$

Since

$$\frac{d_n(\omega, 1)}{\beta_{\omega}^{(n)}} \leq \frac{1}{\beta_{\omega}^{(n-1)}}$$

for $\omega \in \Omega$ and $n \geq 1$, we have that the convergence radius of $\Phi(z)$ is at least $\left(\int_{\Omega} \frac{d\mathbb{P}(\omega)}{\beta(\omega)} \right)^{-1} > 1$. The main result in this paper is the following.

Theorem 4.1. *A complex number λ with $\int_{\Omega} \frac{d\mathbb{P}(\omega)}{\beta(\omega)} < |\lambda| \leq 1$ is an isolated eigenvalue of $\mathcal{L}$ defined on $\mathcal{F}$ if and only if λ^{-1} is a zero of $1 - \Phi(z)$.*

Remark 4.2. (1) The above theorem states that the mean value of n -terms of random beta expansions of 1 determines all isolated eigenvalues of $\mathcal{L}$ outside the closed disk whose radius is $\int_{\Omega} \frac{d\mathbb{P}(\omega)}{\beta(\omega)}$, which can be seen as a natural generalization of Theorem 2.1.

(2) The proof of Theorem 4.1 is given by a generalization of Theorem 2.3 and that of Theorem 2.4 to random beta-maps, which also yields an explicit formula for any eigenfunction and that for a value of any eigenfunctional applied to the indicator function of some interval corresponding to an isolated eigenvalue of $\mathcal{L}$.

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