

Generalized Zariski cancellation problem and principal $\mathbb{G}_a$ -bundles over prevarieties

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Introduction

Generalized Zariski cancellation problem

V, W : varieties over k ($k = \bar{k}$, $\text{char } k = 0$).
Then $V \times \mathbb{A}^1 \simeq W \times \mathbb{A}^1 \Rightarrow V \simeq W$?

■ $\exists$ counterexamples constructed as principal $\mathbb{G}_a$ -bundles $\longrightarrow$

Fact (Danielewski [1])

X : a k -scheme, V, W : affine k -schemes equipped with principal $\mathbb{G}_a$ -bundle structures over X . Then $V \times_k \mathbb{A}^1 \simeq W \times_k \mathbb{A}^1$.

Main Problem

X : a prevariety

V : an affine variety with a principal $\mathbb{G}_a$ -bundle structure over X

W : an affine variety

$V \times \mathbb{A}^1 \simeq W \times \mathbb{A}^1 \iff W$: a principal $\mathbb{G}_a$ -bundle over X ?

Counterexamples for GZCP have been constructed by Danielewski [1], Fieseler [4], Dubouloz [3] and Dryło [2] as principal $\mathbb{G}_a$ -bundles over prevarieties $X'_+ rZ$.

Definition

X' : a variety, Z : a closed subvariety of X' , $r \in \mathbb{N}$. Then

$$X'_+ rZ := X' \sqcup_{X' \setminus Z} \underbrace{X' \sqcup_{X' \setminus Z} \cdots \sqcup_{X' \setminus Z} X'}_r$$

Especially Dryło's counterexamples were constructed in the case X' is a non $\mathbb{A}^1$ -uniruled affine variety.

Counterexample (Dryło [2])

X' : a non $\mathbb{A}^1$ -uniruled affine variety, Z : a principal hypersurface. Then $\exists$ counter examples for GZCP as principal $\mathbb{G}_a$ -bundles over $X = X'_+ Z$.

Main Result

Main Theorem

X : a prevariety with $\exists f: X \rightarrow X'$: quasi-finite dominant morphism

X' : either ① a non $\mathbb{A}^1$ -uniruled affine variety or

② a variety with $\bar{\kappa}(X') \geq 0$

V : an affine variety with a principal $\mathbb{G}_a$ -bundle structure over X

W : an affine variety

$V \times \mathbb{A}^1 \simeq W \times \mathbb{A}^1 \iff W$: a principal $\mathbb{G}_a$ -bundle over X .

Remark

■ prevariety = integral scheme of finite type over k

■ X' : $\mathbb{A}^1$ -uniruled $\iff \exists Y \times \mathbb{A}^1 \rightarrow X'$: a dominant morphism of same dimensional varieties

■ $X = X'_+ rZ$ satisfies the assumption of X in Theorem if X' is non $\mathbb{A}^1$ -uniruled affine variety or $\bar{\kappa}(X') \geq 0$.

$\rightarrow$ Theorem can be applied to Dryło's counterexamples

Proof of Theorem

Step 1: W : an $\mathbb{A}^1$ -bundle over X

$$\begin{array}{ccccc} V \times \mathbb{A}^1 & \xrightarrow{\sim} & W \times \mathbb{A}^1 & & \\ \downarrow & & \downarrow & & \\ V & & W & \xleftarrow{\quad} & U \xrightarrow{\sim} Y \times \mathbb{A}^1 \\ \downarrow & \nearrow \exists q & \downarrow & \nearrow & \downarrow \\ X' & \xleftarrow{f} & X & \xleftarrow{\psi} & Y \end{array}$$

it is enough to show the existence of $q: W \rightarrow X$.

Case ①

1-1: $\exists q': W \rightarrow X'$ ($\leftarrow X'$: non $\mathbb{A}^1$ -uniruled affine)

1-2: q' lifts to $q: W \rightarrow X$ ($\leftarrow f$: quasi-finite)

Case ②

2-1: $\exists Y \times \mathbb{A}^1 \simeq U \subseteq W$: open subset

2-2: $\exists \psi: Y \rightarrow X$ ($\leftarrow \bar{\kappa}(X') \geq 0$)

2-3: $U \simeq Y \times \mathbb{A}^1 \rightarrow Y \rightarrow X \rightarrow X'$ extends to $q': W \rightarrow X'$

2-4: q' lifts to $q: W \rightarrow X$ ($\leftarrow f$: quasi-finite)

Step 2: W : a principal $\mathbb{G}_a$ -bundle over X

L_V, L_W : the line bundles associated to $\mathbb{A}^1$ -bundles V, W

Lemma

W : a principal $\mathbb{G}_a$ -bundle over $X \iff L_W$: trivial.

Lemma

Suppose $V \times \mathbb{A}^m \simeq W \times \mathbb{A}^m$ and the following diagram is commutative.

$$\begin{array}{ccc} V \times \mathbb{A}^m & \xrightarrow{\sim} & W \times \mathbb{A}^m \\ \downarrow & & \downarrow \\ V & & W \\ & \searrow & \swarrow \\ & X & \end{array}$$

Then $L_V \simeq L_W$ as line bundles over X .

Remark: The case V and W : line bundles $\longrightarrow$ proved by Dryło [2].

■ Two lemmas $\longrightarrow$ W : a principal $\mathbb{G}_a$ -bundle over X

References

- [1] W. Danielewski, On a cancellation problem and automorphism groups of affine algebraic varieties, preprint, Warsaw (1989).
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