

The Tate conjecture and the semisimplicity conjecture for t -modules*

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§0. Introduction.

Let l be a prime number. Let k be an algebraic number field and A an abelian variety over k of dimension d . Then the l -adic Tate module

$$V_l(A) \stackrel{\text{def}}{=} (\varprojlim \text{Ker}(l^n \cdot \text{id} : A(\bar{k}) \rightarrow A(\bar{k}))) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

is a $2d$ -dimensional vector space over $\mathbb{Q}_l$ on which $\text{Gal}(\bar{k}/k)$ acts. Thus, fixing a basis of $V_l(A)$, we obtain an l -adic Galois representation

$$\rho_{A,l} : \text{Gal}(\bar{k}/k) \rightarrow GL_{2d}(\mathbb{Q}_l).$$

The following theorem of Faltings is important.

Theorem (0.1).

(i) (Tate conjecture.)

$$\text{Hom}_k(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}_l \simeq \text{Hom}_{\mathbb{Q}_l[\text{Gal}(\bar{k}/k)]}(V_l(A), V_l(A')).$$

(ii) (Semisimplicity conjecture.) $V_l(A)$ is a semisimple $\mathbb{Q}_l[\text{Gal}(\bar{k}/k)]$ -module.

These conjectures can be also formulated for the l -adic Galois representations attached to more general motives, but they are still widely open.

Another problem is: What l -adic Galois representations come from abelian varieties (or motives)? We might hope for characterization of such representations in terms of p -adic theory at the places of k above $p = l$. In the case of abelian varieties, the following partial results are known (Serre, Tate, Raynaud, Deligne,...).

Theorem (0.2).

(i) For each place v of k above l , $\rho_{A,l}|_{\text{Gal}(\bar{k}_v/k_v)}$ is a Hodge-Tate representation, i. e. has a Hodge-Tate decomposition. (In fact, it seems to be known, moreover, to be a potentially semistable representation.)

(ii) Let ρ be an l -adic representation of $\text{Gal}(\bar{k}/k)$ which is potentially abelian. (Namely, the image of $\text{Gal}(\bar{k}/k)$ by ρ admits an abelian open subgroup.) If $\rho|_{\text{Gal}(\bar{k}_v/k_v)}$ is a Hodge-Tate representation for all place v of k above l , then ρ is 'generated' by

*This lecture was given in Japanese.

the representations attached to potentially CM abelian varieties and Artin representations.

In the present article, we consider t -adic Galois representations instead of l -adic Galois representations. A t -adic Galois representation is, by definition, a continuous group homomorphism $\text{Gal}(k^{\text{sep}}/k) \rightarrow GL_n(\mathbb{F}_q((t)))$, where k is a field of characteristic equal to $\text{char}(\mathbb{F}_q)$. (For the definition, we do not have to restrict the characteristic of the field k , but we do not have any interesting theory so far in the case $\text{char}(k) \neq \text{char}(\mathbb{F}_q)$.) Here, the analogues of abelian varieties and motives are Drinfeld modules, Anderson's abelian t -modules, or more general objects, which yield t -adic Galois representations by taking their t -adic Tate modules.

In this new setting, the Tate conjecture has been proved independently by Taguchi [1][2] and the author [3]. See also [4]. In the present article, we consider mainly the semisimplicity conjecture and the problem of characterizing 'geometric' (or 'motivic') t -adic representations.

§1. Pink's restricted modules.

Pink introduced the concept of restricted modules (in 1994) in order to approach the semisimplicity conjecture for t -modules. (In fact, he also gave a proof of the conjecture, different from ours.) Roughly speaking, the category of restricted $k(t)\{\tau\}$ -modules is that of t -motives modulo isogeny. Here, k is a field containing $\mathbb{F}_q$, t is an indeterminate, and the (generally noncommutative) ring $k(t)\{\tau\}$ is defined to be the ring whose underlying abelian group is a (left) $k(t)$ -vector space with basis $\{\tau^i\}_{i=0,1,\dots}$ and whose multiplication rule is:

$$\left(\sum_i f_i \tau^i\right) \left(\sum_j g_j \tau^j\right) = \sum_i \sum_j f_i \sigma^i(g_j) \tau^{i+j},$$

where σ is defined by:

$$\sigma\left(\sum_i c_i t^i\right) = \sum_i c_i^q t^i.$$

Definition (1.1). Let M be a left $k(t)\{\tau\}$ -module.

(i) We say that M is *restricted*, if $\dim_{k(t)} M < \infty$ and

$$\tau_{\text{linear}} : M^{(q)} \stackrel{\text{def}}{=} k(t) \otimes_{\sigma, k(t)} M \rightarrow M, f \otimes x \mapsto f\tau x$$

is an isomorphism.

(ii) Assume M to be restricted. Then we say that M is *étale* (at $t = 0$), if there exists an $O_{k(t)}\{\tau\}$ -submodule $\mathcal{M}$ of M which is finitely generated as an $O_{k(t)}$ -module such that τ_{linear} induces an isomorphism from $\mathcal{M}^{(q)} \stackrel{\text{def}}{=} O_{k(t)} \otimes_{\sigma, O_{k(t)}} \mathcal{M}$ to

$\mathcal{M}$. Here $O_{k(t)} \stackrel{\text{def}}{=} k(t) \cap k[[t]] = k[t]_{(t)}$.

Remark (1.2).

(i) Similarly, we define the concept of restricted and étale restricted $F\{\tau\}$ -modules for each subfield F of $k((t))$ containing $k(t)$ with $\sigma(F) \subset F$. Examples of such F

are: $k((t))$, $Q \stackrel{\text{def}}{=} \text{Frac}(k \otimes_{\mathbb{F}_q} \mathbb{F}_q((t)))$, $Q^h \stackrel{\text{def}}{=} \text{Frac}(k \otimes_{\mathbb{F}_q} \mathbb{F}_q(t)^h)$, etc., where $\mathbb{F}_q(t)^h$ is the algebraic closure of $\mathbb{F}_q(t)$ in $\mathbb{F}_q((t))$.

(ii) In the definition above, the analogue of $(\mathbb{Q}, l, \mathbb{Q}_l)$ is $(\mathbb{F}_q(t), t, \mathbb{F}_q((t)))$. This is only for simplicity, and we can develop our theory for more general setting like [3].

Example (1.3). Let (G, ϕ) be a Drinfeld $\mathbb{F}_q[t]$ -module or an abelian t -module of Anderson's. Then

$$M \stackrel{\text{def}}{=} k(t) \otimes_{k[t]} \text{Hom}_{(\mathbb{F}_q\text{-module schemes}/k)}(G, \mathbb{G}_a)$$

becomes a restricted $k(t)\{\tau\}$ -module. It is étale, unless the ideal (t) is the 'characteristic' of ϕ .

The following proposition gives a relation between restricted modules and t -adic Galois representations.

Proposition (1.4).

We have the following category equivalence:

$$\begin{aligned} (\text{étale restricted } k((t))\{\tau\}\text{-modules}) &\simeq (\text{\textit{t-adic representations of } Gal}(k^{\text{sep}}/k)) \\ M &\mapsto \widehat{V}(M) \stackrel{\text{def}}{=} (k^{\text{sep}}((t)) \otimes_{k((t))} M)^\tau \\ \widehat{D}(V) \stackrel{\text{def}}{=} (k^{\text{sep}}((t)) \otimes_{\mathbb{F}_q((t))} V)^{\text{Gal}(k^{\text{sep}}/k)} &\longleftrightarrow V. \end{aligned}$$

Here τ (resp. $\text{Gal}(k^{\text{sep}}/k)$) acts diagonally on $k^{\text{sep}}((t)) \otimes_{k((t))} M$ (resp. $k^{\text{sep}}((t)) \otimes_{\mathbb{F}_q((t))} V$), and $(\cdot)^\tau$ (resp. $(\cdot)^{\text{Gal}(k^{\text{sep}}/k)}$) means the τ -invariant (resp. $\text{Gal}(k^{\text{sep}}/k)$ -invariant) part. The action of $\text{Gal}(k^{\text{sep}}/k)$ on $\widehat{V}(M)$ (resp. τ on $\widehat{D}(V)$) is induced by its action on $k^{\text{sep}}((t))$.

Definition (1.5). For an étale restricted $k(t)\{\tau\}$ -module M , we write $\widehat{V}(M)$ instead of $\widehat{V}(k((t)) \otimes_{k(t)} M)$, and call it the *(t-adic) Tate module* of M . Similar notation is employed for an étale restricted $F\{\tau\}$ -module. (cf. Remark (1.2)(i).)

The following example explains why we call $\widehat{V}(M)$ Tate module.

Example (1.6). In the case of Example (1.3), we have

$$\widehat{V}(M) \simeq V_t(G)^* = \text{Hom}_{\mathbb{F}_q((t))}(V_t(G), \mathbb{F}_q((t))),$$

where

$$V_t(G) \stackrel{\text{def}}{=} \varprojlim_{\leftarrow} \text{Ker}(\phi_{t^n} : G(\bar{k}) \rightarrow G(\bar{k})) \otimes_{\mathbb{F}_q[[t]]} \mathbb{F}_q((t)).$$

§2. Tate conjecture and semisimplicity conjecture.

From now on, we assume that k is a *finitely generated* field over $\mathbb{F}_q$.

Theorem (2.1). (*Tate conjecture.*)

Let M and M' be étale restricted $k(t)\{\tau\}$ -modules. Then,

$$\mathrm{Hom}_{k(t)\{\tau\}}(M, M') \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q((t)) \simeq \mathrm{Hom}_{\mathbb{F}_q((t))[\mathrm{Gal}(k^{\mathrm{sep}}/k)]}(\widehat{V}(M), \widehat{V}(M')).$$

Theorem (2.2). (*Semisimplicity conjecture.*)

Let M be an étale restricted $k(t)\{\tau\}$ -module, and assume that M is semisimple as a $k(t)\{\tau\}$ -module. Then $\widehat{V}(M)$ is a semisimple $\mathbb{F}_q((t))[\mathrm{Gal}(k^{\mathrm{sep}}/k)]$ -module.

Remark (2.3). In the semisimplicity conjecture, the assumption of semisimplicity of the $k(t)\{\tau\}$ -module M excludes objects like semi-abelian varieties.

The outline of the proof of these theorems is given in the next section.

§3. ‘Geometric’ t -adic Galois representations.

The t -adic representations (of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$) attached to étale restricted $k(t)\{\tau\}$ -modules or, more generally, those attached to étale restricted $Q^h\{\tau\}$ -modules are worth calling *geometric* representations. (See Remark (1.2)(i) for the definition of Q^h and Q .)

Definition (3.1). We say that a t -adic representation of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ is *quasi-geometric*, if it is isomorphic to the t -adic representation attached to an étale restricted $Q\{\tau\}$ -module.

Although we have not yet established any good theory of geometric t -adic representations, we have a good theory of quasi-geometric t -adic representations, as follows.

Remark (3.2). If k is finite, all t -adic representations are quasi-geometric, since Q then coincides with $k((t))$.

Now we have the following diagrams of categories and functors:

$$\begin{array}{ccc} \text{(étale restricted } k(t)\{\tau\}\text{-modules)} & & \\ \begin{array}{c} Q \otimes \cdot \downarrow \\ k(t) \end{array} & & \\ \text{(étale restricted } Q\{\tau\}\text{-modules)} & \rightarrow & \text{(quasi-geometric } t\text{-adic representations)} \\ \begin{array}{c} k((t)) \otimes \cdot \downarrow \\ Q \end{array} & & \cap \\ \text{(étale restricted } k((t))\{\tau\}\text{-modules)} & \simeq & \text{(} t\text{-adic representations).} \end{array}$$

Lemma (3.3).

(i) Let M and M' be étale restricted $k(t)\{\tau\}$ -modules. Then,

$$\mathrm{Hom}_{k(t)\{\tau\}}(M, M') \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q((t)) \simeq \mathrm{Hom}_{Q\{\tau\}}(Q \otimes_{k(t)} M, Q \otimes_{k(t)} M').$$

(ii) Let M be an étale restricted $k(t)\{\tau\}$ -module, and assume that M is semisimple as a $k(t)\{\tau\}$ -module. Then $Q \otimes_{k(t)} M$ is a semisimple $Q\{\tau\}$ -module.

This lemma, which is rather easy to prove, reduces the Tate conjecture (2.1) and the semisimplicity conjecture (2.2) to the following:

Theorem (3.4).(i) *The functor*

$$(\text{étale restricted } Q\{\tau\}\text{-modules}) \xrightarrow{k((t)) \otimes_Q} (\text{étale restricted } k((t))\{\tau\}\text{-modules})$$

is fully faithful.(ii) *The subcategory (quasi-geometric t -adic representations) is stable under taking subquotients in the category (t -adic representations).*

Our proof of this theorem borrows a technique in p -adic Hodge theory. The main point is to construct a commutative ring B , which is a subring of $k^{\text{sep}}((t))$ stable under the actions of τ and $\text{Gal}(k^{\text{sep}}/k)$, satisfying the following properties:

(i) $B^\tau = \mathbb{F}_q((t))$.(ii) $B^{\text{Gal}(k^{\text{sep}}/k)} = Q$.(iii) For each étale restricted $Q\{\tau\}$ -module M , the canonical isomorphism

$$k^{\text{sep}}((t)) \otimes_{\mathbb{F}_q((t))} \widehat{V}(M) \simeq k^{\text{sep}}((t)) \otimes_Q M$$

comes from a (unique) isomorphism

$$B \otimes_{\mathbb{F}_q((t))} \widehat{V}(M) \simeq B \otimes_Q M.$$

Remark (3.5). Roughly speaking, the condition (iii) says that B contains the entries of a ‘period matrix’ of M .

Theorem (3.4)(i) follows directly from the properties of B . In fact, the inverse map of

$$\begin{aligned} \text{Hom}_{Q\{\tau\}}(M, M') &\rightarrow \text{Hom}_{k((t))\{\tau\}}(k((t)) \otimes_Q M, k((t)) \otimes_Q M') \\ &= \text{Hom}_{\mathbb{F}_q((t))[\text{Gal}(k^{\text{sep}}/k)]}(\widehat{V}(M), \widehat{V}(M')) \end{aligned}$$

is defined to map $f \in \text{Hom}_{\mathbb{F}_q((t))[\text{Gal}(k^{\text{sep}}/k)]}(\widehat{V}(M), \widehat{V}(M'))$ to the restriction of $\text{id}_B \otimes f : B \otimes_{\mathbb{F}_q((t))} \widehat{V}(M) \rightarrow B \otimes_{\mathbb{F}_q((t))} \widehat{V}(M')$ to the $\text{Gal}(k^{\text{sep}}/k)$ -invariant parts.

Definition (3.6). For each t -adic representation V of $\text{Gal}(k^{\text{sep}}/k)$, we define

$$D(V) = (B \otimes_{\mathbb{F}_q((t))} V)^{\text{Gal}(k^{\text{sep}}/k)}.$$

From the properties of B , we can easily deduce the following theorem, which completes the proof of Theorem (3.4)(ii).

Theorem (3.7).

Let V be a t -adic representation of $\text{Gal}(k^{\text{sep}}/k)$. Then the following are equivalent:

- (i) V is quasi-geometric;
- (ii) $\dim_Q D(V) = \dim_{\mathbb{F}_q((t))} V$;
- (iii) $k((t)) \otimes_Q D(V) \simeq \widehat{D}(V)$.

In particular, any subquotients of a quasi-geometric representation are again quasi-geometric.

Finally, we mention the construction of the ring B . Fix a proper normal model X of k over $\mathbb{F}_q$, and define Σ to be the set of the points of codimension 1 in X . Let X^{sep} be the normalization of X in k^{sep} , and define Σ^{sep} to be the set of the points of codimension 1 in X^{sep} . Denote by $w_{\bar{x}}$ the additive valuation of k^{sep} associated to $\bar{x} \in \Sigma^{\text{sep}}$ (normalized as $w_{\bar{x}}(k^\times) = \mathbb{Z}$). Define the subring B^+ of $k^{\text{sep}}((t))$ by: $f = \sum a_i t^i \in B^+ \iff$ for all $\bar{x} \in \Sigma^{\text{sep}}$, $\{w_{\bar{x}}(a_i)\}_i$ is bounded below and, for almost all $\bar{x} \in \Sigma^{\text{sep}}$, $w_{\bar{x}}(a_i) \geq 0$ for all i . Here ‘for almost all $\bar{x} \in \Sigma^{\text{sep}}$, ...’ means ‘there exists a finite subset Σ_0 of Σ and, for all $\bar{x} \in \Sigma^{\text{sep}}$ not above Σ_0 , ...’. Next define the subset S of $k^{\text{sep}}((t))$ by

$$S = \{f \in k^{\text{sep}}((t))^\times \mid \sigma(f)f^{-1} \in k \otimes_{\mathbb{F}_q} \mathbb{F}_q((t))\},$$

which turns out to be a multiplicative subset of B^+ . Now the ring B is defined by

$$B = S^{-1}B^+.$$

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