

The generalized Whittaker functions for the discrete series representations of $SU(3, 1)$

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Problem. Let G be a semi-simple Lie group and π_Λ its discrete series representation. What kind of models does π_Λ has? Exactly when the models exist, with how many multiplicity? What explicit form do functions corresponding to the model have? More precisely, let R be a closed subgroup of G . For $\pi_\Lambda \in \hat{G}_d$ and a representation η of R , evaluate the upper bound of

$$\dim_{\mathbb{C}} \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(\pi_\Lambda^*, \text{Ind}_R^G \eta),$$

where π_Λ^* is a contragredient of π_Λ . When the dimension does not equal to zero, write down explicitly the functions describing the intertwiners.

Let $G = NAK$ be the Iwasawa decomposition of G . When R is the maximal unipotent subgroup N of G and η a non-degenerate character of N , $\text{Ind}_R^G \eta$ is the *Gelfand-Graev representation*, hence the problem above is a traditional problem on Whittaker model, considered from various point of view. Here we treat the special unitary group of isometry for the Hermitian form of signature $(3+, 1-)$ realized by

$$G = SU(3, 1) := \{g \in SL(4, \mathbb{C}) \mid {}^t \bar{g} I_{3,1} g = I_{3,1}\}.$$

Whittaker model for discrete series representations of this group G was investigated by Taniguchi [Ta], where he obtained a formula for dimension of the space of the intertwiners and an explicit form of corresponding functions (Whittaker function for π_Λ). His formula tells that the dimension of the Whittaker model not necessarily smaller than one even if the growth condition on the corresponding functions is imposed: The multiplicity one property is not valid for this model. We replace the Gelfand-Graev representation for the *reduced generalized Gelfand-Graev representation* and consider the generalized Whittaker model. That is, we take an irreducible infinite dimensional unitary representation of N , note N is a Heisenberg group, as η and a bigger group containing N as R . We investigate this model and give an explicit form of generalized Whittaker functions. By fixing coordinate on G and explicit realization of representations, we reduce the problem to solving a certain system of difference-differential equations for the coefficient functions of generalized Whittaker functions.

We put some remarks. In the case of the group $SU(2, 1)$, we obtained an explicit form and the multiplicity one result for generalized Whittaker functions for the standard representations previously [I] from a motivation of automorphic forms. And this is just an “étude” for the work on generalized Whittaker functions on $SU(n, 1)$, which will come soon. Main difference from the case of $SU(2, 1)$ is in troublesome combinatorial calculation of K -types S -types.

<Groups and algebras>

We fix a coordinate on subgroups of G as follows,

$$K = \left\{ \begin{pmatrix} k & \\ & \det k^{-1} \end{pmatrix} \mid k \in U(3) \right\}, \quad A = \{a_r := \begin{pmatrix} 1_2 & & \\ & c & s \\ & s & c \end{pmatrix} \mid \begin{array}{l} c = (r + r^{-1})/2, \\ s = (r - r^{-1})/2, \\ r \in \mathbb{R}_{>0} \end{array} \},$$

$$N = \exp \mathfrak{n} = \left\{ \begin{pmatrix} 1 & & \bar{z}_1 & -\bar{z}_1 \\ & 1 & \bar{z}_2 & -\bar{z}_2 \\ -z_1 & -z_2 & \alpha & \beta \\ -z_1 & -z_2 & \alpha & \beta \end{pmatrix} \mid \begin{array}{l} \alpha = 1 - \frac{|z_1|^2}{2} - \frac{|z_2|^2}{2} + it, \\ \beta = 1 + \frac{|z_1|^2}{2} + \frac{|z_2|^2}{2} - it, \\ z_1, z_2 \in \mathbb{C}, t \in \mathbb{R} \end{array} \right\}.$$

Here the Lie algebra $\mathfrak{n}$ of N is given by

$$\mathfrak{n} := \bigoplus_{p=1}^2 (\mathbb{R}X_p + \mathbb{R}Y_p) \oplus \mathbb{R}W,$$

$$X_1 = \begin{pmatrix} & 1 & -1 \\ -1 & & \\ -1 & & \end{pmatrix}, \quad Y_1 = \begin{pmatrix} & -i & i \\ -i & & \\ -i & & \end{pmatrix},$$

$$X_2 = \begin{pmatrix} & 1 & -1 \\ -1 & & \\ -1 & & \end{pmatrix}, \quad Y_2 = \begin{pmatrix} & -i & i \\ -i & & \\ -i & & \end{pmatrix}, \quad W = \begin{pmatrix} & & \\ i & -i \\ i & -i \end{pmatrix},$$

where i denotes the complex unity $\sqrt{-1}$. By natural isomorphisms we identify these groups as

$$K \cong U(3), \quad N \cong H(\mathbb{C}^2).$$

Here $H(\mathbb{C}^2)$ denotes the real Heisenberg group of dimension 5. The center $Z(N)$ of N is of the form

$$Z(N) = \{z_t := \begin{pmatrix} 1_2 & & \\ & 1+it & -it \\ & it & 1-it \end{pmatrix} \mid t \in \mathbb{R}\}$$

The Cartan decomposition of $\mathfrak{g} = \text{Lie } G$ is given by $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, with $\mathfrak{k} = \text{Lie } K$,

$$\mathfrak{p} = \left\{ \begin{pmatrix} O_2 & X \\ {}^t\bar{X} & 0 \end{pmatrix} \mid X \in \mathbb{C}^3 \right\}.$$

The action of the Levi subgroup L of $P := \exp \mathfrak{p}$ on N is naturally extended to that on $\widehat{N}$. By Ston-von Neumann theorem, the unitary dual $\widehat{N}$ of N is exhausted by unitary characters and infinite dimensional irreducible unitary representations. And the infinite dimensional ones ρ are determined by their central characters ψ . Hence the stabilizer S of ρ in L is the centralizer of $Z(N)$ and of the following form

$$S = \{\text{diag}(m, d, d) \in G \mid m \in U(2), d = (\det m)^{-1/2}\},$$

which coincides with the Levi part of P . Using this S , we define the group R as

$$R := S \ltimes N \cong U(2) \ltimes H(\mathbb{C}^2).$$

Let $\mathfrak{t} := \{\text{diag}(ih_1, ih_2, ih_3, ih_4) \mid h_j \in \mathbb{R}, h_1 + \dots + h_4 = 0\}$ be a Cartan subalgebra of $\mathfrak{k}$ and define roots $\beta_{ij} : \mathfrak{t}_{\mathbb{C}} \ni \text{diag}(ih_1, ih_2, ih_3, ih_4) \mapsto t_i - t_j \in \mathbb{C}$. We denote Σ_c and Σ_n the sets of compact and noncompact roots, respectively. In our choice of coordinate,

$$\Sigma_c = \{\beta_{12}, \beta_{13}, \beta_{23}, \beta_{21}, \beta_{31}, \beta_{32}\}, \quad \Sigma_n = \{\beta_{14}, \beta_{24}, \beta_{34}, \beta_{41}, \beta_{42}, \beta_{43}\},$$

and matrix element E_{ij} ($1 \leq i, j \leq 3$) generates the root space $\mathfrak{g}_{\beta_{ij}}$. we put

$$X_{\beta_{ij}} = \begin{cases} -E_{ij} & \text{when } (i, j) = (2, 1), (3, 1), (3, 2); \\ E_{ij} & \text{otherwise,} \end{cases}$$

and take it as a root vector in $\mathfrak{g}_{\beta_{ij}}$. These root vectors decompose with respect to the Iwasawa decomposition as

$$\begin{aligned} X_{\beta_{34}} &= \frac{1}{2}H'_{34} + \frac{1}{2}H + \frac{i}{2}W; & X_{\beta_{43}} &= -\frac{1}{2}H'_{34} + \frac{1}{2}H - \frac{i}{2}W; \\ X_{\beta_{14}} &= X_{\beta_{13}} - \frac{1}{2}X_1 - \frac{i}{2}Y_1; & X_{\beta_{41}} &= X_{\beta_{31}} - \frac{1}{2}X_1 + \frac{i}{2}Y_1; \\ X_{\beta_{24}} &= X_{\beta_{23}} - \frac{1}{2}X_2 - \frac{i}{2}Y_2; & X_{\beta_{42}} &= X_{\beta_{32}} - \frac{1}{2}X_2 + \frac{i}{2}Y_2, \end{aligned}$$

where H'_{34} is a generator $\begin{pmatrix} 0_2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ of $\mathfrak{a} := \log A$. This is used to calculate the action of Schmid operators.

<Representations>

We fix realization of representations of groups.

Parameterization of irreducible K -modules

In this subsection, we recall the Gel'fand-Zetlin basis, which gives a nice realization of irreducible representation of K . The set L_T^+ of $\Sigma_{c,+}$ -dominant T -integral weights is given by $L_T^+ = \{(l, m, n) \in \mathbb{Z}^{\oplus 3} \mid l \geq m \geq n\}$. For a given Σ_c^+ -dominant T -integral weight $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in L_T^+$, let V_λ be a complex vector space spanned by $v(Q)$'s.

$$V_\lambda := \bigoplus_{Q \in GZ(\lambda)} \mathbb{C} v(Q).$$

Here the index set $GZ(\lambda)$ is the set of the Gel'fand-Zetlin schemes with top row λ :

$$GZ(\lambda) := \left\{ Q = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ & \mu_1 & \mu_2 \\ & & k \end{pmatrix} \mid \begin{array}{l} \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3, \\ \mu_1 \geq k \geq \mu_2, \lambda_i, \mu_j, k \in \mathbb{Z} \end{array} \right\}.$$

The $\mathfrak{k}_{\mathbb{C}}$ -module structure defined by

$$\begin{aligned} \tau_\lambda(H'_{14})v(Q) &= kv(Q), & \tau_\lambda(H'_{24})v(Q) &= (|\mu| - k)v(Q), & \tau_\lambda(H'_{34})v(Q) &= (|\lambda| - |\mu|)v(Q), \\ \tau_\lambda(X_{\beta_{23}})v(Q) &= a_2^{1+}(Q)v(Q^{+e_1}) + a_2^{2+}(Q)v(Q^{+e_2}), & \tau_\lambda(X_{\beta_{12}})v(Q) &= a_1^+(Q)v(Q_{+1}), \\ \tau_\lambda(X_{\beta_{32}})v(Q) &= b_2^{1-}(Q)v(Q^{-e_1}) + b_2^{2-}(Q)v(Q^{-e_2}), & \tau_\lambda(X_{\beta_{21}})v(Q) &= b_1^-(Q)v(Q_{-1}) \end{aligned}$$

gives an irreducible K -module $(\tau_\lambda, V_\lambda)$ via the highest weight theory. The coefficients appearing above are given as follows.

$$\begin{aligned}
a_2^{1+}(Q) &= \left(\frac{(\lambda_1 - \mu_1)(\lambda_2 - \mu_1 - 1)(\lambda_3 - \mu_1 - 2)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right)^{1/2} \sqrt{\mu_1 + 1 - k}, \\
a_2^{2+}(Q) &= \left(- \frac{(\lambda_1 + 1 - \mu_2)(\lambda_2 - \mu_2)(\lambda_3 - \mu_2 - 1)}{d_{\mu'}(d_{\mu'} + 1)} \right)^{1/2} \sqrt{k - \mu_2}, \\
a_1^+(Q) &= \sqrt{(\mu_1 - k)(k + 1 - \mu_2)}, \\
b_2^{1-}(Q) &= - \left(\frac{(\lambda_1 + 1 - \mu_1)(\lambda_2 - \mu_1)(\lambda_3 - \mu_1 - 1)}{d_{\mu'}(d_{\mu'} + 1)} \right)^{1/2} \sqrt{\mu_1 - k}, \\
b_2^{2-}(Q) &= - \left(- \frac{(\lambda_1 + 2 - \mu_2)(\lambda_2 + 1 - \mu_2)(\lambda_3 - \mu_2)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right)^{1/2} \sqrt{k + 1 - \mu_2}, \\
b_1^-(Q) &= - \sqrt{(\mu_1 + 1 - k)(k - \mu_2)}.
\end{aligned}$$

And the indices $Q^{\pm e_1}, Q^{\pm e_2}, Q_{\pm 1}$ mean

$$\begin{aligned}
Q^{\pm e_1} &= \begin{pmatrix} \lambda_1 & & \lambda_2 & \lambda_3 \\ & \mu_1 \pm 1 & & \\ & & \mu_2 & \\ & & & k \end{pmatrix}, \quad Q^{\pm e_2} = \begin{pmatrix} \lambda_1 & \lambda_2 & & \lambda_3 \\ & \mu_1 & & \\ & & \mu_2 \pm 1 & \\ & & & k \end{pmatrix}, \\
Q_{\pm 1} &= \begin{pmatrix} \lambda_1 & & \lambda_2 & \lambda_3 \\ & \mu_1 & & \\ & & \mu_2 & \\ & & & k \pm 1 \end{pmatrix},
\end{aligned}$$

respectively. The basis $\{v(Q) \mid Q \in GZ(\lambda)\}$ prescribed above is called the *Gef'fand-Zetlin basis* of $(\tau_\lambda, V_\lambda)$.

Tensor products with $\mathfrak{p}_C$

We regard the 6-dimensional vector space $\mathfrak{p}_C$ as a $\mathfrak{k}_C$ -module via the adjoint representation. Then $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are invariant subspaces, and

$$\mathfrak{p}_+ := \mathbb{C}X_{\beta_{14}} \oplus \mathbb{C}X_{\beta_{24}} \oplus \mathbb{C}X_{\beta_{34}} \cong V_{\beta_{14}}, \quad \mathfrak{p}_- := \mathbb{C}X_{\beta_{41}} \oplus \mathbb{C}X_{\beta_{42}} \oplus \mathbb{C}X_{\beta_{43}} \cong V_{\beta_{43}}.$$

Given an irreducible K -module V_λ Clebsch-Gordan's theorem tells us the following decomposition of $V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_\pm$:

$$V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_+ \cong V_{\lambda+\beta_{14}} \oplus V_{\lambda+\beta_{24}} \oplus V_{\lambda+\beta_{34}}, \quad V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_- \cong V_{\lambda+\beta_{41}} \oplus V_{\lambda+\beta_{42}} \oplus V_{\lambda+\beta_{43}}.$$

The decompositions of $V_\lambda \otimes \mathfrak{p}_C$ induce the following projectors:

$$\begin{aligned}
p^{+\beta_{14}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_C &\rightarrow V_{\lambda+\beta_{14}}, & p^{-\beta_{14}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_C &\rightarrow V_{\lambda-\beta_{14}}, \\
p^{+\beta_{24}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_C &\rightarrow V_{\lambda+\beta_{24}}, & p^{-\beta_{24}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_C &\rightarrow V_{\lambda-\beta_{24}}, \\
p^{+\beta_{34}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_C &\rightarrow V_{\lambda+\beta_{34}}, & p^{-\beta_{34}} : V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_C &\rightarrow V_{\lambda-\beta_{34}},
\end{aligned}$$

In terms of $\{v(Q)\}$, they are expressed as follows:

Proposition 1 *The projectors are described as follows.*

(1)

$$\begin{aligned} p^{-\beta_{14}}(v(Q) \otimes X_{\beta_{41}}) &= A_1^- \sqrt{k - \mu_2} v(\tilde{Q}_{-1}^{-e_1}) - B_1^- \sqrt{\mu_1 + 1 - k} v(\tilde{Q}_{-1}^{-e_2}) \\ p^{-\beta_{14}}(v(Q) \otimes X_{\beta_{42}}) &= A_1^- \sqrt{\mu_1 - k} v(\tilde{Q}^{-e_1}) + B_1^- \sqrt{k + 1 - \mu_2} v(\tilde{Q}^{-e_2}) \\ p^{-\beta_{14}}(v(Q) \otimes X_{\beta_{43}}) &= \sqrt{(\lambda_1 + 1 - \mu_2)(\lambda_1 - \mu_1)} v(\tilde{Q}) \end{aligned}$$

with coefficients

$$A_1^- = \left| \frac{(\lambda_2 - \mu_1)(\lambda_3 - \mu_1 - 1)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\lambda_1 + 1 - \mu_2}, \quad B_1^- = \left| -\frac{(\lambda_2 + 1 - \mu_2)(\lambda_3 - \mu_2)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\lambda_1 - \mu_1}.$$

(2)

$$\begin{aligned} p^{-\beta_{24}}(v(Q) \otimes X_{\beta_{41}}) &= -A_2^- \sqrt{k - \mu_2} v(\tilde{Q}_{-1}^{-e_1}) - B_2^- \sqrt{\mu_1 + 1 - k} v(\tilde{Q}_{-1}^{-e_2}) \\ p^{-\beta_{24}}(v(Q) \otimes X_{\beta_{42}}) &= -A_2^- \sqrt{\mu_1 - k} v(\tilde{Q}^{-e_1}) + B_2^- \sqrt{k + 1 - \mu_2} v(\tilde{Q}^{-e_2}) \\ p^{-\beta_{24}}(v(Q) \otimes X_{\beta_{43}}) &= \sqrt{(\lambda_2 - \mu_2)(\mu_1 + 1 - \lambda_2)} v(\tilde{Q}) \end{aligned}$$

with coefficients

$$A_2^- = \left| -\frac{(\lambda_1 + 1 - \mu_1)(\lambda_3 - \mu_1 - 1)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\lambda_2 - \mu_2}, \quad B_2^- = \left| -\frac{(\lambda_1 + 2 - \mu_2)(\lambda_3 - \mu_2)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\mu_1 + 1 - \lambda_2}.$$

(3)

$$\begin{aligned} p^{-\beta_{34}}(v(Q) \otimes X_{\beta_{41}}) &= -A_3^- \sqrt{k - \mu_2} v(\tilde{Q}_{-1}^{-e_1}) + B_3^- \sqrt{\mu_1 + 1 - k} v(\tilde{Q}_{-1}^{-e_2}) \\ p^{-\beta_{34}}(v(Q) \otimes X_{\beta_{42}}) &= -A_3^- \sqrt{\mu_1 - k} v(\tilde{Q}^{-e_1}) - B_3^- \sqrt{k + 1 - \mu_2} v(\tilde{Q}^{-e_2}) \\ p^{-\beta_{34}}(v(Q) \otimes X_{\beta_{43}}) &= \sqrt{(\mu_2 + 1 - \lambda_3)(\mu_1 + 2 - \lambda_3)} v(\tilde{Q}) \end{aligned}$$

with coefficients

$$A_3^- = \left| -\frac{(\lambda_1 + 1 - \mu_1)(\lambda_2 - \mu_1)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\mu_2 + 1 - \lambda_3}, \quad B_3^- = \left| \frac{(\lambda_1 + 2 - \mu_2)(\lambda_2 + 1 - \mu_2)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\mu_1 + 2 - \lambda_3}$$

(4)

$$\begin{aligned} p^{+\beta_{34}}(v(Q) \otimes X_{\beta_{14}}) &= A_3^+ \sqrt{\mu_1 - k} v(\tilde{Q}_{+1}^{+e_2}) - B_3^+ \sqrt{k - \mu_2 + 1} v(\tilde{Q}_{+1}^{+e_1}) \\ p^{+\beta_{34}}(v(Q) \otimes X_{\beta_{24}}) &= -A_3^+ \sqrt{k - \mu_2} v(\tilde{Q}^{+e_2}) - B_3^+ \sqrt{\mu_1 + 1 - k} v(\tilde{Q}^{+e_1}) \\ p^{+\beta_{34}}(v(Q) \otimes X_{\beta_{34}}) &= \sqrt{(\mu_1 + 1 - \lambda_3)(\mu_2 - \lambda_3)} v(\tilde{Q}) \end{aligned}$$

with coefficients

$$A_3^+ = \left| \frac{(\lambda_1 + 1 - \mu_2)(\lambda_2 - \mu_2)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\mu_1 + 1 - \lambda_3}, \quad B_3^+ = \left| -\frac{(\lambda_1 - \mu_1)(\lambda_2 - \mu_1 - 1)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\mu_2 - \lambda_3}.$$

□

Here we denote Gel'fand-Zetlin schemata with top row $\lambda \pm \beta$

$$\begin{pmatrix} & \lambda \pm \beta & \\ \mu_1 \pm 1 & & \mu_2 \pm 1 \\ & k \pm 1 & \end{pmatrix}$$

by $\tilde{Q} \in GZ(\lambda \pm \beta)$. Note $\beta_{14}, \beta_{24}, \beta_{34}$ is $(2, 1, 1)$, $(1, 2, 1)$, $(1, 1, 2)$ respectively. And other schemata mean as follows.

$$\begin{aligned} \tilde{Q}_{\pm 1}^{\pm e_1} &= \begin{pmatrix} & \lambda \pm \beta & \\ \mu_1 \pm 2 & & \mu_2 \pm 1 \\ & k \pm 2 & \end{pmatrix}, \quad \tilde{Q}_{\pm 1}^{\pm e_2} = \begin{pmatrix} & \lambda \pm \beta & \\ \mu_1 \pm 1 & & \mu_2 \pm 2 \\ & k \pm 2 & \end{pmatrix}, \\ \tilde{Q}^{\pm e_1} &= \begin{pmatrix} & \lambda \pm \beta & \\ \mu_1 \pm 2 & & \mu_2 \pm 1 \\ & k \pm 1 & \end{pmatrix}, \quad \tilde{Q}^{\pm e_2} = \begin{pmatrix} & \lambda \pm \beta & \\ \mu_1 \pm 1 & & \mu_2 \pm 2 \\ & k \pm 1 & \end{pmatrix}. \end{aligned}$$

Representations of S

By identifying the group S with $U(2)$, for each dominant weight $\mu' = (\mu'_1, \mu'_2)$, relations

$$\sigma_{\mu'}(H'_{14} - H'_{24} - H'_{34})w_{k'} = |\mu'|w_{k'}, \quad \sigma_{\mu'}(H'_{14} - H'_{24})w_{k'} = (2k' - |\mu'|)w_{k'},$$

$$\sigma_{\mu'}(X_{\beta_{12}})w_{k'} = \sqrt{(\mu'_1 - k')(k' + 1 - \mu'_2)}w_{k'}, \quad \sigma_{\mu'}(X_{\beta_{21}})w_{k'} = \sqrt{(\mu'_1 + 1 - k')(k' - \mu'_2)}w_{k'}$$

define a representation $\sigma_{\mu'}$ of S on $W_{\mu'} := \bigoplus_{k'=\mu'_2}^{\mu'_1} \mathbb{C}w_{k'}$.

The Fock representation of n

Here we realize the infinite dimensional unitary representation ρ with central character $\psi_s : Z(N) \ni z_t \mapsto e^{\sqrt{-1}st} \in \mathbb{C}^{(1)}$, $s \in \mathbb{R} \setminus \{0\}$, on $\mathbb{C}[z_1, z_2]$ by

$$\begin{aligned} \rho_{\psi_s} : H(\mathbb{C}_J^2) &\rightarrow \text{Aut}(\mathcal{F}_J), \\ \rho_{\psi_s}(X_i) &:= \sqrt{s} \left(\frac{\partial}{\partial z_i} + z_i \right), \quad \rho_{\psi_s}(Y_i) := -\sqrt{-s} \left(\frac{\partial}{\partial z_i} - z_i \right), \\ \rho_{\psi_s}(W) &:= \sqrt{-1}s, \end{aligned}$$

when s is positive. We choose the monomials $f_{j_1, j_2} := z_1^{j_1} z_2^{j_2}$, $j_i = 0, 1, 2, \dots$ of two variables, abbreviated by f_j , as a base of $\mathbb{C}[z_1, z_2]$.

Representations of R with nontrivial central characters

By natural identification $R = S \ltimes N$ is isomorphic to $U(2) \ltimes H(\mathbb{C}^2)$ and can be regarded as a subgroup of $\widetilde{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4)$. From the theory of Weil representations, we have the canonical extension

$$\omega_\psi \times \rho_\psi : \widetilde{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4) \rightarrow \text{Aut}(\mathbb{C}[z_1, z_2]).$$

Let $\tilde{R}$ be the pullback $\tilde{R} := \tilde{S} \ltimes N \cong \tilde{U}(2) \ltimes H(\mathbb{R}^4)$ of R by the covering

$$pr \times id : \widetilde{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4) \rightarrow Sp_2(\mathbb{R}) \ltimes H(\mathbb{R}^4).$$

Then tensoring an odd character $\tilde{\chi}_{1/2}$ of $\tilde{U}(2)$ to $(\omega_\psi \times \rho_\psi)|_{\tilde{R}}$, we have a representation of R

$$\tilde{\chi}_{1/2} \otimes (\omega_\psi \times \rho_\psi)|_{\tilde{R}} : R = S \ltimes N \rightarrow \text{Aut}(\mathbb{C}[z_1, z_2]).$$

A result of Wolf ([Wolf] Prop 5.7.) says that all representations of R which come from infinite dimensional representation of $H(\mathbb{C}^2)$ are exhausted by the representations of the form of this representation tensored by representations of $U(2)$. That is

$$\hat{R}_{\text{ctchr} \neq 1} = \{ \sigma_{\mu'} \otimes \tilde{\chi}_{1/2} \otimes (\omega_{\psi} \times \rho_{\psi})|_{\tilde{R}} \mid \sigma_{\mu'} \in \hat{U}(2) \}.$$

We denote this representation by $(\eta, \mathbb{C}[z_1, z_2])$.

The action of $\tilde{S}$ on $\mathbb{C}[z_1, z_2]$ through ω_{ψ} is given infinitesimally as follows

$$\begin{aligned} \omega_{\psi}(H'_{14} - H'_{24} - H'_{34})f_j &= -(j_1 + j_2 + 2)f_j, & \omega_{\psi}(H'_{14} - H'_{24})f_j &= -(j_1 - j_2)f_j, \\ \omega_{\psi}(X_{\beta_{12}})f_j &= -j_1 f_{j-e_1+e_2}, & \omega_{\psi}(X_{\beta_{21}})f_j &= -j_2 f_{j+e_1-e_2}. \end{aligned}$$

Here is a diagram explaining the above construction

$$\begin{array}{ccc} \tilde{R} = \tilde{S} \ltimes N & \tilde{Sp}_2(\mathbb{R}) \ltimes H(\mathbb{R}^4) & \xrightarrow{\omega_{\psi} \times \rho_{\psi}} \text{Aut}(\mathbb{C}[z_1, z_2]) \\ \downarrow & \downarrow \text{pr} \times \text{id} & \\ R = S \ltimes N & \longrightarrow & Sp_2(\mathbb{R}) \ltimes H(\mathbb{R}^4). \end{array}$$

The discrete series representations of G

By a theorem of Harish-Chandra, there is a one-to-one correspondence between Σ -regular $\Sigma_{c,+}$ -dominant T -integral weight $\Lambda \in \Xi$ and equivalence class of discrete series representations $\pi_{\Lambda} \in \hat{G}_d$ of G . The parameter set $\Xi = \{ \Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} \mid \Lambda_1 > \Lambda_2 > \Lambda_3, \Lambda_1 \Lambda_2 \Lambda_3 \neq 0 \}$ decomposes into four disjoint subsets $\Xi_J (J = I, II, III, IV)$ correspond to positive root systems $\Sigma_I^+ := \Sigma_{c,+} \cup \{ \beta_{14}, \beta_{24}, \beta_{34} \}$, $\Sigma_{II}^+ := \Sigma_{c,+} \cup \{ \beta_{14}, \beta_{24}, \beta_{43} \}$, $\Sigma_{III}^+ := \Sigma_{c,+} \cup \{ \beta_{14}, \beta_{42}, \beta_{43} \}$, $\Sigma_{IV}^+ := \Sigma_{c,+} \cup \{ \beta_{41}, \beta_{42}, \beta_{43} \}$. By the inner product induced from the Killing form we can see

$$\begin{aligned} \Xi_I^+ &= \{ (\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} \mid \Lambda_1 > \Lambda_2 > \Lambda_3 > 0 \}, \\ \Xi_{II}^+ &= \{ (\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} \mid \Lambda_1 > \Lambda_2 > 0 > \Lambda_3 \}, \\ \Xi_{III}^+ &= \{ (\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} \mid \Lambda_1 > 0 > \Lambda_2 > \Lambda_3 \}, \\ \Xi_{IV}^+ &= \{ (\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{Z}^{\oplus 3} \mid 0 > \Lambda_1 > \Lambda_2 > \Lambda_3 \}. \end{aligned}$$

Representations parameterized by Ξ_I^+ (resp. Ξ_{IV}^+) are called the holomorphic discrete series representations (resp. the antiholomorphic discrete series representations). In the remaining case, discrete series representations whose Harish-Chandra parameters Λ 's belong to Ξ_{II}^+, Ξ_{III}^+ are the large discrete series representations in the sense of Vogan [Vo].

<The space of generalized Whittaker functions of the discrete series >

Under the setting above, our main concern $I_{\pi, \eta} := \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(\pi_{\Lambda}^*, \text{Ind}_R^G \eta)$ is called *the space of the algebraic generalized Whittaker functionals*. Specifying a K -type of π

$$\text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(\pi_{\Lambda}^*, \text{Ind}_R^G \eta) \ni l \mapsto \iota_{\tau}^*(l) \in \text{Hom}_K(\tau_{\Lambda}^*, \text{Ind}_R^G \eta|_K),$$

where $\iota_{\tau} : \tau_{\Lambda} \hookrightarrow \pi$, we define a function F through next identification $\text{Hom}_K(\tau_{\Lambda}^*, \text{Ind}_R^G \eta|_K) \cong (\text{Ind}_R^G \eta|_K \otimes \tau_{\Lambda})^K$. The latter space $(C_{\eta}^{\infty}(R \backslash G) \otimes_{\mathbb{C}} V_{\Lambda})^K$ is defined by

$$C_{\eta, \tau_{\Lambda}}^{\infty}(R \backslash G / K) := \left\{ \varphi : G \rightarrow \mathbb{C}[z] \otimes_{\mathbb{C}} V_{\Lambda} \mid \begin{array}{l} \varphi \text{ is a } C^{\infty}\text{-function satisfying} \\ \varphi(r g k) = \eta(r) \tau_{\Lambda}(k)^{-1} \cdot \varphi(g), \\ \forall r \in R, \forall g \in G, \forall k \in K \end{array} \right\}.$$

We call the function $F_\eta^\tau \in C_{\eta, \tau_\lambda}^\infty(R \backslash G/K)$ representing $\iota_\tau^*(l)$ the algebraic generalized Whittaker function associated to the discrete series representation π_Λ with K -type τ_λ . By definition, $l(v^*)(g) = \langle v^*, F(g) \rangle_K$, $v^* \in V_\tau^*$. Here $\langle \cdot, \cdot \rangle_K$ means the canonical pairing of K -modules V_τ^* and V_τ .

Yamashita's fundamental result tells that the algebraic generalized Whittaker functions F are characterized by a system of differential equations.

Proposition 2 ([Ya] Theorem 2.4.) *Let π_Λ be a discrete series representation of G with Harish-Chandra parameter $\Lambda \in \Xi_J$, and λ be the Blattner parameter $\Lambda + \rho_J - 2\rho_c$ of π_Λ . Assume Λ is far from walls, then the image of $\text{Hom}_{(\mathfrak{g}, K)}(\pi_\Lambda^*, \text{Ind}_R^G \eta)$ in $C_{\eta, \tau_\lambda}^\infty(R \backslash G/K)$ by the correspondence above is characterized by*

$$(D) : \quad \mathcal{D}_{\eta, \tau_\lambda}^{-\beta} \cdot F = 0 \quad (\forall \beta \in \Sigma_J^+ \cap \Sigma_n).$$

Here the differential operators

$$\mathcal{D}_{\eta, \tau_\lambda}^{-\beta} : C_{\eta, \tau_\lambda}^\infty(R \backslash G/K) \rightarrow C_{\eta, \tau_\lambda - \beta}^\infty(R \backslash G/K).$$

are defined by $\mathcal{D}_{\tau_\lambda}^{-\beta} \varphi(g) := p^{-\beta}(\nabla_{\tau_\lambda} \varphi(g))$, $\nabla_{\tau_\lambda} \varphi := \sum_{i=1}^6 R_{X_i} \varphi \otimes X_i$. Here $\{X_i \ (i = 1, \dots, 6)\}$ is an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form on $\mathfrak{g}$ and $R_X \varphi$ means the right differential of function φ by $X \in \mathfrak{g}$: $R_X \varphi(g) = \frac{d}{dt} \varphi(g \exp tX)|_{t=0}$. We call the space

$$\text{Wh}_\eta^\tau(\pi_\Lambda) := \{F \in C_{\eta, \tau_\lambda}^\infty(R \backslash G/K) | l(v^*) = \langle v^*, F(\cdot) \rangle_K, l \in I_{\pi, \eta}, v^* \in V_\lambda^*\}.$$

the generalized Whittaker model for the representation π_Λ of G with K -type τ and the elements in this space the generalized Whittaker functions associated to the representation π_Λ with K -type τ .

<Difference-differential equations for coefficients>

Radial part of Schmid operators

For the representation $(\eta, \mathbb{C}[z])$ of R and for any finite dimensional K -module V , we denote the space of the smooth $\mathbb{C}[z] \otimes_{\mathbb{C}} V$ -valued functions on A by

$$C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V) := \{\phi : A \rightarrow W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V \mid C^\infty\text{-function}\}.$$

Let

$$\begin{aligned} \text{res}_A & : C_{\eta, \tau_\lambda}^\infty(R \backslash G/K) \rightarrow C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V_\lambda) \\ \text{res}_{A, \pm} & : C_{\eta, \tau_\lambda \otimes \text{Ad } \mathfrak{p}_\pm}^\infty(R \backslash G/K) \rightarrow C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V_\lambda \otimes_{\mathbb{C}} \mathfrak{p}_\pm) \end{aligned}$$

be the restriction maps to A . Then we define the radial part $R(\nabla_{\eta, \tau_\lambda}^\pm)$ of $\nabla_{\eta, \tau_\lambda}^\pm$ on the image of res_A by

$$R(\nabla_{\eta, \tau_\lambda}^\pm) \cdot (\text{res}_A \varphi) = \text{res}_{A, \pm}(\nabla_{\eta, \tau_\lambda}^\pm \cdot \varphi).$$

Let us denote by ϕ and ∂ the restriction to A of $\varphi \in C_{\eta, \tau_\lambda}^\infty(R \backslash G/K)$ and the generator H of $\mathfrak{a}$, respectively, $\partial \phi = (H \cdot \varphi)|_A$. We remark $\partial = r \frac{d}{dr}$: the Euler operator in variable r . By using the Iwasawa decomposition of root vectors, we have next proposition.

Proposition 3 Let ϕ be the above element in $C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V_\lambda)$. Then the radial part $R(\nabla_{\eta, \tau_\lambda}^+)$ of $\nabla_{\eta, \tau_\lambda}^+$ is given by

$$(i) \quad R(\nabla_{\eta, \tau_\lambda}^+). \phi = \frac{1}{2} \{ \partial - \sqrt{-1} r^2 \eta(W) - 6 \} . (\phi \otimes X_{\beta_{34}}) + \frac{1}{2} (\tau_\lambda \otimes \text{Ad}_{p_+})(H'_{34}). (\phi \otimes X_{\beta_{34}}) \\ - \frac{1}{2} r \eta(X_1 - \sqrt{-1} Y_1). (\phi \otimes X_{\beta_{14}}) - (\tau_\lambda \otimes \text{Ad}_{p_+})(X_{\beta_{31}}). (\phi \otimes X_{\beta_{14}}) \\ - \frac{1}{2} r \eta(X_2 - \sqrt{-1} Y_2). (\phi \otimes X_{\beta_{24}}) - (\tau_\lambda \otimes \text{Ad}_{p_+})(X_{\beta_{32}}). (\phi \otimes X_{\beta_{24}}).$$

Similarly for the radial part $R(\nabla_{\eta, \tau_\lambda}^-)$ of $\nabla_{\eta, \tau_\lambda}^-$, we have

$$(ii) \quad R(\nabla_{\eta, \tau_\lambda}^-). \phi = \frac{1}{2} \{ \partial + \sqrt{-1} r^2 \eta(W) - 6 \} . (\phi \otimes X_{\beta_{43}}) - \frac{1}{2} (\tau_\lambda \otimes \text{Ad}_{p_-})(H'_{34}). (\phi \otimes X_{\beta_{43}}) \\ - \frac{1}{2} r \eta(X_1 + \sqrt{-1} Y_1). (\phi \otimes X_{\beta_{41}}) - (\tau_\lambda \otimes \text{Ad}_{p_-})(X_{\beta_{13}}). (\phi \otimes X_{\beta_{41}}) \\ - \frac{1}{2} r \eta(X_2 + \sqrt{-1} Y_2). (\phi \otimes X_{\beta_{42}}) - (\tau_\lambda \otimes \text{Ad}_{p_-})(X_{\beta_{23}}). (\phi \otimes X_{\beta_{42}}).$$

□

Compatibility of S -type and K -type

Here we note the compatibility of the action of S from left hand side and the action of K or M from right hand side on the function $\phi = \text{res}_A \varphi$, $\varphi \in C_{\eta, \tau_\lambda}^\infty(R \backslash G/K)$. If we write $\phi = \varphi|_A \in C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V_\lambda)$ as

$$\phi(a_r) = \sum_{k'=0}^{d_{\mu'}} \sum_{j_i=0}^{\infty} \sum_{Q \in GZ(\lambda)} c_{j,k}^{k', \mu}(a_r) \left((w_{k'}^{\mu'} \otimes f_j) \otimes v(Q) \right)$$

in terms of basis $\{w_k^{\mu'} | k = 0, \dots, d_{\mu'}\}$, $\{f_j | j \in \mathbb{N}^2\}$ and $\{v(Q) | Q \in GZ(\lambda)\}$ of $W_{\mu'}$, $\mathbb{C}[z_1, z_2]$ and V_λ respectively, the compatibility of S -action and K -action implies of the vanishing of many coefficients $c_{j,k}^{k', \mu}$. Actually by calculating $\phi(mam^{-1})$, $m \in S = M$, $a \in A$ in two ways, we have next lemma.

Lemma 4 (1) There is linear relations between indices of bases

$$j_1 = -k - k' - |\mu|/2 + (|\lambda|/2 - 1), \quad j_2 = k + k' + 3|\mu|/2 + (|\lambda|/2 - 1 - |\mu'|).$$

And there are relations between coefficient functions

$$-(j_1 + 1)c_{j+e_1-e_2, k}^{k', \mu} = \sqrt{(\mu'_1 - k' + 1)(k' - \mu'_2)} c_{j, k}^{k'-1, \mu} + \sqrt{(\mu_1 - k + 1)(k - \mu_2)} c_{j, k-1}^{k', \mu}, \\ -(j_2 + 1)c_{j-e_1+e_2, k}^{k', \mu} = \sqrt{(\mu'_1 - k')(k' - \mu'_2 + 1)} c_{j, k}^{k'+1, \mu} + \sqrt{(\mu_1 - k)(k - \mu_2 + 1)} c_{j, k+1}^{k', \mu}.$$

(2) If above relations are not satisfied, then the image of res_A in $C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V_\lambda)$ is zero. □

Difference-differential equations

Because an algebraic generalized Whittaker function F is determined by its A -radial part $\phi = F|_A$, and ϕ is determined by the coefficient functions $c_{j,k}^{k', \mu}(a_r)$, we write down the A -radial part $R(\mathcal{D}_{\eta, \tau_\lambda}^{-\beta})$ of the β -shift operators $\mathcal{D}_{\eta, \tau_\lambda}^{-\beta}$ in terms of coefficient functions of ϕ .

Proposition 5 Let ϕ be any function in $C^\infty(A; W_{\mu'} \otimes_{\mathbb{C}} \mathbb{C}[z] \otimes_{\mathbb{C}} V_\lambda)$ which is the A -radial part of $\varphi \in C_{\eta, \tau_\lambda}^\infty(R \backslash G/K)$. Then for an arbitrary noncompact root β , the action of the A -radial part $R(\mathcal{D}_{\eta, \tau_\lambda}^{-\beta})$ of the β -shift operator is given as follows:

$$R(\mathcal{D}_{\eta, \tau_\lambda}^{-\beta})\phi(a_r) = \sum c_{j,k}^{k', \mu}[-\beta](a_r) \left((w_{k'}^{\mu'} \otimes f_j) \otimes v(\tilde{Q}) \right),$$

with

$$\begin{aligned} 2c_{j,k}^{k', \mu}[-\beta_{14}](a_r) &= \sqrt{(\lambda_1 - \mu_1)(\lambda_1 + 1 - \mu_2)} \left\{ \partial - 6 - sr^2 + |\lambda| - 2\lambda_1 + 2 - |\mu| \right\} c_{j,k}^{k', \mu}(a_r) \\ &- 2\sqrt{s} \left| \frac{(\mu_1 + 1 - \lambda_2)(\mu_1 + 2 - \lambda_3)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\lambda_1 + 1 - \mu_2} \\ &\times \left(\sqrt{k + 1 - \mu_2(j_1 + 1)} rc_{j+e_1, k+1}^{k', \mu+e_1}(a_r) + \sqrt{\mu_1 + 1 - k(j_2 + 1)} rc_{j+e_2, k}^{k', \mu+e_1}(a_r) \right) \\ &- 2\sqrt{s} \left| \frac{(\lambda_2 - \mu_2)(\mu_2 + 1 - \lambda_3)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\lambda_1 - \mu_1} \\ &\times \left(-\sqrt{\mu_1 - k(j_1 + 1)} rc_{j+e_1, k+1}^{k', \mu+e_2}(a_r) + \sqrt{k - \mu_2(j_2 + 1)} rc_{j+e_2, k}^{k', \mu+e_2}(a_r) \right) \end{aligned}$$

$$\begin{aligned} 2c_{j,k}^{k', \mu}[-\beta_{24}](a_r) &= \sqrt{(\lambda_2 - \mu_2)(\mu_1 + 1 - \lambda_2)} \left\{ \partial - 6 - sr^2 + |\lambda| - 2\lambda_2 + 2 + 2 - |\mu| \right\} c_{j,k}^{k', \mu}(a_r) \\ &+ 2\sqrt{s} \left| \frac{(\lambda_1 - \mu_1)(\mu_1 + 2 - \lambda_3)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\lambda_2 - \mu_2} \\ &\times \left(\sqrt{k + 1 - \mu_2(j_1 + 1)} rc_{j+e_1, k+1}^{k', \mu+e_1}(a_r) + \sqrt{\mu_1 + 1 - k(j_2 + 1)} rc_{j+e_2, k}^{k', \mu+e_1}(a_r) \right) \\ &- 2\sqrt{s} \left| \frac{(\lambda_1 + 1 - \mu_2)(\mu_2 + 1 - \lambda_3)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\mu_1 + 1 - \lambda_2} \\ &\times \left(-\sqrt{\mu_1 - k(j_1 + 1)} rc_{j+e_1, k+1}^{k', \mu+e_2}(a_r) + \sqrt{k - \mu_2(j_2 + 1)} rc_{j+e_2, k}^{k', \mu+e_2}(a_r) \right) \end{aligned}$$

$$\begin{aligned} 2c_{j,k}^{k', \mu}[-\beta_{34}](a_r) &= \sqrt{(\mu_1 - \lambda_3 + 2)(\mu_2 + 1 - \lambda_3)} \left\{ \partial - 6 - sr^2 + |\lambda| - 2\lambda_3 + 4 + 2 - |\mu| \right\} c_{j,k}^{k', \mu}(a_r) \\ &+ 2\sqrt{s} \left| \frac{(\lambda_1 - \mu_1)(\mu_1 + 1 - \lambda_2)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\mu_2 + 1 - \lambda_3} \\ &\times \left(\sqrt{k + 1 - \mu_2(j_1 + 1)} rc_{j+e_1, k+1}^{k', \mu+e_1}(a_r) + \sqrt{\mu_1 + 1 - k(j_2 + 1)} rc_{j+e_2, k}^{k', \mu+e_1}(a_r) \right) \\ &+ 2\sqrt{s} \left| \frac{(\lambda_1 + 1 - \mu_2)(\lambda_2 - \mu_2)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\mu_1 + 2 - \lambda_3} \\ &\times \left(-\sqrt{\mu_1 - k(j_1 + 1)} rc_{j+e_1, k+1}^{k', \mu+e_2}(a_r) + \sqrt{k - \mu_2(j_2 + 1)} rc_{j+e_2, k}^{k', \mu+e_2}(a_r) \right) \end{aligned}$$

$$\begin{aligned} 2c_{j,k}^{k', \mu}[-\beta_{43}](a_r) &= \sqrt{(\mu_2 - \lambda_3)(\mu_1 + 1 - \lambda_3)} \left\{ \partial - 6 + sr^2 - |\lambda| + 2\lambda_3 + 2 + |\mu| \right\} c_{j,k}^{k', \mu}(a_r) \\ &+ 2\sqrt{s} \left| \frac{(\lambda_1 + 1 - \mu_1)(\mu_1 - \lambda_2)}{d_{\mu'}(d_{\mu'} + 1)} \right|^{1/2} \sqrt{\mu_2 - \lambda_3} \end{aligned}$$

$$\begin{aligned}
& \times \left(\sqrt{k - \mu_2 r} c_{j-e_1, k-1}^{k', \mu-e_1}(a_r) + \sqrt{\mu_1 - k r} c_{j-e_2, k}^{k', \mu-e_1}(a_r) \right) \\
& + 2\sqrt{s} \left| \frac{(\lambda_1 + 2 - \mu_2)(\lambda_2 + 1 - \mu_2)}{(d_{\mu'} + 1)(d_{\mu'} + 2)} \right|^{1/2} \sqrt{\mu_1 + 1 - \lambda_3} \\
& \times \left(-\sqrt{\mu_1 + 1 - k r} c_{j-e_1, k-1}^{k', \mu-e_2}(a_r) + \sqrt{k + 1 - \mu_2 r} c_{j-e_2, k}^{k', \mu-e_2}(a_r) \right)
\end{aligned}$$

□

<An explicit formula >

By solving the system of difference-differential equations given above for coefficient functions, we can obtain an explicit form of the generalized Whittaker functions F .

The case of holomorphic discrete series

Here we treat the holomorphic discrete series π_Λ , $\Lambda \in \Xi_I^+$. In this case $\Sigma_I^+ \cup \Sigma_n = \{\beta_{14}, \beta_{24}, \beta_{34}\}$. Hence the system (D) characterizing the generalized Whittaker function F associated to π_Λ with the minimal K -type turns into the system of difference-differential equations for coefficient functions

$$\begin{cases} c_{j,k}^{k', \mu}[-\beta_{14}](a_r) = 0 \\ c_{j,k}^{k', \mu}[-\beta_{24}](a_r) = 0 \\ c_{j,k}^{k', \mu}[-\beta_{34}](a_r) = 0. \end{cases}$$

This reduces to an ordinary differential equation of first order

$$\{\partial - sr^2 - |\lambda| + 2\mu_1\} c_{j,k}^{k', \mu}(a_r) = 0,$$

and we obtain

$$c_{j,k}^{k', \mu}(a_r) = (\text{const.}) \cdot r^{|\lambda| - 2\mu_1} e^{sr^2/2}.$$

Theorem 6 When $\Lambda \in \Xi_I$, π_Λ has multiplicity one property if and only if

$$-k - k' - |\mu|/2 + (|\lambda|/2 - 1) \in \mathbb{Z}_{\geq 0}, \quad k + k' + 3|\mu|/2 + (|\lambda|/2 - 1 - |\mu'|) \in \mathbb{Z}_{\geq 0}.$$

Under this condition, the minimal K -type generalized Whittaker model $Wh_\eta^{\tau_\lambda}(\pi_\Lambda)$ of π_Λ has a basis $F_\eta^{\tau_\lambda}$ whose A -radial part is given by

$$F(a_r) = \sum_{k'=0}^{d_{\mu'}} \sum_{Q \in GZ(\lambda)} \sum_{j \in SK(\mu', \lambda)} r^{|\lambda| - 2\mu_1} e^{sr^2/2} \cdot \left((w_{k'}^{\mu'} \otimes f_j) \otimes v(Q) \right),$$

where the indices j run through nonnegative integers satisfying the constraint condition in lemma 4. □

The case of large discrete series

In this case $\Sigma_{II}^+ \cup \Sigma_n = \{\beta_{14}, \beta_{24}, \beta_{43}\}$ and we have

$$\begin{cases} c_{j,k}^{k', \mu}[-\beta_{14}](a_r) = 0 \\ c_{j,k}^{k', \mu}[-\beta_{24}](a_r) = 0 \\ c_{j,k}^{k', \mu}[-\beta_{43}](a_r) = 0 \end{cases}$$

for characterizing system of difference-differential equations of coefficient functions of generalized Whittaker functions. This system can be solved when the Gel'fand-Zetlin scheme is of the extremal form

$$Q = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ & \lambda_2 & \mu_2 \\ & & \lambda_2 \end{pmatrix}.$$

Actually when $k = \mu_1 = \lambda_2$, from the first line and the second one we have a two term relation

$$(1) \quad \begin{aligned} & \{\partial - sr^2 - 2 - \lambda_1 + \lambda_3 - \mu_2\} c_{j, \lambda_2}^{k', \mu}(a_r) \\ &= 2\sqrt{s} \sqrt{\frac{(\lambda_2 + 1 - \mu_2)(\mu_2 + 1 - \lambda_3)}{(\lambda_1 + 1 - \mu_2)(\lambda_2 - \mu_2)}} (j_1 + 1) r c_{j+e_2, \lambda_2}^{k', \mu+e_2}(a_r). \end{aligned}$$

On the other hand the third line turns into

$$\begin{aligned} & \{\partial + sr^2 - 4 - \lambda_1 + \lambda_3 + \mu_2\} c_{j, \lambda_2}^{k', \mu}(a_r) \\ &= -2\sqrt{s} \sqrt{\frac{\lambda_1 + 2 - \mu_2}{(\lambda_2 + 2 - \mu_2)(\mu_2 - \lambda_3)}} \left\{ -r c_{j-e_1, \lambda_2-1}^{k', \mu-e_2}(a_r) + \sqrt{\lambda_2 + 1 - \mu_2} r c_{j-e_2, \lambda_2}^{k', \mu-e_2}(a_r) \right\}. \end{aligned}$$

Here use the relation caused by the compatibility of S -action and K -action. For $k' = \mu'_2$ the second relation in lemma4 is of the form

$$-(j_1 + 1) c_{j+e_1-e_2, k}^{\mu'_2, \mu} = \sqrt{(\mu_1 - k + 1)(k - \mu_2)} c_{j, k-1}^{\mu'_2, \mu}.$$

By this we can raise the k parameter and obtain

$$(2) \quad \begin{aligned} & \{\partial + sr^2 - 3 - \lambda_1 + \lambda_3 + \mu_2\} c_{j+e_2, \lambda_2}^{\mu'_2, \mu+e_2}(a_r) \\ &= -2\sqrt{s} \sqrt{\frac{\lambda_1 + 1 - \mu_2}{(\lambda_2 + 1 - \mu_2)(\mu_2 + 1 - \lambda_3)}} \frac{j_1 + \lambda_2 - \mu_2}{\sqrt{\lambda_2 - \mu_2}} r c_{j, \lambda_2}^{\mu'_2, \mu}(a_r). \end{aligned}$$

From these equations (1) and (2), we at last obtain the differential equation

$$\begin{aligned} & \left[\partial^2 - 2(\lambda_1 - \lambda_3 + 3)\partial - \{s^2 r^4 + 2\mu_2 sr^2 + (\mu_2 - 1)^2 - (\lambda_1 - \lambda_3 + 3)^2\} \right] c_{j, \lambda_2}^{\mu'_2, \mu}(a_r) \\ &= -4s \frac{(j_1 + \lambda_2 - \mu_2)(j_2 + 1)}{\lambda_2 - \mu_2} r^2 c_{j, \lambda_2}^{\mu'_2, \mu}(a_r). \end{aligned}$$

After some variable changes we have an explicit form of extremal coefficient functions.

Theorem 7 When $\Lambda \in \Xi_{II}$, the A -radial part of the minimal K -type generalized Whittaker function

$$F(a_r) = \sum_{k'=0}^{d_{\mu'}} \sum_{Q \in GZ(\lambda)} \sum_{j \in SK(\mu', \lambda)} c_{j, k}^{k', \mu}(a_r) \cdot \left((w_{k'}^{\mu'} \otimes f_j) \otimes v(Q) \right)$$

for large discrete series representation π_Λ has extremal coefficient functions

$$c_{j, \lambda_2}^{\mu'_2, \mu}(a_r) = r^{\lambda_1 - \lambda_3 + 2} \{ c_1(\mu_2) \cdot W_{\kappa, \frac{\mu_2 - 1}{2}}(sr^2) + c_2(\mu_2) \cdot M_{\kappa, \frac{\mu_2 - 1}{2}}(sr^2) \},$$

where $\kappa = -\frac{\mu_2}{2} - \frac{(j_1 + \lambda_2 - \mu_2)(j_2 + 1)}{\lambda_2 - \mu_2}$, $W_{\kappa, m}, M_{\kappa, m}$ are the classical Whittaker functions and $c_1(\mu_2), c_2(\mu_2)$ are constants depending only on μ_2 . Other coefficient functions are determined recursively by difference-differential relations between them. $\square$

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