

On the cohomology of finite Chevalley groups and free loop spaces

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1 Introduction

Let p, ℓ be distinct primes and let q be a power of p . We denote by $\mathbb{F}_q$ the finite field with q -elements. Let G be a connected compact Lie group. There exists a reductive finite algebraic group $G(\mathbb{C})$ associated with G , called the complexification of G . One may consider $G(\mathbb{C})$ as $\mathbb{C}$ -rational points of the reductive integral algebraic group scheme $G_{\mathbb{Z}}$. Taking the $\mathbb{F}_q$ -rational points of $G_{\mathbb{Z}}$, we have the finite Chevalley group $G(\mathbb{F}_q)$. Denote by $\bar{\mathbb{F}}_q$ the algebraic closure of the finite field $\mathbb{F}_q$, so that

$$\bar{\mathbb{F}}_p = \bigcup_q \mathbb{F}_q.$$

We may consider the the finite Chevalley group $G(\mathbb{F}_q)$ as the fixed point set $G(\bar{\mathbb{F}}_p)^{\phi^q}$ where

$$\phi^q : G(\bar{\mathbb{F}}_p) \rightarrow G(\bar{\mathbb{F}}_p)$$

is the Frobenius map induced by the Frobenius homomorphism $\phi^q : \bar{\mathbb{F}}_p \rightarrow \bar{\mathbb{F}}_p$ sending x to x^q .

In [4], Quillen computed the mod ℓ cohomology of finite general linear group $GL_n(\mathbb{F}_q)$. The finite general linear group $GL_n(\mathbb{F}_q)$ is the finite Chevalley group associated with the unitary group $U(n)$. Quillen's computation could be explained by the the following Theorem 1.1 due to Friedlander [1, Theorem 12.2], Friedlander–Mislin [2, Theorem 1.4].

We fix a connected compact Lie group G . Let $BG^\wedge$ be the Bousfield–Kan $\mathbb{Z}/\ell$ -completion of the classifying space BG of the connected compact Lie group G . We denote by $\text{fib}(\alpha)$ the homotopy fibre of a map α . For the sake of notational simplicity, we write A, V for $H^*(BG; \mathbb{Z}/\ell)$, $H^*(G; \mathbb{Z}/\ell)$, respectively.

Theorem 1.1 There exist maps

$$D : BG(\bar{\mathbb{F}}_p) \rightarrow BG^\wedge$$

and

$$\phi^q : BG^\wedge \rightarrow BG^\wedge$$

satisfying the following three conditions:

(1) The induced homomorphism

$$D^* : H^*(BG^\wedge; \mathbb{Z}/\ell) \rightarrow H^*(BG(\bar{\mathbb{F}}_p); \mathbb{Z}/\ell)$$

is an isomorphism.

(2) $\phi^q \circ D \simeq D \circ \phi^q$ where

$$\phi^q : BG(\bar{\mathbb{F}}_p) \rightarrow BG(\bar{\mathbb{F}}_p)$$

is the Frobenius map induced by the Frobenius homomorphism $\phi^q : \bar{\mathbb{F}}_p \rightarrow \bar{\mathbb{F}}_p$.

(3) The induced map

$$\text{fib}(D_q) \rightarrow \text{fib}(\Delta)$$

induces an isomorphism

$$H^*(\text{fib}(\Delta); \mathbb{Z}/\ell) \rightarrow H^*(\text{fib}(D_q); \mathbb{Z}/\ell),$$

where the above map is induced by the following homotopy commutative diagram.

$$\begin{array}{ccc} BG(\mathbb{F}_q) & \longrightarrow & BG^\wedge \\ D_q \downarrow & & \downarrow \Delta \\ BG^\wedge & \xrightarrow{1 \times \phi^q \circ \Delta} & BG^\wedge \times BG^\wedge, \end{array}$$

where $D_q = D \circ i_q$, $i_q = BG(\mathbb{F}_q) \rightarrow BG(\bar{\mathbb{F}}_p)$ is the map induced by the inclusion of $\mathbb{F}_q$ into $\bar{\mathbb{F}}_p$ and $\Delta : BG^\wedge \rightarrow BG^\wedge \times BG^\wedge$ is the diagonal map.

On the one hand, there exists the Eilenberg–Moore spectral sequence converging to

$$gr H^*(BG(\mathbb{F}_q); \mathbb{Z}/\ell)$$

with the E_2 -term

$$\text{Tor}_{A \otimes A}(A, A).$$

On the other hand, for the free loop space

$$\mathcal{L}BG = \{\lambda : I = [0, 1] \rightarrow BG \mid \lambda(1) = \lambda(0)\},$$

we have the following fibre square:

$$\begin{array}{ccc} \mathcal{L}BG & \longrightarrow & BG \\ \pi_0 \downarrow & & \downarrow \Delta \\ BG & \xrightarrow{\Delta} & BG \times BG, \end{array}$$

where π_0 is the evaluation map at 0, so that $\pi_0(\lambda) = \lambda(0)$. There exists the Eilenberg-Moore spectral sequence

$$\mathrm{Tor}_{A \otimes A}(A, A) \Rightarrow \mathrm{gr} H^*(\mathcal{L}BG; \mathbb{Z}/\ell).$$

If $H_*(G; \mathbb{Z})$ has no ℓ -torsion and if $\ell \nmid q - 1$, then $A = H^*(BG; \mathbb{Z}/\ell)$ is a polynomial algebra and the induced homomorphism

$$\phi^{q*} : A \rightarrow A$$

is the identity homomorphism. Hence, the above E_2 -terms are the same polynomial tensor exterior algebra $A \otimes V$ and both spectral sequences collapse at the E_2 -level. Thus, the mod ℓ cohomology of the free loop space of the classifying space BG is isomorphic to the mod ℓ cohomology of the finite Chevalley group $G(\mathbb{F}_q)$. Even if $H^*(G; \mathbb{Z})$ has ℓ -torsion, E_2 -terms of the above spectral sequences are the same. Observing this phenomenon, Tezuka in [5] asked the following:

Conjecture 1.2 If $\ell \nmid q - 1$ (resp. $4 \nmid q - 1$) when ℓ is odd (resp. even), there exists an ring isomorphism between $H^*(BG(\mathbb{F}_q); \mathbb{Z}/\ell)$ and $H^*(\mathcal{L}BG; \mathbb{Z}/\ell)$.

In conjunction with this conjecture, in this paper, we give an outline of the proof of the following result:

Theorem 1.3 There exists an integer b such that, for $q = p^{ab}$ where a is an arbitrary positive integer, there exists an isomorphism between Leray-Serre spectral sequences associated with the map

$$D_q : BG(\mathbb{F}_q) \rightarrow BG^\wedge$$

and the diagonal map

$$\Delta : BG \rightarrow BG \times BG.$$

So, we have an isomorphism of graded $\mathbb{Z}/\ell$ -algebras

$$\mathrm{gr} H^*(BG(\mathbb{F}_q); \mathbb{Z}/\ell) = \mathrm{gr} H^*(\mathcal{L}BG; \mathbb{Z}/\ell).$$

Remark 1.4 Although we give an example of the integer b in Theorem 1.3 in §2 as a function of $\dim G$ and $\dim V = H^*(G; \mathbb{Z}/\ell)$, it is not at all the best possible.

When we want to show that the cohomology of a space X is isomorphic to the cohomology of another space Y , we usually try to construct a chain of maps

$$X = X_0 \xleftarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longleftarrow \cdots \longleftarrow X_n \xrightarrow{f_n} X_{n+1} = Y$$

such that maps f_k 's induce isomorphisms in mod ℓ cohomology. For example, Theorem 1.1 is proved by this method. However, when we try to prove Theorem 1.3 or Conjecture 1.2, we can not construct such a chain of maps. On the one hand, since the rational cohomology of finite group is trivial, we have

$$H^*(BG(\mathbb{F}_q); \mathbb{Q}) = \mathbb{Q}.$$

On the other hand, the rational cohomology of the free loop space is easy to compute and we have

$$\begin{aligned} H^*(\mathcal{L}BG; \mathbb{Q}) &= H^*(BG; \mathbb{Q}) \otimes H^*(G; \mathbb{Q}) \\ &= \mathbb{Q}[y_1, \dots, y_n] \otimes \Lambda(x_1, \dots, x_n), \end{aligned}$$

where n is the rank of the connected compact Lie group G . If there exists such a chain of maps, then they also induce isomorphisms of Bockstein spectral sequences. This contradicts the above observation on the rational (and integral) cohomology of $BG(\mathbb{F}_q)$ and $\mathcal{L}BG$. Thus, we construct maps which induce monomorphisms of Leray-Serre spectral sequences. By comparing the image of Leray-Serre spectral sequences, we construct an isomorphism between Leray-Serre spectral sequences. This isomorphism could not be realized by a chain of maps.

In §2, we define the integer b as a function of $\dim G$ and $\dim V = \dim H^*(G; \mathbb{Z}/\ell)$. In §3, we give a proof of Theorem 1.3 assuming Lemmas 3.2, 3.3 and Proposition 2.2.

Acknowledgement. Since there exists no map realizing the isomorphism between $H^*(BG(\mathbb{F}_q); \mathbb{Z}/\ell)$ and $H^*(\mathcal{L}BG; \mathbb{Z}/\ell)$, it is difficult to believe the existence of such isomorphism for arbitrary connected compact Lie group G . It is my pleasure to thank Prof. M. Tezuka for informing me of amazing Conjecture 1.2 in this workshop in August, 2003. Since 2001, I have been participating this workshop “Cohomology Theory of Finite Groups and Related Topics”. Not only I learned Conjecture 1.2 in this workshop, but also this workshop had a great impact on my mathematics. So, I would like to thank Prof. Y. Sasaki for organizing this workshop every two years for many years. The author is partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C) 19540105.

2 The integer b

In this section, we define the integer b in Theorem 1.3. We define the integer b as

$$b = e_1^{\dim G} e_2$$

and we define e_1, e_2 in this section.

Recall that $V = H^*(G; \mathbb{Z}/\ell)$. We have isomorphisms

$$V = H^*(\text{fib}(D_q); \mathbb{Z}/\ell) = H^*(\text{fib}(\Delta); \mathbb{Z}/\ell) = H^*(\Omega BG^\wedge; \mathbb{Z}/\ell).$$

Denote by $GL(V)$ be the group of automorphisms of V .

Firstly, we define the integer e_1 . Let q' be a power of q . The inclusion of $\mathbb{F}_q$ into $\mathbb{F}_{q'}$ induces maps

$$i : BG(\mathbb{F}_q) \rightarrow BG(\mathbb{F}_{q'})$$

and

$$j : \text{fib}(D_q) \longrightarrow \text{fib}(D_{q'}).$$

Suppose that $q' = q^e$ and $e = \ell m$ where m is the order of

$$\phi^{q*} : H^*(\Omega BG^\wedge; \mathbb{Z}/\ell) \rightarrow H^*(\Omega BG^\wedge; \mathbb{Z}/\ell)$$

as an element in $GL(V)$ (see the proof of Lemma 1.3 in [2]). Consider the induced homomorphism

$$j_* : \tilde{H}_*(\text{fib}(D_q); \mathbb{Z}/\ell) \rightarrow \tilde{H}_*(\text{fib}(D_{q'}); \mathbb{Z}/\ell),$$

where $\tilde{H}_*$ (resp. $\tilde{H}^*$) is the reduced homology (resp. reduced cohomology). Recall Lemma 1.3 in [2], we see that there hold the following:

- (1) If $j_*(y) = 0$ for $\deg y < \deg x$ and if x is primitive, then $j_*(x) = 0$.
- (2) If $j_*(y) = 0$ for $\deg y < \deg x$, then $j_*(x)$ is primitive.

Hence, if $q' = q^{e^2}$ and if $j_*(y) = 0$ for $\deg y < k$, then $j_*(x) = 0$ for $\deg x \leq k$. Let $e_1 = (\ell |GL(V)|)^{2 \dim G}$. Then, the induced homomorphism

$$\tilde{H}^*(\text{fib}(D_{q^{e_1}}); \mathbb{Z}/\ell) \rightarrow \tilde{H}^*(\text{fib}(D_q); \mathbb{Z}/\ell)$$

is zero. Therefore, we have the following lemma.

Lemma 2.1 The induced homomorphisms

$$\tilde{H}^*(\text{fib}(D_{p^{e_1^{k+1}}}); \mathbb{Z}/\ell) \rightarrow \tilde{H}^*(\text{fib}(D_{p^{e_1^k}}); \mathbb{Z}/\ell)$$

are zeros homomorphisms for $k = 0, \dots, \dim G - 1$.

Secondly, we define the integer e_2 . We consider the fibre square

$$\begin{array}{ccc} \text{fib}(D_q) \times_{BG(\mathbb{F}_q)} \text{fib}(D_q) & \longrightarrow & \text{fib}(D_q) \\ \downarrow & & \downarrow \\ \text{fib}(D_q) & \longrightarrow & BG(\mathbb{F}_q) \end{array}$$

and the induced map

$$1 \times \phi^q : \text{fib}(D_q) \times_{BG(\mathbb{F}_q)} \text{fib}(D_q) \rightarrow \text{fib}(D_q) \times_{BG(\mathbb{F}_q)} \text{fib}(D_q).$$

We assume the following proposition.

Proposition 2.2 The local coefficients of $\text{fib}(D_q) \times_{BG(\mathbb{F}_q)} \text{fib}(D_q) \rightarrow \text{fib}(D_q)$ are trivial and the E_2 -term of the Leray-Serre spectral sequence for

$$H^*(\text{fib}(D_q) \times_{BG(\mathbb{F}_q)} \text{fib}(D_q); \mathbb{Z}/\ell)$$

is given by

$$H^*(\text{fib}(D_q); \mathbb{Z}/\ell) \otimes H^*(\Omega BG^\wedge; \mathbb{Z}/\ell) = V \otimes V.$$

Hence, we have that

$$\dim H^*(\text{fib}(D_q) \times_{BG(\mathbb{F}_q)} \text{fib}(D_q); \mathbb{Z}/\ell) \leq \dim(V \otimes V).$$

Let $e_2 = |GL(V \otimes V)|$. Then, we have the following proposition.

Proposition 2.3 The induced homomorphism

$$(1 \times \phi^q)^{ae_2*} : \tilde{H}^*(\text{fib}(D_q) \times_{BG(\mathbb{F}_q)} \text{fib}(D_q); \mathbb{Z}/\ell) \rightarrow \tilde{H}^*(\text{fib}(D_q) \times_{BG(\mathbb{F}_q)} \text{fib}(D_q); \mathbb{Z}/\ell)$$

is the identity homomorphism for any positive integer a .

3 Proof of Theorem 1.3

Let X be a space and let $f : X \rightarrow X$ be a self-map of X with non-empty fixed point set X^f . Let A' be a subspace of the fixed point set X^f . We choose a base-point $*$ in A' . Let

$$A = \{\lambda : I \rightarrow X \mid \lambda(1) \in A'\}.$$

and let F be the homotopy fibre of the inclusion of A' into X , say

$$F = \{\lambda : I \rightarrow X \mid \lambda(0) = *, \lambda(1) \in A'\}.$$

We assume that X is simply connected and that F is connected. Let

$$\mathcal{L}_f X = \{\lambda : I \rightarrow X \mid \lambda(1) = f(\lambda(0))\}$$

and we call it the twisted loop space of f . When $f = 1$, the identity map, we denote $\mathcal{L}_1 X$ by $\mathcal{L}X$. This is the free loop space of X . There is an evaluation map

$$\pi_0 : \mathcal{L}_f X \rightarrow X$$

at 0, say $\pi_0(\lambda) = \lambda(0)$.

Firstly, we define a map

$$\varphi : \mathcal{L}_f X \times_X \mathcal{L}_f X \rightarrow \mathcal{L}X,$$

where

$$\mathcal{L}_f X \times_X \mathcal{L}_f X = \{(\lambda_1, \lambda_2) \in \mathcal{L}_f X \times \mathcal{L}_f X \mid \pi_0(\lambda_1) = \pi_0(\lambda_2)\}.$$

The map φ is defined by

$$\begin{cases} \varphi(\lambda_1, \lambda_2)(t) = \lambda_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \varphi(\lambda_1, \lambda_2)(t) = \lambda_2(2 - 2t) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $\lambda_1(1) = f(\lambda_1(0))$, $\lambda_2(1) = f(\lambda_2(0))$ and $\lambda_1(0) = \lambda_2(0)$, this map is well-defined.

Next, we define a map from A to $\mathcal{L}_f X$, say $\psi : A \rightarrow \mathcal{L}_f X$ by

$$\begin{cases} \psi(\lambda)(t) = \lambda(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \phi(\lambda)(t) = f(\lambda(2 - 2t)) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $\lambda(1) = f(\lambda(1))$, this map is also well-defined.

Now, we consider the following diagram:

$$\begin{array}{ccccc} \mathcal{L}_f X & \xleftarrow{p_1} & \mathcal{L}_f X \times_X \mathcal{L}_f X & \xrightarrow{\varphi} & \mathcal{L}X \\ \downarrow \pi_0 & & \downarrow \pi_0 & \nearrow & \downarrow \pi_0 \\ X & \xleftarrow{=} & X & \xrightarrow{1 \times \psi} & X \\ & & & \searrow & \\ & & & & \mathcal{L}_f X \times_X A \\ & & & \nearrow & \downarrow \pi_0 \\ & & & & X, \end{array}$$

where

$$\mathcal{L}_f X \times_X A = \{(\lambda_1, \lambda_2) \in \mathcal{L}_f X \times A \mid \pi_0(\lambda_1) = \pi_0(\lambda_2)\},$$

and p_1 is the projection onto the first factor. Let us denote by $E_r(\xi)$ the Leray-Serre spectral sequence associated with a fibration $\xi \rightarrow X$. Then we have the following diagram of spectral sequences:

$$\begin{array}{ccccc} E_r(\mathcal{L}_f X) & \xrightarrow{p_1^*} & E_r(\mathcal{L}_f X \times_X \mathcal{L}_f X) & \xleftarrow{\varphi^*} & E_r(\mathcal{L}X) \\ & & \downarrow 1 \times \psi^* & & \\ & & E_r(\mathcal{L}_f X \times_X A). & & \end{array}$$

We denote by $\psi : F \rightarrow \Omega X$ the restriction of $\psi : \mathcal{L}_f X \rightarrow A$ to fibres. Let us consider a sufficient condition for the induced homomorphism

$$\psi^* : \tilde{H}^*(\Omega X; \mathbb{Z}/\ell) \rightarrow \tilde{H}^*(F; \mathbb{Z}/\ell)$$

to be zero. Let

$$F \times_{A'} F = \{(\lambda_1, \lambda_2) \in F \times F \mid \pi_1(\lambda_1) = \pi_1(\lambda_2)\},$$

where $\pi_1 : F \rightarrow A'$ is the evaluation map at $1 \in I$. We denote by $\varphi : \Omega X \times \Omega X \rightarrow \Omega X$ the restriction of $\varphi : \mathcal{L}_f X \times_X \mathcal{L}_f X \rightarrow \mathcal{L}_f X$ to fibres. The map $\psi : F \rightarrow X$ factors through

$$F \xrightarrow{\Delta} F \times_{A'} F \xrightarrow{1 \times f} F \times_{A'} F \xrightarrow{\varphi} \Omega X.$$

It is clear that the composition $\varphi \circ \Delta$ is null homotopic since an obvious null homotopy h_s is given by

$$\begin{cases} h_s(t) = \lambda(2st) & (0 \leq t \leq \frac{1}{2}), \\ h_s(t) = \lambda(2s - 2st) & (\frac{1}{2} \leq t \leq 1). \end{cases}$$

Thus, we have the following lemma.

Lemma 3.1 If the induced homomorphism

$$(1 \times f)^* : H^*(F \times_{A'} F; \mathbb{Z}/\ell) \rightarrow H^*(F \times_{A'} F; \mathbb{Z}/\ell)$$

is the identity homomorphisms, then the induced homomorphism

$$\psi^* : \tilde{H}^*(\Omega X; \mathbb{Z}/\ell) \rightarrow \tilde{H}^*(F; \mathbb{Z}/\ell)$$

is zero and

$$\begin{aligned} \text{Im}(1 \times \psi)^* \circ p_1^* &= \text{Im}(1 \times \psi)^* \circ \varphi^* = H^*(X; \mathbb{Z}/\ell) \otimes H^*(\Omega X; \mathbb{Z}/\ell) \otimes \mathbb{Z}/\ell \\ &\subset H^*(X; \mathbb{Z}/\ell) \otimes H^*(\Omega X; \mathbb{Z}/\ell) \otimes H^*(F; \mathbb{Z}/\ell) = E_2(\mathcal{L}_f X \times_X A) \end{aligned}$$

in $E_2(\mathcal{L}_f X \times_X A)$.

We also need the following lemmas in the proof of Theorem 1.3.

Lemma 3.2 Let $\alpha : A' \rightarrow X$ be a map. If $H^i(\text{fib}(\alpha); \mathbb{Z}/\ell) = 0$ for $i > k$ and if there exists a sequence of maps

$$\begin{array}{ccccccc}
 A'_0 & \longrightarrow & A'_1 & \longrightarrow & A'_2 & \longrightarrow & \cdots \longrightarrow A'_k \xrightarrow{=} A' \\
 \downarrow \alpha_0 & & \downarrow \alpha_1 & & \downarrow & & \downarrow \alpha_k \swarrow \alpha \\
 X & \xrightarrow{=} & X & \xrightarrow{=} & X & \xrightarrow{=} & \cdots \longrightarrow X
 \end{array}$$

such that the induced map

$$\text{fib}(\alpha_i) \rightarrow \text{fib}(\alpha_{i+1})$$

induces a zero homomorphism

$$\tilde{H}^*(\text{fib}(\alpha_{i+1}); \mathbb{Z}/\ell) \rightarrow \tilde{H}^*(\text{fib}(\alpha_i); \mathbb{Z}/\ell)$$

for $i = 0, \dots, k-1$, then the map α induces a monomorphism $E_r(Y) \rightarrow E_r(Y \times_X A)$ of Leray-Serre spectral sequences for arbitrary fibration $Y \rightarrow X$.

Lemma 3.3 Let

$$E'_r \xrightarrow{\rho'_r} E_r \xleftarrow{\rho''_r} E''_r$$

be homomorphisms of spectral sequences. Suppose that

- (1) $\text{Im } \rho'_2 = \text{Im } \rho''_2$,
- (2) ρ'_r is a monomorphism for $r \geq 2$.

Then, there exists an isomorphism of spectral sequences $\tau : E''_r \rightarrow E'_r$.

Now, we prove Theorem 1.3 assuming Lemmas 3.2, 3.3 and Proposition 2.2.

Proof of Theorem 1.3 Let $q_0 = p^{e_1^{\dim G}}$, $q = q^{ae_2}$ ($a \geq 1$). Let X be the mapping cylinder of $D_{q_0} : BG(\mathbb{F}_{q_0}) \rightarrow BG^\wedge$, that is,

$$X = BG^\wedge \cup (BG(\mathbb{F}_{q_0}) \times I) / \sim$$

where $D_{q_0}(x) \sim (x, 0)$. Choose a homotopy $H : BG(\mathbb{F}_q) \times I \rightarrow BG^\wedge$ between $\phi^q \circ D_{q_0}$ and D_{q_0} , so that

$$\begin{aligned}
 H(x', 0) &= \phi^q \circ D_{q_0}(x'), \\
 H(x', 1) &= D_{q_0}(x').
 \end{aligned}$$

Let f be the map defined by

$$\begin{aligned} f(x) &= \phi^q(x) && \text{for } x \in BG^\wedge, \\ f(x', t) &= H(x', 2t) && (0 \leq t \leq \tfrac{1}{2}), \\ f(x', t) &= (x', 2t - 1) && (\tfrac{1}{2} \leq t \leq 1). \end{aligned}$$

Let $A' = BG(\mathbb{F}_{q_0}) \times \{1\} \subset X$. By definition, we have $A' \subset X^f$.

Let $A'_i = BG(\mathbb{F}_{p^{e_i}})$. Then, by Lemmas 2.1, 3.2, we have a monomorphism

$$E_r(\mathcal{L}_f X) \xrightarrow{(1 \times \psi)^* \circ p_1^*} E_r(\mathcal{L}_f X \times_X A).$$

By Proposition 2.3, the induced homomorphism $((1 \times \phi^{q_0})^*)^{ae_2}$ is the identity homomorphism. Hence, so is $1 \times f^*$. By Lemma 3.1, we obtain

$$\text{Im}(1 \times \psi)^* \circ p_1^* = \text{Im}(1 \times \psi)^* \circ \varphi^*$$

in $E_2(\mathcal{L}_f X \times_X A)$.

Therefore, using Lemma 3.3, we obtain an isomorphism between $E_r(\mathcal{L}_f X)$ and $E_r(\mathcal{L}X)$. $\square$

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