

Geometric properties of certain analytic functions with real coefficients

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Abstract

Let $\mathcal{T}$ be the class of analytic functions with real coefficients in the open unit disk $\mathbb{U}$. For $f(z)$ belonging to the class $\mathcal{T}$, some sufficient conditions for starlikeness and convexity are discussed. Furthermore, for $f(z)$ in the class $\mathcal{T}$, we prove the starlikeness of $f(z)$ having property $\operatorname{Re}\{f'(z)\} > 0$.

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1 Introduction

Let $\mathcal{A}$ be the class of functions

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.

We denote by $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ the subclasses of $\mathcal{A}$ whose members map $\mathbb{U}$ onto domain which are univalent, starlike, convex and close-to-convex.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order α ($\alpha < 1$) in $\mathbb{U}$ if and only if

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathbb{U}).$$

Similarly, $f(z) \in \mathcal{A}$ is said to be convex of order α ($\alpha < 1$) in $\mathbb{U}$ if and only if

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in \mathbb{U}).$$

We shall denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{A}$ whose members satisfy (1.2) and (1.3), respectively.

It is known that for $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*$, $\mathcal{K}(\alpha) \subset \mathcal{K}$ and that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$, $\mathcal{K}(0) \equiv \mathcal{K}$. Chichra [2] showed that for $f(z) \in \mathcal{A}$ and $\alpha \geq 0$ the following implication holds in $\mathbb{U}$:

$$(1.4) \quad \operatorname{Re}\{f'(z) + \alpha f''(z)\} > 0 \Rightarrow \operatorname{Re} f'(z) > 0.$$

On the other hand, Singh and Singh [13], Mocanu [5] have the following results for $f(z) \in \mathcal{A}$, respectively.

$$(1.5) \quad \operatorname{Re}\{f'(z) + z f''(z)\} > -\frac{1}{4} \Rightarrow \operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > 0,$$

$$(1.6) \quad \operatorname{Re} \left\{ f'(z) + \frac{1}{2} z f''(z) \right\} > 0 \Rightarrow \operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > 0.$$

Furthermore, Salagean [11] defined $\mathcal{N}$ the class of functions with negative coefficient, that is,

$$(1.7) \quad \mathcal{N} = \left\{ f(z) \in \mathcal{A} \mid f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0 \right\}$$

and obtained the following implications that are that if $f(z) \in \mathcal{N}$ in $\mathbb{U}$, then

$$(1.8) \quad \operatorname{Re}\{f'(z) + z f''(z)\} > -1 \Rightarrow \operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > 0,$$

$$(1.9) \quad \operatorname{Re}\{f'(z) + z f''(z)\} > 0 \Rightarrow \operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \frac{1}{2}.$$

2 Preliminaries

Recently, we prove the following Lemma in [9].

Lemma 1. [9, Nunokawa et al] *Let $f(z) \in \mathcal{A}$ and suppose that*

$$(2.1) \quad \operatorname{Re}\{f'(z) + \alpha f''(z)\} > -\frac{\alpha}{2} \quad \text{in } \mathbb{U}$$

for some α ($\alpha > 0$). Then we have $\operatorname{Re} f'(z) > 0$ in $\mathbb{U}$.

Next lemma was given by Nunokawa in 1993.

Lemma 2. [8] *Let $p(z)$ be analytic in $\mathbb{U}$, $p(0) = 1$, $p(z) \neq 0$ in $\mathbb{U}$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\pi\alpha}{2} \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\alpha}{2}$$

where $\alpha > 0$. Then we have $\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$ where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \text{ when } \arg p(z_0) = \frac{\pi\alpha}{2}$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \text{ when } \arg p(z_0) = -\frac{\pi\alpha}{2}$$

where

$$p(z_0)^{\frac{1}{\alpha}} = \pm i\alpha, \text{ and } a > 0.$$

Let us define $\mathcal{T}$ the class of analytic functions with real coefficients, that is,

$$(2.2) \quad \mathcal{T} = \left\{ f(z) \in \mathcal{A} \mid f(z) = z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathbb{R} \right\}$$

where $\mathbb{R}$ is the set of real numbers. Then it follows that

$$\mathcal{N} \subset \mathcal{T} \subset \mathcal{A}.$$

In [9], we have the following theorem.

Theorem A. [9] Let $f(z) \in \mathcal{T}$ and suppose that

$$\operatorname{Re}\{f'(z) + \alpha z f''(z)\} > 0 \quad \text{in } \mathbb{U}$$

where $\alpha \geq 1$. Then we have

$$1 + \operatorname{Re} \left\{ \frac{z f''(z)}{f'(z)} \right\} > \frac{\alpha - 1}{\alpha} \quad \text{in } \mathbb{U},$$

or $f(z)$ is convex of order $\frac{\alpha - 1}{\alpha}$.

Remark 1. Putting $\alpha = 1$ in Theorem 1, we have

$$\begin{aligned} f(z) &\in \mathcal{T}, \quad \operatorname{Re}\{f'(z) + z f''(z)\} > 0 \\ &\Rightarrow 1 + \operatorname{Re} \left\{ \frac{z f''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathbb{U}) \\ &\Rightarrow \operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \frac{1}{2} \quad (z \in \mathbb{U}). \end{aligned}$$

Let $\mathcal{P}'$ be the subclass of $\mathcal{A}$ whose members $f(z)$ satisfy $\operatorname{Re} f'(z) > 0$ in $\mathbb{U}$. It is well-known that $\mathcal{P}'$ is a subclass of $\mathcal{C}$ whose elements are close-to-convex in $\mathbb{U}$.

3 Main results

Theorem 1. Let $p(z) = 1 + p_1z + p_2z^2 + \dots$ be analytic in $\mathbb{U}$ and all coefficients p_i are real numbers. Suppose that

$$(3.1) \quad \operatorname{Re}\{p(z) + \alpha zp'(z)\} > 0 \quad \text{in } \mathbb{U}$$

where $\alpha \geq 1$. Then we have

$$(3.2) \quad 1 + \operatorname{Re} \left\{ \frac{zp'(z)}{p(z)} \right\} > 0 \quad \text{in } \mathbb{U}.$$

Proof. Using assumption (3.1) and Lemma 1, we have

$$\operatorname{Re}\{p(z)\} > 0 \quad \text{in } \mathbb{U}.$$

Therefore, we have

$$(3.3) \quad \left| \arg p(z) + \arg \left(1 + \alpha \frac{zp'(z)}{p(z)} \right) \right| < \frac{\pi}{2} \quad \text{in } \mathbb{U}.$$

for a sufficiently small and positive ϵ , there exists a point $z_1 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\epsilon \quad \text{for } |z| < |z_1|$$

and

$$|\arg p(z_1)| = \frac{\pi}{2}\epsilon,$$

then from Lemma 2, we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = i\epsilon k$$

where

$$k \geq 1 \quad \text{when } \arg p(z_1) = \frac{\pi}{2}\epsilon$$

and

$$k \leq -1 \quad \text{when } \arg p(z_1) = -\frac{\pi}{2}\epsilon.$$

Then it follows that for the case $\arg p(z_1) = \frac{\pi}{2}\epsilon$, we have

$$(3.4) \quad \begin{aligned} \arg \left(1 + \alpha \frac{z_1 p'(z_1)}{p(z_1)} \right) &= \arg(1 + i\alpha\epsilon k) \\ &= \tan^{-1} \alpha\epsilon k \geq \tan^{-1} \alpha\epsilon > 0. \end{aligned}$$

And for the case $\arg p(z_1) = -\frac{\pi}{2}\epsilon$, we also have

$$(3.5) \quad \begin{aligned} \arg \left(1 + \alpha \frac{z_1 p'(z_1)}{p(z_1)} \right) &= \arg(1 + i\alpha\epsilon k) \\ &= \tan^{-1} \alpha\epsilon k \leq \tan^{-1}(-\alpha\epsilon) < 0. \end{aligned}$$

From the assumption of Theorem 1, the image domains of the open unit disk $\mathbb{U}$ under the mapping $w = p(z)$ and $w = 1 + \alpha \frac{zp'(z)}{p(z)}$ are symmetric with respect to the real axis. Therefore, from above properties (3.4) and (3.5), it shows that the image domains of the open unit disk $\mathbb{U}$ under the mapping $w = p(z)$ and $w = 1 + \alpha \frac{zp'(z)}{p(z)}$ are the same side of the complex plane which is divided into two parts by the real axis.

Now then, if there exists a point $z_0 \in \mathbb{U}$ such that

$$\left| \arg \left(1 + \alpha \frac{zp'(z)}{p(z)} \right) \right| < \frac{\pi}{2} \quad \text{for } |z| < |z_0|$$

and

$$\left| \arg \left(1 + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) \right| = \frac{\pi}{2},$$

then for the case

$$\arg \left(1 + \alpha \frac{z_0 p'(z)}{p(z)} \right) = \frac{\pi}{2}$$

we have $\arg p(z_0) > 0$. This contradicts (3.3) and for the case

$$\arg \left(1 + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right) = -\frac{\pi}{2},$$

we have $\arg p(z_0) < 0$. This contradicts (3.3) and therefore, we have

$$1 + \alpha \operatorname{Re} \frac{zp'(z)}{p(z)} > 0 \quad \text{in } \mathbb{U}.$$

□

Letting $p(z) = f'(z)$, we have Theorem A. Furthermore, putting $p(z) = \frac{f(z)}{z}$ for $f(z) \in \mathcal{A}$, we have

Corollary 1. *Let $f(z) \in \mathcal{T}$ and suppose that*

$$\operatorname{Re} \left\{ (1 - \alpha) \frac{f(z)}{z} + \alpha f'(z) \right\} > 0 \quad (z \in \mathbb{U})$$

and $\alpha \geq 1$. Then we have

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{\alpha - 1}{\alpha} \quad (z \in \mathbb{U}),$$

that is, $f(z)$ is starlike of order $\frac{\alpha - 1}{\alpha}$.

Remark 2. In 1962, Krzyz [3] gave an example of a function $f(z) \in \mathcal{P}'$ such that $f(z) \notin \mathcal{S}^*$

However, in the case of $f(z) \in \mathcal{T}$, we have

Theorem 2. Let $f(z) \in \mathcal{T}$. If $\operatorname{Re} f'(z) > 0$ in $\mathbb{U}$, then we have $f(z) \in \mathcal{S}^*$.

Proof. Putting $\alpha = 1$ in Corollary 1, we prove Theorem 2. □

Using our results, we have many starlike functions and convex functions.

Example 1. Let $f(z) \in \mathcal{T}$ and $\alpha \geq 1$. If

$$f'(z) + \alpha z f''(z) = \frac{1+z}{1-z},$$

then we have

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2}{n(1+(n-1)\alpha)} z^n \in \mathcal{K}\left(\frac{\alpha-1}{\alpha}\right).$$

Example 2. Putting $\alpha = 1$ in Example 1, we have

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2}{n^2} z^n \in \mathcal{K}, \quad |f(z)| < \frac{\pi^2 - 3}{3} = 2.289 \dots$$

Example 3. Let $f(z) \in \mathcal{T}$ and $\alpha \geq 1$. If

$$(1-\alpha) \frac{f(z)}{z} + \alpha f'(z) = \frac{1+z}{1-z},$$

then we have

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2}{1+(n-1)\alpha} z^n \in \mathcal{S}^*\left(\frac{\alpha-1}{\alpha}\right).$$

Example 4. Letting $\alpha = 1$ in Example 3, we have

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2}{n} z^n \in \mathcal{S}^*.$$

Next result is well-known. Let

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1)$$

that is, Libera transform. If $f(z) \in \mathcal{P}'$, then $F(z) \in \mathcal{P}'$.

A natural question arise, that is, If $f(z) \in \mathcal{P}'$, is the Libera transform of $f(z)$ starlike in

$\mathbb{U}$?

Singh and Singh [12] answered.

Theorem B. ([12]) *If $f(z) \in \mathcal{P}'$, then the function $F(z)$, defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1)$$

belongs to $\mathcal{S}^$ for all c ($-1 < c \leq 0$).*

We consider the next question, that is,

" If $f(z) \in \mathcal{T}$ and $\operatorname{Re} f'(z) > 0$, is the Libera transform of $f(z)$ convex in $\mathbb{U}$?"

Theorem 3. *$f(z) \in \mathcal{T}$ and $\operatorname{Re} f'(z) > 0$, then the function*

$$(3.6) \quad F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1)$$

belongs to $\mathcal{K}(-c)$ for all c ($-1 < c \leq 0$).

Proof. By differentiating (3.6), we have

$$F'(z) + \frac{1}{c+1} z F''(z) = f'(z).$$

Therefore,

$$\operatorname{Re} \left\{ F'(z) + \frac{1}{c+1} z F''(z) \right\} = \operatorname{Re} f'(z) > 0$$

and $\frac{1}{c+1} \geq 1$ ($-1 < c \leq 0$). Using Theorem A, we have

$$1 + \operatorname{Re} \left\{ \frac{z F''(z)}{F'(z)} \right\} > -c \quad (0 \leq -c < 1).$$

That is, $F(z) \in \mathcal{K}(-c)$. □

Putting $c = 0$ in Theorem 3, we have

Corollary 2. *If $f(z) \in \mathcal{T}$ and $\operatorname{Re} f'(z) > 0$, then the function*

$$g(z) = \int_0^z \frac{f(t)}{t} dt$$

belongs to $\mathcal{K}$, that is, $g(z) \in \mathcal{K}$.

To prove our next result, we prepare the following lemma due to Owa and Nunokawa [10].

Lemma 3. [10] *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$, $p'(0) = \dots = p^{(n-1)}(0) = 0$. If*

$$\operatorname{Re}\{p(z) + \alpha zp'(z)\} > \beta \quad (z \in \mathbb{U}),$$

then

$$\operatorname{Re}\{p(z)\} > \beta + (1 - \beta) \left\{ 2 \int_0^1 \frac{1}{1 + \rho^{n\operatorname{Re}(\alpha)}} d\rho - 1 \right\} \quad (z \in \mathbb{U})$$

where $\alpha \neq 0$, $\operatorname{Re}(\alpha) \geq 0$ and $\beta < 1$.

Letting $\beta = 0$, $n = 1$ in Lemma 3, and applying Theorem 3, we can prove next Theorem.

Theorem 4. *If $f(z) \in \mathcal{T}$ and $\operatorname{Re}f'(z) > 0$, let the function $F(z)$ given by (3.6), then we have*

$$\operatorname{Re}F'(z) > 2 \int_0^1 \frac{1}{1 + \rho^{\frac{1}{c+1}}} d\rho - 1 > 0.$$

Putting $c = 0$ in Theorem 4, we have

Corollary 3. *If $f(z) \in \mathcal{T}$ and $\operatorname{Re}f'(z) > 0$, and let the function*

$$g(z) = \int_0^z \frac{f(t)}{t} dt,$$

then we have

$$\operatorname{Re}g'(z) > 2 \log 2 - 1.$$

Letting $c = 1$ in Theorem 4, we can get

Corollary 4. *If $f(z) \in \mathcal{T}$ and $\operatorname{Re}f'(z) > 0$, and let the function*

$$s(z) = \frac{2}{z} \int_0^z f(t) dt,$$

then we have

$$\operatorname{Re}s'(z) > 3 - 4 \log 2.$$

References

- [1] S. D. Bernardi, *The radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. **24**(1970), 312 - 318.

- [2] P. N. Chichra, *New subclasses of the class of close-to-convex functions*, Proc. Amer. Math. Soc. **62**(1977), 37 - 43.
- [3] J. Krzyz, *A counterexample concerning univalent functions*, Mat. Fiz. Chem. **2**(1962), 57 - 58.
- [4] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16**(1965), 755 - 758.
- [5] P. T. Mocanu, *New extensions of a theorem R. Singh and S. Singh*, Mathematica (Cluj). **37**(60)(1995), 171 - 182.
- [6] P. T. Mocanu, *On starlikeness of Libera transform*, Mathematica (Cluj). **28**(51)(1986), 153-155.
- [7] M. Nunokawa, *On properties of Non-Caratheodory functions*, Proc. Japan. Acad. Ser. A **68**(1992), 152 - 153.
- [8] M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan. Acad. Ser. A **69**(1993), 234 - 237.
- [9] M. Nunokawa, S. Owa, J. Nishiwaki and H. Saitoh, *Sufficient conditions for starlikeness and convexity of analytic functions with real coefficients*, (to appear) Southeast Asian Bull. Math.
- [10] S. Owa and M. Nunokawa, *Applications of subordination theorem*, Jour. Math. Anal and Appl. **188**(1)(1994), 219 - 226.
- [11] G. S. Salagean, *Sufficient conditions for starlikeness and convexity of analytic functions with negative coefficients*, Mathematica. Tome. **47**(70), 2(2005), 209 - 216.
- [12] S. Singh and R. Singh, *Starlikeness of close-to-convex functions*, Indian J. pure appl. Math. **13**(2)(1982), 190 - 194.
- [13] R. Singh and S. Singh, *Convolution properties of a class of starlike functions*, Proc. Amer. Math. Soc. **106**(1)(1989), 145 - 152.